GENERALIZED GREEN FUNCTIONS
AND UNIPOTENT CLASSES
FOR FINITE REDUCTIVE GROUPS, I

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To George Lusztig on his sixtieth birthday

Abstract. The algorithm of computing generalized Green functions of a reductive group $G$ contains some unknown scalars occurring from the $\mathbb{F}_q$-structure of irreducible local systems on unipotent classes of $G$. In this paper, we determine such scalars in the case where $G = SL_n$ with Frobenius map $F$ of split type or non-split type. In the case where $F$ is of non-split type, we use the theory of graded Hecke algebras due to Lusztig.

§0. Introduction

Let $G$ be a connected reductive group defined over a finite field $\mathbb{F}_q$ with Frobenius map $F$. In [L1], Lusztig classified the irreducible characters of finite reductive groups $G^F$ in the case where the center of $G$ is connected. Later in [L5], he extended his results to the disconnected center case. In the course of the classification, in particular in the connected center case, he defined almost characters of $G^F$, which forms an orthonormal basis of the space $\mathcal{V}(G^F)$ of class functions of $G^F$ different from the basis consisting of irreducible characters. They are defined as explicit linear combinations of irreducible characters, and the transition matrix between these two bases are almost diagonal. So, the determination of the character values of irreducible characters of $G^F$ is equivalent to that of almost characters.

On the other hand, Lusztig founded in [L3] the theory of character sheaves, and showed that the characteristic functions of character sheaves form an orthonormal basis of $\mathcal{V}(G^F)$. He conjectured that those functions coincide, up to scalar, with almost characters (with an appropriate generalization of almost characters if the center is disconnected). Lusztig’s
conjecture was proved by the author in [S3] in the case where the center is connected. It was also proved for certain groups with disconnected center, i.e., for $Sp_{2n}$ and (under a suitable modification for a disconnected group) $O_{2n}$ with $\text{ch} F_q \neq 2$ by Waldspurger [W], for $SL_n$ by the author [S4] (with $\text{ch} F_q$ not too small), and independently, for $SL_n$ and $SU_n$ by Bonnafé [B] (with $q$ not too small).

If Lusztig's conjecture is established, the computation of irreducible characters of $G^F$ is reduced to the computation of characteristic functions of character sheaves, and to the determination of scalars involved in Lusztig's conjecture. In [L3], Lusztig proved that the computation of the characteristic functions of character sheaves are reduced to the computation of generalized Green functions of various reductive subgroups of $G^F$. Then he showed that there exists a general algorithm of computing generalized Green functions. More precisely, he showed that generalized Green functions can be expressed as an explicit linear combination of various characteristic functions $\chi_{C', \mathcal{E}'}$ of the $G$-equivariant local system $\mathcal{E}'$ on a unipotent class $C'$ in $G$. Up to scalar, $\chi_{C', \mathcal{E}'}$ can be easily described in terms of the irreducible character of the component group $A_G(u) = Z_G(u)/Z_G^0(u)$ for $u \in C'^F$ corresponding to $\mathcal{E}'$. However, this scalar depends on the choice of the isomorphism $F^* \mathcal{E} \simeq \mathcal{E}$ for a cuspidal pair $(C, \mathcal{E})$ on a Levi subgroup $L$ of a parabolic subgroup $P$ of $G$, and on the intersection cohomology complex $K$ induced from $\mathcal{E} \boxtimes \mathcal{Q}_l$ on $C \times Z_L^0$ (see (1.2.2)).

The purpose of this paper is to determine these scalars occurring in the computation of generalized Green functions. In the case of Green functions, this problem is equivalent to determining a representative $u \in C'^F$ such that the action of $F$ on the $l$-adic cohomology group $H^m(B_u, \mathcal{Q}_l)$ can be described explicitly, where $B_u$ is the variety of Borel subgroups of $G$ containing $u$, and $m/2 = \dim B_u$. It was shown in [S1], [S2] and [BS] that there exists a unipotent element $u \in C'^F$, in the case where $G^F$ is of split type, and $G$ is not of type $E_8$, such that $F$ acts on $H^m(B_u, \mathcal{Q}_l)$ by a scalar multiplication $q^{m/2}$. Such a unipotent element is called a split element. Even in the remaining cases, the action of $F$ can be described, and by using this, Green functions of exceptional groups ($F_4$, $E_6$, $E_7$ and $E_8$) were computed explicitly by [S1], [BS] for a good characteristic case. The case $G_2$ had been computed by Springer [Spr] in an earlier stage. (Green functions of exceptional groups in certain bad characteristic case were computed by Malle [M] by a direct computation.)

In the case of generalized Green functions, one has to consider the
cohomology group $H_c^m(\mathcal{P}_u, \hat{\mathcal{E}})$, where $\mathcal{P}_u$ is a certain subvariety of parabolic subgroups of $G$ conjugate to $P$, and $\hat{\mathcal{E}}$ is a local system on $\mathcal{P}_u$ determined from the cuspidal pair $(C, \mathcal{E})$ on a Levi subgroup $L$ of $P$, and $m/2 = \dim \mathcal{P}_u$. We need to describe the action of $F$ on such cohomology groups. This problem is reduced to the case where $G$ is simply connected, and simple modulo center. In this paper, we discuss the case where $G = SL_n$ with $F$ of split type or non-split type. In the case where $F$ is of split type, the Frobenius action was determined by investigating the action of $F$ on $H^*_c(B, \mathcal{Q}_l)$ by making use of the Frobenius action in the case of $SL_n$ with $SL_{n-1}$, which is a natural generalization of the method in the case of $GL_n$. In the case of $GL_n$ with $F$ of non-split type, the Frobenius action was determined by investigating the action of $F$ on $H^*_c(B, \mathcal{Q}_l)$ by making use of the Frobenius action in the case of $GL_n$.

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§1. Preliminaries

1.1. Let $G$ be a connected reductive algebraic group over a field $k$, where $k$ is an algebraic closure of a finite field $\mathbb{F}_q$ of characteristic $p$. Let $C$ be a unipotent conjugacy class in $G$, and $\mathcal{E}$ an irreducible local system on $C$ which is $G$-equivariant for the conjugation action. $\mathcal{E}$ is called a cuspidal local system on $C$ if the following condition is satisfied: for any proper parabolic subgroup $P$ of $G$ with Levi decomposition $P = LU_P$ and for any unipotent element $u \in L$, we have $H_c^i(uU_P \cap C, \mathcal{E}) = 0$, where $\delta = \dim C - \dim(\text{class of } u \text{ in } L)$ (cf. [L2, 2.4]). It is known by [L3, V, 23.1], that if $p$ is almost good then the above condition is equivalent to the condition that $H_c^i(uU_P \cap C, \mathcal{E}) = 0$ for any $i$ (i.e., $\mathcal{E}$ is strongly cuspidal). We also say that $(C, \mathcal{E})$ is a cuspidal pair in $G$. 
Let $N_G$ be the set of pairs $(C', \mathcal{E}')$ up to $G$-conjugacy, where $C'$ is a unipotent class in $G$ and $\mathcal{E}'$ is a $G$-equivariant irreducible local system on $C$. We also denote by $M_G$ the set of triples $(L, C, \mathcal{E})$ up to $G$-conjugacy, where $L$ is a Levi subgroup of a parabolic subgroup of $G$, and $\mathcal{E}$ is a cuspidal local system on a unipotent class $C$ of $L$. In [L2, 6.5], Lusztig has shown that there exists a natural bijection

\[(1.1.1) \quad N_G \simeq \prod_{(L, C, \mathcal{E}) \in M_G} (N_G(L)/L)^\wedge,\]

which is called the generalized Springer correspondence between unipotent classes and irreducible characters of various Coxeter groups. (For a finite group $H$, we denote by $H^\wedge$ the set of irreducible characters of $H$.) Note that $N_G(L)/L$ is a Coxeter group with standard generators whenever $(L, C, \mathcal{E}) \in M_G$.

1.2. We describe the generalized Springer correspondence more precisely. Take $(L, C, \mathcal{E}) \in M_G$. Let $Z_0^L$ be the connected center of $L$, and put $
abla = C \cdot (Z_0^L)_{\text{reg}} \subset \nabla = C \cdot Z_0^L$, where

\[(Z_0^L)_{\text{reg}} = \{z \in Z_0^L \mid Z_G^0(z) = L\}.\]

We define a diagram

\[(1.2.1) \quad \nabla \xrightarrow{\alpha_1} \nabla \xrightarrow{\beta_1} \nabla \xrightarrow{\pi} Y,\]

where

\[Y = \bigcup_{x \in G} x\nabla_{\text{reg}}x^{-1} \subset G;\]

\[\nabla = \{(g, xL) \in G \times (G/L) \mid x^{-1}gx \in \nabla_{\text{reg}}\};\]

\[\nabla = \{(g, x) \in G \times G \mid x^{-1}gx \in \nabla\},\]

and

\[\alpha_1(g, x) = x^{-1}gx, \quad \beta_1(g, x) = (g, xL), \quad \pi(g, xL) = g.\]

Then $Y$ is a smooth, irreducible subvariety of $G$, and $\pi$ is a principal covering of $Y$ with group $W = N_G(L)/L$. There is a canonical local system $\nabla$ on $Y$ satisfying the property that $\beta_1^*\nabla = \alpha_1^*(\mathcal{E} \boxtimes \tilde{\mathcal{Q}}_l)$, where $\mathcal{E} \boxtimes \tilde{\mathcal{Q}}_l$ is the inverse image of $\mathcal{E}$ under the natural map $\nabla = C \times Z_0^L \to C$. We define an intersection cohomology complex $K$ by

\[(1.2.2) \quad K = IC(\nabla, \pi_1 \nabla)[\dim Y].\]
and regard it as a perverse sheaf on $G$ by extending by 0 outside of $\overline{\mathcal{Y}}$. Lusztig showed that $K$ is a $G$-equivariant semisimple perverse sheaf on $G$, and that $\text{End} \, K \simeq \mathbb{Q}_l[\mathcal{W}]$. It follows that $K$ can be decomposed as

\begin{equation}
K \simeq \bigoplus_{E \in \mathcal{W}^\wedge} V_E \otimes K_E,
\end{equation}

where $K_E$ is a simple perverse sheaf on $G$ such that $V_E = \text{Hom}(K_E, K)$ is an irreducible $\mathcal{W}$-module corresponding to $E \in \mathcal{W}^\wedge$.

Let $G_{\text{uni}}$ be the unipotent variety of $G$. Then $K[-d]|_{G_{\text{uni}}}$ turns out to be a $G$-equivariant semisimple perverse sheaf on $G_{\text{uni}}$, where $d = \dim Z^0_L = \dim Y - \dim (Y \cap G_{\text{uni}})$. Hence it is decomposed as

\begin{equation}
K[-d]|_{G_{\text{uni}}} = \bigoplus_{(C', \mathcal{E}') \in \mathcal{N}_G} V_{(C', \mathcal{E}')} \otimes \text{IC}(\overline{C'}, \mathcal{E}')[\dim C'],
\end{equation}

where $V_{(C', \mathcal{E}')} \leq \text{dim} C'$ is the multiplicity space for the simple perverse sheaf $\text{IC}(\overline{C'}, \mathcal{E}')[\dim C']$ on $G_{\text{uni}}$. Comparing (1.2.3) with (1.2.4), we see that for each $E \in \mathcal{W}^\wedge$, there exists a pair $(C', \mathcal{E}') \in \mathcal{N}_G$ such that

\begin{equation}
K_E|_{G_{\text{uni}}} \simeq \text{IC}(\overline{C'}, \mathcal{E}')[\dim C' + \dim Z^0_L].
\end{equation}

The correspondence $E \mapsto (C', \mathcal{E}')$ gives a bijection $\coprod_{(L, C, \mathcal{E})} (\mathcal{N}_G(L)/L)^\wedge \to \mathcal{N}_G$ in (1.1.1).

**1.3.** We now consider the $\mathbb{F}_q$-structure on $G$. So assume that $G$ is defined over $\mathbb{F}_q$ with Frobenius endomorphism $F : G \to G$. Then $F$ acts naturally on the set $\mathcal{N}_G$ and $\mathcal{M}_G$ by $(C', \mathcal{E}') \mapsto (F^{-1}C', F^*\mathcal{E}')$, $(L, C, \mathcal{E}) \mapsto (F^{-1}L, F^{-1}C, F^*\mathcal{E})$, and the map in (1.1.1) is compatible with $F$-action. Now assume that $(L, C, \mathcal{E}) \in \mathcal{M}_G$ is $F$-stable. Then we may choose $(L, C, \mathcal{E})$, as a representative of its $G$-conjugacy class, such that $L$ is an $F$-stable Levi subgroup of an $F$-stable parabolic subgroup $P$ of $G$, with $FC = C$, $F^*\mathcal{E} \simeq \mathcal{E}$. We choose an isomorphism $\varphi_0 : F^*\mathcal{E} \simeq \mathcal{E}$ which induces a map of finite order on the stalk of $\mathcal{E}$ at any point of $C_F$. Since the diagram in (1.2.1), and so the construction of the complex $K$ is compatible with $\mathbb{F}_q$-structure, $\varphi_0$ induces a natural isomorphism $\varphi : F^*K \simeq K$. We consider the characteristic function $\chi_{K, \varphi}$ of $K$. The restriction of $\chi_{K, \varphi}$ on $G_{\text{uni}}$ gives a $G^F$-invariant function on $G_{\text{uni}}^F$, which is the generalized Green function $Q_{L, C, \mathcal{E}, \varphi_0}$ (cf. [L3, II]).

Here $F$ acts naturally on $\mathcal{W}$, which induces a Coxeter group automorphism of order, say $c$. We consider the semidirect product $\mathcal{W} = \mathcal{W} \rtimes (\mathbb{Z}/c\mathbb{Z})$. 

https://www.cambridge.org/core/terms. https://doi.org/10.1017/S00277630000009338
If an irreducible representation $V_E$ of $\mathcal{W}$ is $F$-stable, it can be extended to an irreducible representation of $\mathcal{W}$, in $c$ different ways. Assume that $E \in \mathcal{W}^\wedge$ is $F$-stable. Then the corresponding $(C', \mathcal{E}') \in \mathcal{N}_G$ is also $F$-stable, and we have $F^* K_E \simeq K_E$. A choice of an isomorphism $\varphi_E : F^* K_E \simeq K_E$ induces a bijection $\sigma_E : V_E \to V_E$, which makes $V_E$ into an irreducible $\mathcal{W}$-module $\widetilde{V}_E$. We choose $\varphi_E$ so that $\widetilde{V}_E$ turns out to be a preferred extension of $V_E$ (cf. [L3, IV, (17.2)]. By making use of $\varphi_E : F^* K_E \simeq K_E$, we shall define an isomorphism $\psi : F^* \mathcal{E}' \simeq \mathcal{E}'$ as follows; By (1.2.5), we have $\mathcal{H}^{a_0}(K_E)|_{C'} = \mathcal{E}'$ for $a_0 = -\dim Z_L^0 - \dim C'$. We define $\psi$ so that $q^{(a_0+r)/2}\psi$ corresponds to the map defined by $\varphi_E : F^* \mathcal{H}^{a_0}(K_E) \simeq \mathcal{H}^{a_0}(K_E)$, where

$$r = \dim Y = \dim G - \dim L + \dim (C \times Z_L^0),$$

and so

$$a_0 + r = (\dim G - \dim C') - (\dim L - \dim C').$$

We define a function $Y_j$ on $G^F_{\text{uni}}$ for each $j = (C', \mathcal{E}') \in \mathcal{N}_G^F$ by

$$Y_j(g) = \begin{cases} \text{Tr}(\psi, g) & \text{if } g \in C'^F, \\ 0 & \text{if } g \notin C'^F. \end{cases}$$

Then $\{Y_j \mid j \in \mathcal{N}_G^F\}$ gives rise to a basis of the space of $G^F$-invariant functions on $G^F_{\text{uni}}$. Now the computation of $\chi_{K, \varphi}$ is reduced to the computation of $\chi_{K_E, \varphi_E}$ for each $F$-stable irreducible character $\mathcal{E}$ of $\mathcal{W}$. We denote $\chi_{K_E, \varphi_E}$ by $X_j$ if $E$ corresponds to $j = (C', \mathcal{E}')$ under the generalized Springer correspondence. In [L3, V], Luzitg gave a general algorithm of expressing $X_i$ as an explicit linear combination of various $Y_j$. Thus the computation of $\chi_{K, \varphi}$ is reduced to the computation of $Y_j$.

We shall describe the functions $Y_j$. Let us choose $u \in C'^F$, and put $A_G(u) = Z_G(u)/Z_G^0(u)$. Then $F$ acts naturally on $A_G(u)$, and the set of $G$-equivariant simple local systems on $C'$ is in bijective correspondence with the set of $F$-stable irreducible characters of $A_G(u)$. Let us denote by $\rho$ the irreducible character of $A_G(u)$ corresponding to $\mathcal{E}'$. Let $\sigma$ be the restriction of $F$ on $A_G(u)$. Then $\rho$ can be extended to an irreducible character of the semidirect product $\widetilde{A}_G(u) = A_G(u) \rtimes \langle \sigma \rangle$. We choose an extension $\widetilde{\rho}$ of $\rho$. $\mathcal{E}'_u$ has a structure of $A_G(u)$-module affording the character $\rho$, which is extended to the $\widetilde{A}_G(u)$-module affording $\widetilde{\rho}$. We choose an isomorphism $\psi_0 : F^* \mathcal{E}' \simeq \mathcal{E}'$ by the condition that $\psi_0$ induces an isomorphism on $\mathcal{E}'_u$ corresponding to the action of $\sigma$ on $\widetilde{\rho}$. 


Since $\mathcal{E}'$ is a simple local system, there exists $\gamma \in \mathcal{Q}_\mathcal{E}$ (depending on the choice of $\varphi_0$, $u$ and $\tilde{\rho}$) such that $\psi = \gamma \psi_0$. We define functions $Y_j^0$ on the set $G^F_{\text{uni}}$ in a similar way as $Y_j$, but replacing $\psi$ by $\psi_0$. Then clearly we have $Y_j = \gamma Y_j^0$. We note that the functions $Y_j^0$ are described in an explicit way as follows. The set of $G^F$-conjugacy classes in $C^F_0$ is in bijective correspondence with the set of $F$-twisted conjugacy classes in $A_G(u)$. We denote by $u_\alpha$ a representative in the $G^F$-conjugacy class contained in $C^F_0$ corresponding to an $F$-twisted conjugacy class in $A_G(u)$ containing $a$. Then we have

$$Y_j^0(g) = \begin{cases} \tilde{\rho}(a\sigma) & \text{if } g \text{ is } G^F\text{-conjugate to } u_\alpha, \\ 0 & \text{if } g \notin C^F. \end{cases}$$

It follows from the above discussion that the computation of generalized Green functions is reduced to the determination of the scalar constant for each pair $(C', \mathcal{E}') \in \mathcal{N}^F_C$. Let us choose $v \in C^F$, and let $\rho_0$ be the $F$-stable irreducible character of $A_L(v)$ corresponding to $\mathcal{E}$. Then as in the discussion above, the isomorphism $\varphi_0 : F^* \mathcal{E} \cong \mathcal{E}$ is given by choosing an extension $\tilde{\rho}_0$ of $\rho_0$ to the semidirect product $\tilde{A}_L(v) = A_L(v) \rtimes \langle \sigma \rangle$. Thus $\gamma$ is determined by $v$, $\tilde{\rho}_0$, $u$, $\tilde{\rho}$, which we denote by $\gamma = \gamma(v, \tilde{\rho}_0, u, \tilde{\rho})$. The purpose of this paper is to describe the constants $\gamma(v, \tilde{\rho}_0, u, \tilde{\rho})$ explicitly.

1.4. In order to make the Frobenius action more explicit, we shall consider the following varieties. Put

$$(1.4.1) \quad \mathcal{P}_u = \{gP \in G/P \mid g^{-1}ug \in CU_P\},$$

$$\mathcal{P}_u = \{g \in G \mid g^{-1}ug \in CU_P\},$$

and consider the diagram

$$(1.4.2) \quad C \quad \xleftarrow{\alpha} \quad \mathcal{P}_u \quad \xrightarrow{\beta} \quad \mathcal{P}_u$$

with

$$\alpha : g \mapsto C\text{-component of } g^{-1}ug \in CU_P, \quad \beta : g \mapsto gP.$$ 

We define a local system $\mathcal{E}$ on $\mathcal{P}_u$ by the property that $\alpha^* \mathcal{E} = \beta^* \mathcal{E}$. Then it is known by [L3, 24.2.5] that

$$(1.4.3) \quad \mathcal{H}_u^{a_0}(K) \simeq H_\mathcal{E}^{a_0 + r}(\mathcal{P}_u, \mathcal{E}).$$

It is also known by [L2, 1.2 (b)] that $\dim \mathcal{P}_u \leq (a_0 + r)/2$. Since the left hand side of (1.4.3) is non-zero by (1.2.5), we see that

$$(1.4.4) \quad \dim \mathcal{P}_u = (a_0 + r)/2.$$
Since $P$ is $F$-stable, $\mathcal{P}_u, \hat{\mathcal{P}}_u$ are $F$-stable, and the diagram in (1.4.2) is compatible with Frobenius maps. Moreover, the isomorphism $\varphi_0$ induces an isomorphism $\varphi_0 : F^* \hat{\mathcal{E}} \cong \hat{\mathcal{E}}$. This induces a linear map $\Phi$ on $V = H^F_{c_0} a_r(\mathcal{P}_u, \hat{\mathcal{E}})$. By (1.4.3), $W$ acts on $V$. Also $Z_G(u) \rho$ acts naturally on $V$, where $Z_G(u)$ acts trivially on it. Then it induces an action of $A_G(u)$, which commutes with the action of $W$. Let $\rho$ be an $F$-stable irreducible character of $A_G(u)$ corresponding to $\mathcal{E}'$ as in 1.3, and $V_{\rho}$ the $\rho$-isotypic part of $V$. Then $\Phi$ leaves $V_{\rho}$ stable. The previous discussion shows that $V_{\rho}$ can be identified with $\hat{V}_E \otimes \mathcal{E}'_u$, and $\Phi|_{V_{\rho}}$ coincides with $\sigma_E \otimes q^{(q_0+r)/2}$. Thus the map $\psi$ can be described by investigating $\Phi$ on $H^F_{c_0+1}(\mathcal{P}_u, \hat{\mathcal{E}})_{\rho}$.

1.5. We show that the description of the mixed structure $\psi : F^* \mathcal{E}' \to \mathcal{E}'$ on $C'$ is reduced to the case where $G$ is simply connected, almost simple. In fact, let $\pi : G \to G' = G/Z_G^0$ be the natural homomorphism. Then $\pi$ induces a bijection between $\mathcal{M}_G$ (resp. $N_G$) and $\mathcal{M}_{G'}$ (resp. $N_{G'}$) which commutes with their $\mathbb{F}_q$-structures. Hence we may assume that $G$ is semisimple. Let $\tilde{\pi} : \tilde{G} \to G$ be the simply connected covering of $G$. Then $(L, C, \mathcal{E}) \mapsto (\tilde{\pi}^{-1}(L), C, \tilde{\pi}^* \mathcal{E})$ gives a bijection between the set $\mathcal{M}_G$ and the subset of $\mathcal{M}_{\tilde{G}}$ on which ker $\tilde{\pi}$ acts trivially. Hence the mixed structure $\varphi_0 : F^* \tilde{\pi}^* \mathcal{E} \to \tilde{\pi}^* \mathcal{E}$ for the pair $(C, \tilde{\pi}^* \mathcal{E})$ on $\tilde{G}$ determines the mixed structure for the pair $(C, \mathcal{E})$ on $G$. Similarly, $\tilde{\pi}$ induces a bijection between the set $N_G$ and the subset of $N_{\tilde{G}}$ on which ker $\tilde{\pi}$ acts trivially, and so the mixed structure of the pair $(C', \mathcal{E}')$ on $G$ is determined by the mixed structure of the pair $(C', \tilde{\pi}^* \mathcal{E}')$ on $G$. The procedure of determining the mixed structure of $(C', \mathcal{E}')$ from that of $(C, \mathcal{E})$ is parallel for $G$.

It follows from the above discussion that we may assume $G$ is simply connected, semisimple. Then $G$ is isomorphic to the direct product of simply connected, almost simple groups, with $F$-action. Now it is easy to see that we are reduced to the case where $G \simeq G_1 \times \cdots \times G_r$, with $G_i$ a copy of $G_1$, and $F$ acts on $G$ as a cyclic permutation of all the factors. Then $G_1$ is $F^r$-stable, and the set $\mathcal{M}_{G_1}^F$ is in bijective correspondence with the set $\mathcal{M}_{G_1}^{F'}$, via the correspondence $(L, C, \mathcal{E}) \leftrightarrow (L_1, C_1, \mathcal{E}_1)$, where

\[ L = L_1 \times F^{-r+1}(L_1) \times \cdots \times F^{-1}(L_1), \]
\[ C = C_1 \times F^{-r+1}(C_1) \times \cdots \times F^{-1}(C_1), \]
\[ \mathcal{E} = \mathcal{E}_1 \boxtimes F^{-r-1} \mathcal{E}_1 \boxtimes \cdots \boxtimes F^r \mathcal{E}_1. \]

Moreover, $C_1^{F'} \simeq C^F$ via $v_1 \mapsto v = (v_1, F(v_1), \ldots, F^{r-1}(v_1))$. Then $\varphi_0 : \mathcal{E}' \mapsto \mathcal{E}'$
$F^*\mathcal{E} \cong \mathcal{E}$ is determined by $\varphi_1 : F^r*\mathcal{E}_1 \cong \mathcal{E}_1$ as

$$(\varphi_0)_v = (\varphi_1)_{v_1} \otimes (\varphi_1)^{F_{r-1}(v_1)} \otimes \cdots \otimes (\varphi_1)^{F(v_1)}$$
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on $\mathcal{E}_v = (\mathcal{E}_1)_{v_1} \otimes (\mathcal{E}_1)^{F_{r-1}(v_1)} \otimes \cdots \otimes (\mathcal{E}_1)^{F(v_1)}$. Similarly, the mixed $F$-structure of $(C', \mathcal{E}') \in \mathcal{N}_G$ is described by the mixed $F^r$-structure of $(C'_1, \mathcal{E}'_1) \in \mathcal{N}_{G_1}$.

Thus, the determination of the mixed structure of $(C', \mathcal{E}')$ is reduced to the case where $G$ is an $F$-stable, simply connected, almost simple group.

1.6. Assume that $G$ is almost simple and simply connected. Let $\mathfrak{g} = \text{Lie} G$ be the Lie algebra of $G$. We further assume that $p$ is good for $G$ unless $G$ is of type $A$, and that $p > n$ if $G = SL_n$. Then by [BR], there exists a logarithm map $\log : G \to \mathfrak{g}$ satisfying the following properties; $\log$ is an $\text{Ad}(G)$-equivariant morphism and $\log(1) = 0$, $d(\log)_1 : \mathfrak{g} \to \mathfrak{g}$ is the identity map. In particular, for any closed subgroup $H$ of $G$, $\log(H) \subset \text{Lie} H \subset \mathfrak{g}$. Moreover, $\log|_{\mathfrak{g}_{\text{uni}}}$ turns out to be an isomorphism $\mathfrak{g}_{\text{uni}} \to \mathfrak{g}_{\text{nil}}$, where $\mathfrak{g}_{\text{nil}}$ is the nilpotent variety of $\mathfrak{g}$.

Let $\mathcal{L}$ be an irreducible $G$-local system on a nilpotent orbit $\mathcal{C}$ in $\mathfrak{g}$. The notion of cuspidal local system on $\mathcal{C}$ is defined in a similar way as in the case of groups, i.e., $\mathcal{L}$ is said to be cuspidal or $(\mathcal{C}, \mathcal{L})$ is a cuspidal pair if for any proper parabolic subalgebra $\mathfrak{p}_1$ of $\mathfrak{g}$ with nilpotent radical $\mathfrak{n}_1$ and any $y \in \mathfrak{p}_1$, we have $H^i_c((y + \mathfrak{n}_1) \cap \mathcal{C}, \mathcal{L}) = 0$ for any $i$. Then it is easily checked (cf. [L4]) that $\log^*$ gives a bijection between the set of cuspidal pairs in $G$ and the set of cuspidal pairs in $\mathfrak{g}$.

Let $(L, C, \mathcal{E}) \in \mathcal{M}_G$, and $(\mathcal{C}, \mathcal{L})$ the corresponding cuspidal pair in $L = \text{Lie} L$, where $C = \log^{-1}(C)$, $\mathcal{E} = \log^* \mathcal{L}$. We put $\mathfrak{p} = \text{Lie}(P)$ and $\mathfrak{n}_P = \text{Lie} U_P$. Let $\mathcal{C}' = \log(C')$ be a nilpotent orbit in $\mathfrak{g}$. For each $y \in \mathcal{C}'$, put

$$(1.6.1) \quad \mathcal{P}_y = \{gP \in G/P \mid \text{Ad}(g)^{-1}y \in C + \mathfrak{n}_P\},$$

$$\hat{\mathcal{P}}_y = \{g \in G \mid \text{Ad}(g)^{-1}y \in C + \mathfrak{n}_P\}.$$ 

Then by using a similar diagram as in (1.4.2), one can define a local system $\hat{\mathcal{L}}$ on $\mathcal{P}_y$. It is easy to see that $\log$ gives an isomorphism $\hat{\mathcal{P}}_u \cong \hat{\mathcal{P}}_y$ with $y = \log(u)$, and so induces an isomorphism $\mathcal{P}_u \cong \mathcal{P}_y$. Then we have $\log^* \hat{\mathcal{L}} = \hat{\mathcal{E}}$. It follows that we have a canonical isomorphism

$$(1.6.2) \quad H^{a_0+r}_c(\mathcal{P}_u, \hat{\mathcal{E}}) \cong H^{a_0+r}_c(\mathcal{P}_y, \hat{\mathcal{L}}).$$
In the case where $G$ has an $\mathbb{F}_q$-structure with Frobenius map $F$, $\mathfrak{g}$ has also an action of $F$, and we may assume that log is $F$-equivariant. Then the isomorphism (1.6.2) is compatible with $\mathbb{F}_q$-structures. We denote by the same symbol $\Phi$ the linear map on $H_c^{a_0+r}(\mathcal{P}_y, \mathcal{L})$ obtained as in the case of $H_c^{a_0+r}(\mathcal{P}_u, \mathcal{L})$. Hence the linear map $q^{(a_0+r)/2} \psi$ on $\mathcal{E}_u$ can be described in terms of the Frobenius action $\Phi$ on $H_c^{a_0+r}(\mathcal{P}_y, \mathcal{L})$.

§2. Graded Hecke algebras

2.1. The graded Hecke algebra $H$ was introduced by Lusztig [L7], which is a degenerate version of affine Hecke algebras. In this section, following [L7] we review the definition of $H$ and its representations on equivariant $K$-homology groups. In [L7], $H$ is constructed as an algebra over $\mathbb{C}$, but here we regard it as the algebra over $\mathbb{Q}_l$ so that one can relate it to $l$-adic cohomology groups.

Let $\Phi$ be a root system with a set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_m\}$ and $W$ the Weyl group of $\Phi$ with corresponding simple reflections $\{s_1, \ldots, s_m\}$. We assume that the root lattice $\mathbb{Z}\Phi$ is embedded in a vector space $\mathfrak{h}^*$ over $\mathbb{Q}_l$. The action of $W$ on $\mathbb{Z}\Phi$ makes $\mathfrak{h}^*$ into a $W$-module. (Hence $\mathfrak{h}^*$ has a direct sum decomposition, one summand being $W$-invariant, the other having $\Pi$ as a basis.) Let $S$ be the symmetric algebra of $\mathfrak{h}^* \otimes \mathbb{Q}_l$. We denote $r = (0,1) \in \mathfrak{h}^* \otimes \mathbb{Q}_l$, so that $S = S(\mathfrak{h}^*) \otimes \mathbb{Q}_l[r]$. $W$ acts naturally on $S$ so that $r$ is left invariant by $W$. We denote by $\xi \mapsto \omega \xi$ the action of $W$ on $S$. Let $c_1, \ldots, c_m$ be integers $\geq 2$ such that $c_i = c_j$ whenever $s_i$ and $s_j$ are conjugate in $W$. Let $e$ be the neutral element of $W$. Lusztig showed in [L7, Theorem 6.3] that there is a unique structure of associative $\mathbb{Q}_l$-algebra on the $\mathbb{Q}_l$-vector space $H = S \otimes \mathbb{Q}_l[W]$ with unit $1 \otimes e$ such that

(i) $\xi \mapsto \xi \otimes e$ is an algebra homomorphism $S \rightarrow H$,

(ii) $w \mapsto 1 \otimes w$ is an algebra homomorphism $\mathbb{Q}_l[W] \rightarrow H$,

(iii) $(\xi \otimes e) \cdot (1 \otimes w) = \xi \otimes w$, $(\xi \in S, w \in W)$,

(iv) $(1 \otimes s_i)(\xi \otimes e) - (s_i \xi \otimes e)(1 \otimes s_i) = c_i r\frac{\xi - s_i \xi}{\alpha_i} \otimes e$, $(\xi \in S, 1 \leq i \leq m)$.

$H$ is called a graded Hecke algebra attached to $W$ with parameters $c_i$. It follows from (iv) that $r$ is in the center of $H$.

2.2. The discussion in [L7] is concerned with algebraic groups over $\mathbb{C}$. Hence the equivariant $K$-homology is defined for the varieties over $\mathbb{C}$. Since
we treat algebraic groups over finite fields, we need to construct the equivariant $K$-homology based on the $l$-adic cohomology groups. Fortunately, the basic properties established in Section 1 in [L7] work well also for our situation, by a suitable modification. We give some comments below.

Let $G$ be an affine algebraic group over $k$, and let $X$ be a $k$-variety on which $G$ acts algebraically. As in [L7], for each integer $m \geq 1$, there exists a smooth irreducible variety $\Gamma$ with free $G$-action such that $\Gamma \to G \setminus \Gamma$ has a locally trivial principal $G$-fibration, and that $H^i(\Gamma, \mathbb{Q}_l) = 0$ for $i = 1, \ldots, m$. (As in [L7, 1.1], we embed $G$ as a closed subgroup of $GL_r$, and consider the embedding

$$
(2.2.1) \quad G \subset GL_r \times \{e\} \subset GL_r \times GL_{r'} \subset GL_{r+r'}.
$$

Then $\Gamma = (\{e\} \times GL_{r'}) \setminus GL_{r'+r'}$ for large $r'$ ($2r' \geq m + 2$), with the left action of $G$ on $\Gamma$, satisfies the required condition.) For a $G$-variety $X$, we consider $\Gamma X = G \setminus (\Gamma \times X)$ (the quotient by the diagonal action of $G$). Then for an $G$-equivariant local system $\mathcal{L}$ on $X$, there exists a unique local system $\mathcal{L}$ on $\Gamma X$ such that $\pi^*(\mathcal{L}) = p^*\mathcal{L}$, where $\pi : \Gamma \times X \to G \setminus (\Gamma \times X)$ is a natural map, and $p : \Gamma \times X \to X$ is a projection. Then as in [L7], we define

$$
H^j_G(X, \mathcal{L}) = H^j(\Gamma X, \mathcal{L}), \quad H^j_G(X, \mathcal{L}) = H^{2d-j}_e(\Gamma X, \mathcal{L}^*)^*,
$$

where $d = \dim(\Gamma X)$, and the upper-script $^*$ denotes the dual local system or the dual vector space. (We understand that $H^j_G(X, \mathcal{L}) = H^j(X, \mathcal{L})$ and $H^j_G(X, \mathcal{L}) = H^{2d-j}_e(\Gamma X, \mathcal{L}^*)^*$ in the case where $G = \{e\}$.) We write them as $H^j_G(X)$, $H^j_G(X)$ if $\mathcal{L}$ is a constant sheaf $\mathbb{Q}_l$. Also we write $H^j_G(X)$, $H^i(X)$ instead of $H^j_G(X, \mathbb{Q}_l)$, $H^i(X, \mathbb{Q}_l)$.

By cup-product, $H^*_G(X) = \bigoplus j H^j_G(X)$ becomes a graded $\mathbb{Q}_l$-algebra with 1, and

$$
H^*_G(X, \mathcal{L}) = \bigoplus j H^j_G(X, \mathcal{L}), \quad H^*_G(X, \mathcal{L}) = \bigoplus j H^j_G(X, \mathcal{L})
$$

become graded $H^*_G(X)$-modules.

We write $H^*_G, H^*_G$ instead of $H^*_G(\text{point}), H^*_G(\text{point})$. Then the map $X \to \text{point}$ defines a $\mathbb{Q}_l$-algebra homomorphism $\varepsilon : H^*_G \to H^*_G(X)$ preserving the grading. Via the map $\varepsilon$, $H^*_G(X, \mathcal{L}), H^*_G(X, \mathcal{L})$ can be regarded also as $H^*_G$-modules.
2.3. Let $T$ be a torus and $X(T)$ be its character group. The arguments in 1.10 in [L7] do not hold in that form. We modify them as follows. In the case where $T \cong G_m$ is the one dimensional torus, it can be verified directly by the definition that $H^*_T \cong \mathbb{Q}_l[x]$, a polynomial ring with one variable, with $x \in H^2_T$. Since $H^{*}_{G \times G'} \cong H^*_G \otimes H^*_{G'}$, we see that $H^*_T \cong S(V^*)$, the symmetric algebra of a $\mathbb{Q}_l$-vector space $V^* = \mathbb{Q}_l \otimes \mathbb{Z} X(T)$. In particular, we have

$$H^{2j}_T \cong S^j(V^*), \quad H^{2j+1}_T = 0,$$

and we may identify $H^*_T$ with $V^*$. ($S^j(V^*)$ denotes the degree $j$-part of $S(V^*)$.)

For $\chi \in X(T)$, let $k_\chi$ be the $T$-module $k$ with the $T$-action by $(t, z) \mapsto \chi(t)z$. Let $i : \{0\} \hookrightarrow k$, $\pi : k \rightarrow \{0\}$ be the obvious maps. Then $\pi^*$ is an isomorphism, and the composition

$$H^*_T(\{0\}) \xrightarrow{i_!} H^*_T(k_\chi) \xrightarrow{\pi^*-1} H^*_T(\{0\})$$

is $H^*_T$-linear of degree 2. Since $H^*_T(\{0\}) \cong H^*_T$, as $H^*_T$-modules, $(\pi^*)^{-1} \circ i_!$ is given by multiplication by an element $c(\chi) \in H^2_T$ (cf. [L7, 1.10]). The map $c : X(T) \rightarrow H^2_T = V^*$, $\chi \mapsto c(\chi)$ gives an injective group homomorphism.

Assume that $G$ is an algebraic group such that $G^0$ is a torus $T$. Then $W = G/G^0$ acts naturally on $H^*_T$, preserving the grading (see [L7, 1.9]). $W$ acts also on $X(T)$, and we have

(2.3.1) The map $c : X(T) \rightarrow H^2_T = V^*$ is $W$-equivariant.

In fact, take $\Gamma$ on which $G$ acts freely. Then, for a representative $\tilde{w} \in G$ of $w \in W$, the map $\Gamma \times k_\chi \rightarrow \Gamma \times k_{w(\chi)}$, $(g, x) \mapsto (\tilde{w}g, x)$ induces a map $f_w : T \backslash (\Gamma \times k_\chi) \rightarrow T \backslash (\Gamma \times k_{w(\chi)})$, which makes the following diagram commutative.

$$
\begin{array}{ccc}
H^*_T(\{0\}) & \xrightarrow{i_!} & H^*_T(k_\chi) \\
\downarrow{w} & & \downarrow{(f_w^*)^{-1}} \\
H^*_T(\{0\}) & \xrightarrow{i_!} & H^*_T(k_{w(\chi)}) \\
\end{array}
\begin{array}{ccc}
& & \\
& & \\
H^*_T(\{0\}) & \xrightarrow{i_!} & H^*_T(\{0\}) \\
\end{array}
\begin{array}{ccc}
& & \\
& & \\
H^*_T(k_\chi) & \xrightarrow{\pi^*-1} & H^*_T(k_{w(\chi)}) \\
\downarrow{(f_w^*)^{-1}} & & \downarrow{(\pi^*)^{-1}} \\
H^*_T(\{0\}) & \xrightarrow{i_!} & H^*_T(\{0\}) \\
\end{array}
$$

(2.3.1) follows from this.

It follows from (2.3.1) that we have

(2.3.2) $H^*_T \cong S(\mathbb{Q}_l \otimes \mathbb{Z} X(T))$
as graded $W$-modules.

We don’t know whether the counter part of 1.11 in [L7] holds in our setting. However, the following related fact holds.

**Lemma 2.4.** Assume that $G$ is a connected algebraic group. Let $G_r$ be a maximal reductive subgroup of $G$, and $T$ a maximal torus of $G_r$. Let $W = N_{G_r}(T)/T$ be the Weyl group of $G_r$. Then $W$ acts naturally on $H^*_T$, and the natural map $H^*_G \rightarrow H^*_T$ (cf. [L7, 1.4 (g)]) induced from the inclusion $T \hookrightarrow G$ gives an isomorphism

$$H^*_G \sim \rightarrow (H^*_T)^W.$$

**Proof.** By [L7, 1.4 (h)], we know that $H^*_G \sim \rightarrow H^*_G$. Hence it is enough to show the lemma in the case where $G$ is reductive. Assume that $G = G_r$. Let $m$ be a large integer and let $\Gamma$ be an irreducible, smooth variety with a free $G$-action such that $H^i(\Gamma) = 0$ for $1 \leq i \leq m$. We consider the map $f : T\backslash \Gamma \rightarrow G\backslash \Gamma$, which is a locally trivial fibration with fibre isomorphic to $T\backslash G$. We have a spectral sequence

$$H^p(G \backslash \Gamma, R^q f_\ast \mathbb{Q}_l) \Rightarrow H^{p+q}(T\backslash \Gamma). \quad (2.4.1)$$

The map $f$ is $W$-equivariant with respect to the trivial action of $W$ on $G\backslash \Gamma$, and the left action of $W$ on $T\backslash \Gamma$, and so $R^q f_\ast \mathbb{Q}_l$ has a structure of $W$-sheaf, which induces an action of $W$ on $H^p(G \backslash \Gamma, R^q f_\ast \mathbb{Q}_l)$. $W$ acts naturally on $H^{p+q}(T\backslash \Gamma)$, and by taking the $W$-invariant parts in (2.4.1), we have a spectral sequence

$$H^p(G \backslash \Gamma, R^q f_\ast \mathbb{Q}_l)^W \Rightarrow H^{p+q}(T\backslash \Gamma)^W. \quad (2.4.2)$$

Since $f$ is a locally trivial fibration, $R^q f_\ast \mathbb{Q}_l$ is a local system with fibre $H^q(T\backslash \Gamma)$. We may assume that $\Gamma = (\{e\} \times GL_{r'})\backslash GL_{r+r'}$ as in 2.2. Then $f$ is $GL_{r+r'}$-equivariant, and so $R^q f_\ast \mathbb{Q}_l$ is a $GL_{r+r'}$-local system on the space $G\backslash \Gamma$ (with respect to the right action of $GL_{r+r'}$). $GL_{r+r'}$ acts transitively on $G\backslash \Gamma$ with a stabilizer of a point isomorphic to $G \times GL_{r'}$. Since $G$ is connected, we see that $R^q f_\ast \mathbb{Q}_l$ is a constant sheaf $H^q(T\backslash \Gamma)$. It follows that

$$H^p(G \backslash \Gamma, R^q f_\ast \mathbb{Q}_l) \sim H^p(G \backslash \Gamma) \otimes H^q(T\backslash G) \sim$$

and we have

$$H^p(G \backslash \Gamma, R^q f_\ast \mathbb{Q}_l)^W \sim H^p(G \backslash \Gamma) \otimes H^q(T\backslash G)^W.$$
since $W$ acts trivially on $H^p(G\backslash \Gamma)$. It is known that $H^*(T\backslash G)$ is a graded regular $W$-module, and

$$H^q(T\backslash G)^W = \begin{cases} \bar{Q}_l & \text{if } q = 0, \\ 0 & \text{otherwise}. \end{cases}$$

Hence the spectral sequence (2.4.2) collapses, and we have

(2.4.3) \[ H^p(G\backslash \Gamma) \simeq H^p(T\backslash \Gamma)^W. \]

This isomorphism is induced from the natural map $H^p(G\backslash \Gamma) \rightarrow H^p(T\backslash \Gamma)$. Since $H^p_G = H^p(G\backslash \Gamma)$, and $H^p_T = H^p(T\backslash \Gamma)$ by definition, the lemma follows from (2.4.3). \hfill \Box

For later discussion, we note the following.

**Corollary 2.5.** Assume that $G$ is connected reductive, and let $T$, $W$ be as before. Let $L$ be a Levi subgroup of a parabolic subgroup of $G$ containing $T$. Assume further that $L$ contains a cuspidal pair as in 1.1. Put $\mathcal{W} = N_G(Z_L^0)/L = N_G(L)/L$. Then the image of the natural map $H^*_Z \rightarrow H^*_Z \mathcal{W}$ coincides with $(H^*_Z)^\mathcal{W}$.

**Proof.** The inclusions $Z_L^0 \hookrightarrow T \hookrightarrow G$ induces the maps $H^*_Z \rightarrow H^*_T \rightarrow H^*_Z$. Put $V^* = \bar{Q}_l \otimes \mathbb{Z} X(T), V^*_1 = \bar{Q}_l \otimes \mathbb{Z} X(Z_L^0)$. Then by (2.3.2), the map $H^*_T \rightarrow H^*_Z$ is nothing but the natural map $\varphi : S(V^*) \rightarrow S(V^*_1)$ obtained from the restriction map $X(T) \rightarrow X(Z_L^0)$. Now $W$, $\mathcal{W}$ acts naturally on $S(V^*)$, $S(V^*_1)$, respectively. Since $\mathcal{W} \simeq N_W(W_L)/W_L$, $\varphi$ induces a map $\bar{\varphi} : S(V^*)^\mathcal{W} \rightarrow S(V^*_1)^\mathcal{W}$. By [L7, Proposition 2.6], $Z_L^0$ coincides with a maximal torus of a certain connected reductive subgroup $H$ of $G$, and $\mathcal{W}$ is regarded as the Weyl group of $H$. Thus in view of Lemma 2.4, it is enough to show that $\bar{\varphi}$ is surjective. This is equivalent to the fact that $V_1/\mathcal{W} \rightarrow V/W$ is a closed embedding, where $V$ is the dual space of $V^*$ which is identified with the Lie algebra of the torus $T_{Q_l}$ over $\bar{Q}_l$, and similarly for $V_1$. But by using the classification of the triple $(L, C, E) \in \mathcal{M}_G$, it is checked that $V_1/\mathcal{W} \rightarrow V/W$ is a closed embedding. Thus the corollary follows. \hfill \Box

**2.6.** 1.12 (a), (b) in [L7] were deduced by using 1.11 there. Here we show the corresponding facts by using 2.3 as follows.
(2.6.1) Let $G$ be an algebraic group such that $G^0$ is a central torus in $G$. Then we have

$$H^*_G \simeq H^*_{G^0}.$$  

In fact, by [L7, 1.9 (a)], we have

$$H^*_G \simeq (H^*_{G^0})^{G^0}.$$  

But $H^*_{G^0} \simeq S(V^*)$ with $V^* = H^2_{G^0}$, and the action of $G/G^0$ on $S(V^*)$ is determined by the action of $G/G^0$ on $X(G^0)$ by (2.3.2). By our assumption, $G/G^0$ acts trivially on $X(G^0)$, and so on $S(V^*)$. This implies that $H^*_G \simeq S(V^*) \simeq H^*_{G^0}$, and (2.6.1) follows.

(2.6.2) In the same setting as above, let $E$ be an irreducible representation of $G/G^0$ over $\mathbb{Q}_l$. Then we have

$$H^G_\ast (\text{point, } E \otimes E^*) \simeq H^*_{G^0}.$$  

The proof is similar to [L7, 1.12 (b)], by making use of (2.6.1).

2.7. We return to the setting in 1.1, and consider a connected reductive algebraic group $G$, and its Lie algebra $\mathfrak{g}$. We further assume that $G$ is almost simple, simply connected. Let $\mathbb{G}_m$ be the multiplicative group of $k$. Then $G$ acts on $\mathfrak{g}$ by the adjoint action, and $G \times \mathbb{G}_m$ acts on $\mathfrak{g}$ by $(g_1, t) : x \mapsto t^{-2} \text{Ad}(g_1)x$. For $x \in \mathfrak{g}$, we denote by $Z_G(x)$ the stabilizer of $x$ in $G$, and by $M_G(x)$ the stabilizer of $x$ in $G \times \mathbb{G}_m$. Hence

$$M_G(x) = \{(g_1, t) \in G \times \mathbb{G}_m \mid \text{Ad}(g_1)x = t^2x\}.$$  

We assume that $p$ is large enough so that Jacobson-Morozov’s theorem and Dynkin-Kostant theory hold for $\mathfrak{g}$, (e.g., $p > 3(h-1)$, where $h$ is the Coxeter number of $W$, [C, 5.5]). Then, for each nilpotent element $y \in \mathfrak{g}$, there exists a Lie algebra homomorphism $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$, and elements $y^-$, $h \in \mathfrak{g}$ such that

$$y = \phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y^- = \phi \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \phi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

Thus we have $[h, y] = 2y$, $[h, y^-] = -2y^-$, $[y, y^-] = h$. Moreover, we have a decomposition $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$, where $\mathfrak{g}_i$ is the $i$-eigenspace of $\text{ad} h : \mathfrak{g} \to \mathfrak{g}$. In particular, note that $y \in \mathfrak{g}_2$, $y^- \in \mathfrak{g}_{-2}$. One can define an
algebra homomorphism \( \rho' : \mathbb{G}_m \to \text{Aut} \mathfrak{g} \) by \( \rho'(t)z = tz \) for \( z \in \mathfrak{g} \). Since the identity component of \( \text{Aut} \mathfrak{g} \) coincides with \( \text{ad} G = G/Z_G \), \( \rho'(\mathbb{G}_m) \) is a one-dimensional torus in \( \text{ad} G \). By taking the identity component of \( \pi^{-1}(\rho'(\mathbb{G}_m)) \) for \( \pi : G \to \text{ad} G \), one obtains a one parameter subgroup \( \rho : \mathbb{G}_m \to G \) such that \( \rho' = \pi \circ \rho \).

We put

\[
Z_G(\phi) = Z_G(y) \cap Z_G(y^-),
\]
\[
M_G(\phi) = \{(g_1, t) \in G \times \mathbb{G}_m \mid \text{Ad}(g_1)y = t^2y, \text{Ad}(g_1)y^- = t^{-2}y^-\}.
\]

It is known that \( Z_G(\phi) \) is a maximal reductive subgroup of \( Z_G(y) \). It is easy to check that \( (g_1, t) \mapsto (g_1 \rho(t), t) \) gives isomorphisms of algebraic groups

\[
(2.7.1) \quad Z_G(y) \times \mathbb{G}_m \xrightarrow{\sim} M_G(y), \quad Z_G(\phi) \times \mathbb{G}_m \xrightarrow{\sim} M_G(\phi).
\]

Hence \( M_G(\phi) \) is also a maximal reductive subgroup of \( M_G(\phi) \). It also follows from (2.7.1) that the embedding \( Z_G(y) \hookrightarrow M_G(y) \) by \( g_1 \mapsto (g_1, 1) \) induces an isomorphism

\[
Z_G(y)/Z_G^0(y) \xrightarrow{\sim} M_G(y)/M_G^0(\phi).
\]

This implies that the \( G \)-orbit of \( x \in \mathfrak{g} \) is also a \( G \times \mathbb{G}_m \)-orbit, and a \( G \)-local system on a nilpotent \( G \)-orbit in \( \mathfrak{g} \) is automatically a \( G \times \mathbb{G}_m \)-local system. In later discussions, we use the notation \( M(y), M^0(y) \), etc. instead of \( M_G(y), M^0_G(\phi), \) etc. by omitting the subscript \( G \) if there is no fear of confusion.

2.8. Under the setting in 1.1, let \( \mathfrak{p}, \mathfrak{l}, \mathfrak{n}_P \) be the Lie algebras of \( P, L, U_P \) so that \( \mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}_P \). Let \( \mathfrak{z} \) be the Lie algebra of \( Z_L^0 \). We assume that \( (L, \mathcal{C}, \mathcal{E}) \in \mathcal{M}_G \), and let \( (\mathcal{C}, \mathcal{L}) \) be the corresponding cuspidal pair on \( \mathfrak{l} \) (cf. 1.6). Let

\[
(2.8.1) \quad \hat{\mathfrak{g}} = \{(x, gP) \in \mathfrak{g} \times G/P \mid \text{Ad}(g^{-1})x \in \mathcal{C} + \mathfrak{z} + \mathfrak{n}_P\},
\]

and \( \pi : \hat{\mathfrak{g}} \to \mathfrak{g} \) be the first projection. \( G \times \mathbb{G}_m \) acts on \( \hat{\mathfrak{g}} \) by \( (g_1, t) : (x, gP) \mapsto (t^{-2} \text{Ad}(g_1)x, g_1gP) \), and \( \pi \) is \( G \times \mathbb{G}_m \)-equivariant. We consider the diagram

\[
\begin{array}{c}
\mathcal{C} \xleftarrow{\alpha} \hat{\mathfrak{g}} = \{(x, g) \in \mathfrak{g} \times G \mid \text{Ad}(g^{-1})x \in \mathcal{C} + \mathfrak{z} + \mathfrak{n}_P\} \xrightarrow{\beta} \hat{\mathfrak{g}},
\end{array}
\]

where \( \alpha(x, g) = \text{pr}_C(\text{Ad}(g^{-1})x), \beta(x, g) = (x, gP) \). Here \( \alpha, \beta \) are \( G \times \mathbb{G}_m \)-equivariant with respect to the action of \( G \times \mathbb{G}_m \) on \( \mathcal{C} \) given by \( (g_1, t) :
\( x \mapsto t^{-2}x \), and the action of it on the middle term given by \((g_1, t) : (x, g) \mapsto (t^{-2} \Ad(g_1)x, g_1g)\). Since \( \mathcal{L} \) is an \( L \)-local system, there exists a unique local system \( \hat{\mathcal{L}} \) on \( \hat{g} \) such that \( \alpha^* \mathcal{L} = \beta^* \hat{\mathcal{L}} \). By 2.7, \( \mathcal{L} \) is \( L \times \mathbb{G}_m \)-equivariant, and so is \( G \times \mathbb{G}_m \)-equivariant with respect to the above action. Hence \( \hat{\mathcal{L}} \) turns out to be \( G \times \mathbb{G}_m \)-equivariant.

Let \( \hat{\mathcal{L}}^* \) be the dual local system of \( \hat{\mathcal{L}} \), and consider \( K = \pi_!(\hat{\mathcal{L}}^*) \). Then it is shown in [L7, 3.4] that \( K \) is a \( G \times \mathbb{G}_m \)-equivariant perverse sheaf on \( g \) with a canonical \( W \) action, where \( \delta = \dim(g/\mathfrak{l}) + \dim(C + z) \).

Let \( X \) be an algebraic variety with a given morphism \( m : X \to g \). We consider the fibre product \( \hat{X} = X \times_g \hat{g} \) with the cartesian diagram

\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{\hat{m}} & \hat{g} \\
\pi' \downarrow & & \downarrow \pi \\
X & \xrightarrow{m} & g
\end{array}
\]

(2.8.2)

Then \( m^*K \) is a complex with \( W \)-action, and it induces a natural \( W \)-action on the cohomologies

\[
(2.8.3) \quad \mathbb{H}^j_c(X, m^*K) \simeq \mathbb{H}^j_c(X, \pi'_! \hat{m}^* \hat{\mathcal{L}}^*) \simeq H^j_c(\hat{X}, \hat{m}^* \hat{\mathcal{L}}^*).
\]

We further assume that \( X \) is a \( G' \)-variety, where \( G' \) is a connected closed subgroup of \( G \times \mathbb{G}_m \), and that \( m \) is compatible with \( G' \)-actions. If we choose a smooth irreducible variety \( \Gamma \) with a free \( G' \)-action as in 2.2, the cartesian diagram (2.8.2) is lifted to the cartesian diagram

\[
\begin{array}{ccc}
\Gamma \hat{X} & \xrightarrow{\Gamma \hat{m}} & \Gamma \hat{g} \\
\Gamma \pi' \downarrow & & \downarrow \Gamma \pi \\
\Gamma X & \xrightarrow{\Gamma m} & \Gamma g
\end{array}
\]

As in 2.2, we have a local system \( \Gamma \hat{\mathcal{L}}^* \) on \( \Gamma \hat{g} \), and a perverse sheaf (up to shift) \( \Gamma \mathcal{K} \) on \( \Gamma g \) which inherits a \( W \)-action from \( K \). Since \( \Gamma \mathcal{K} = (\Gamma \pi)! (\Gamma \hat{\mathcal{L}}^*) \), as in (2.8.3) we have natural \( W \)-actions on cohomologies

\[
\mathbb{H}^j_c(\Gamma X, (\Gamma m)^* (\Gamma \mathcal{K})) \simeq \mathbb{H}^j_c(\Gamma X, (\Gamma \pi')_! (\Gamma \hat{m})^* \Gamma \hat{\mathcal{L}}^*) \simeq H^j_c(\Gamma \hat{X}, (\Gamma \hat{m})^* \Gamma \hat{\mathcal{L}}^*).
\]

Hence we have an action of \( W \) on the equivariant homology

\[
H^{2d-j}_c(X, \hat{\mathcal{L}}^*) = H^{2d-j}_c(\Gamma \hat{X}, \Gamma \hat{\mathcal{L}}^*),
\]

where \( d = \dim(\Gamma \hat{X}) \). (Here we write \( \hat{m}^* \hat{\mathcal{L}}^* \), \( (\Gamma \hat{m})^* \Gamma \hat{\mathcal{L}}^* \), etc. as \( \hat{\mathcal{L}}^* \), \( \Gamma \hat{\mathcal{L}}^* \), etc. by abbreviation.)
2.9. We fix an element $x_0 \in \mathcal{C}$ and a Lie algebra homomorphism $\phi_0 : \mathfrak{sl}_2 \rightarrow \mathfrak{t}$ such that $\phi_0 \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = x_0$. As in [L7, 2.3 (b)], we have

\begin{equation}
Z_L^0(\phi_0) = Z_L^0.
\end{equation}

It follows that $Z_L^0(\phi_0)$ is central in $Z_L(\phi)$. Hence by (2.7.1), we see that

\begin{equation}
M_L^0(\phi_0) \simeq Z_L^0 \times G_m, \text{ and } M_L^0(\phi_0) \text{ is contained in the center of } M_L(\phi).
\end{equation}

Put $\mathfrak{h}^* = \mathcal{Q}_t \otimes_{\mathbb{Z}} X(Z_L^0)$. The $\mathfrak{h}^*$ is a $\mathcal{Q}_t$-space of dim$_{\mathcal{Q}} \mathfrak{h}^* = \dim_{\mathbb{k}} \mathfrak{h}$, on which $\mathcal{W}$ acts naturally. We define a symmetric algebra $S$ over $\mathcal{Q}_t$ by

\begin{equation}
S = S(\mathfrak{h}^* \oplus \mathcal{Q}_t) = S(\mathfrak{h}^*) \otimes \mathcal{Q}_t[r],
\end{equation}

where $\mathcal{Q}_t[r]$ is the polynomial ring with an indeterminate $r$ corresponding to $(0, 1) \in \mathfrak{h}^* \oplus \mathcal{Q}_t$. We now consider the equivariant cohomology $H_{G \times G_m}(\mathfrak{g})$. As in [L7, Proposition 4.2], we have an isomorphism

\begin{equation}
H_{G \times G_m}(\mathfrak{g}) \simeq S
\end{equation}

as graded algebras. In particular, $H_{G \times G_m}^j(\mathfrak{g}) = 0$ for odd $j$. For the proof, the argument in [L7] implies that

\[ H_{G \times G_m}^* (\mathfrak{g}) \simeq H_{M_L(\phi_0)}^* . \]

Then by using (2.6.1) and (2.9.2), combined with (2.3.2), we have

\[ H_{M_L(\phi_0)}^* \simeq H_{M_L^0(\phi_0)}^* \simeq H_{Z_L^0 \times G_m}^* = S. \]

Hence (2.9.3) follows.

Let $\tilde{X}$ be a $G'$-variety ($G'$ is a connected closed subgroup of $G \times G_m$), with a given $G'$-equivariant morphism $\tilde{m} : \tilde{X} \rightarrow \mathfrak{g}$. $\tilde{m}^* \mathcal{L}$ is a $G'$-local system on $\tilde{X}$, which we denote by $\mathcal{L}$ by abbreviation. Now $\tilde{m}^*$ induces an algebra homomorphism $H_{G'}^{*}(\mathfrak{g}) \rightarrow H_{G'}^{*}(\tilde{X})$. By combining the natural homomorphism $H_{G \times G_m}^*(\mathfrak{g}) \rightarrow H_{G'}^*(\mathfrak{g})$ (cf. [L7, 1.4 (g)]), we have a homomorphism $H_{G \times G_m}^*(\mathfrak{g}) \rightarrow H_{G'}^*(\tilde{X})$. Since $H_{G'}^*(\tilde{X}, \mathcal{L})$ is a $H_{G'}^*(\tilde{X})$-module by 2.2, $H_{G'}^*(\tilde{X}, \mathcal{L})$ has a structure of a left $H_{G \times G_m}^*(\mathfrak{g})$-module. Thus by (2.9.3), $H_{G'}^*(\tilde{X}, \mathcal{L})$ turns out to be an $S$-module.
2.10. Let \( \pi : \mathfrak{g} \to \mathfrak{g} \) be as in 2.8. Then for each \( y \in \mathfrak{g}_{\text{nil}} \), \( \pi^{-1}(y) \) coincides with \( \mathcal{P}_y \) in (1.6.1). The variety \( \mathcal{X} = \{ y \} \) is invariant under the action of \( M^0(y) \subset G \times G_m \). Let \( G' \) be a connected closed subgroup of \( M^0(y) \). By applying 2.8 to the inclusion \( m : \mathcal{X} \hookrightarrow \mathfrak{g} \) together with \( \mathcal{X} = \mathcal{P}_y \), we see that \( H^*_G(\mathcal{P}_y, \hat{\mathcal{L}}) \) has a natural \( \mathcal{W} \)-action. By applying 2.9 for \( e \mathcal{X} = \mathcal{X} \), \( H^*_G(\mathcal{P}_y, \hat{\mathcal{L}}) \) has a natural \( \mathbf{S} \)-action. It also has a structure of \( H^*_G \)-module by 2.2.

We consider the graded Hecke algebra \( H = \mathbf{S} \otimes \mathcal{Q}_l[\mathcal{W}] \) as defined in 2.1, where \( \mathbf{S} \) is as in 2.9, with a natural action of the Coxeter group \( \mathcal{W} \). Lusztig proved the following theorem.

**Theorem 2.11.** (Lusztig [L7, Theorem 8.13]) There is a unique \( H \)-module structure on \( H^*_M(\mathcal{P}_y, \hat{\mathcal{L}}) \) such that the actions of \( \mathbf{S} \) and \( \mathcal{W} \) are given as in 2.10. (The integers \( c_i \) are determined according to the cuspidal pair \( (C, \mathcal{L}) \). See [L7, 2.13] for explicit values for \( c_i \).) Moreover, the \( H \)-module structure commutes with the \( H^*_M(\mathcal{P}_y, \hat{\mathcal{L}}) \)-module structure on \( H^*_M(\mathcal{P}_y, \hat{\mathcal{L}}) \).

**Remark 2.12.** The arguments used in [L7] to prove the theorem are valid also for our setting in almost all cases, by taking 2.3–2.7 into account. We give further comments on the discrepancies of the arguments.

(a) In [L7, 4.3], the property of the image \( H^*_G \mathbf{G} \mathbf{m} \to H^*_M(\mathcal{P}_y, \hat{\mathcal{L}}) \) is used. For this we appeal to Corollary 2.5.

(b) In the proof of Proposition 7.2 in [L7], a property of simply connected space is used, which is not valid in the positive characteristic case. As in 7.1, we consider a connected algebraic group \( M \), and an \( M \)-variety \( X \), \( M \)-equivariant local system \( \mathcal{E} \) on \( X \). Let \( \Gamma \) be an irreducible, smooth variety with a free \( M \)-action as before. Let \( f : M \setminus (\Gamma \times X) \to M \setminus \Gamma \) be the locally trivial fibration. We consider the Leray-Serre spectral sequence

\[
H^p_c(M \setminus \Gamma, R^q f_!(\Gamma \mathcal{E}^*)) \Rightarrow H^{p+q}_c(M \setminus (\Gamma \times X), \Gamma \mathcal{E}^*). 
\]

We show that

\[
(2.12.1) \quad E^{p,q}_2 = H^p_c(\Gamma \times X, R^q f_!(\Gamma \mathcal{E}^*)) = H^p_c(M \setminus \Gamma) \otimes H^q_\mathcal{E}(X, \mathcal{E}^*). 
\]

(In [L7], this is obtained as a consequence of the fact that \( M \setminus \Gamma \) can be chosen to be simply connected.) We consider the cartesian diagram

\[
\begin{array}{ccc}
\Gamma \times X & \xrightarrow{\pi} & M \setminus (\Gamma \times X) \\
\downarrow f & & \downarrow f \\
\Gamma & \xrightarrow{\pi} & M \setminus \Gamma.
\end{array}
\]
Now $f_! \mathcal{E}^*$ on $M \setminus (\Gamma \times X)$ satisfies the property that $\mathcal{Q}_l \boxtimes \mathcal{E}^* = \pi^*(f_! \mathcal{E}^*)$. By the base change theorem, we have $\pi^* R^q f_!(f_! \mathcal{E}^*) \simeq R^q f_! \pi^*(f_! \mathcal{E}^*)$. It is easy to see that $R^q f_!(\mathcal{Q}_l \boxtimes \mathcal{E}^*)$ is an $M$-equivariant constant sheaf, and $R^q f_!(f_! \mathcal{E}^*)$ is obtained from it as the unique quotient. Thus, $R^q f_!(f_! \mathcal{E}^*)$ is also a constant sheaf with the stalk $H^q_M(X, \mathcal{E}^*)$. This implies (2.12.1).

Once this is established, the other parts in the proof of Proposition 7.2 work without change.

2.13. We return to the setting in 2.10. Let $T(y)$ be a maximal torus of $M^0(y)$ and $W(y)$ the Weyl group of a maximal reductive subgroup of $M^0(y)$ with respect to $T(y)$. Then by (2.3.2) and Lemma 2.4, $H^*_M(y)$ can be identified with $S(V^*)^W(y)$, where $V^* = \mathcal{Q}_l \boxtimes Z X(T(y))$. Hence $H^*_M(y)$ may be regarded as the coordinate ring of an affine algebraic variety (over $\mathcal{Q}_l$) $V = V/W(y)$, where $V$ is the dual space of $V^*$. Then for each $v \in V_1$, one obtains an algebra homomorphism $H^*_M(y) \to \mathcal{Q}_l$, $f \mapsto f(v)$. We denote the thus obtained $H^*_M(y)$-module $\mathcal{Q}_l$ by $(\mathcal{Q}_l)_v$. It is known by [L7, 8.6] that $H^*_M(y; \mathcal{L})$ is a finitely generated projective $H^*_M(y)$-module. It follows that $H^*_M(y; \mathcal{L})$ may be regarded as a space of sections of algebraic vector bundle $E$ over $V_1$, where the fibre of $E$ at $v \in V_1$ is given by

$$(2.13.1) \quad E_v = (\mathcal{Q}_l)_v \otimes H^*_M(y; \mathcal{L}).$$

Put $\overline{M}(y) = M(y)/M^0(y)$. Then the finite group $\overline{M}(y)$ acts on $H^*_M(y)$ as a $\mathcal{Q}_l$-algebra automorphism, and acts on $H^*_M(y; \mathcal{L})$ compatible with the action of $H^*_M(y)$. Also this action of $\overline{M}(y)$ on $H^*_M(y)$ commutes with the action of $H$. The action of $\overline{M}(y)$ on $H^*_M(y)$ induces an action of $\overline{M}(y)$ on $V_1$, and $E$ turns out to be an $\overline{M}(y)$-equivariant vector bundle over $V_1$. For each $v \in V_1$, we denote by $\overline{M}(y, v)$ the stabilizer of $v$ in $\overline{M}(y)$. Then $\overline{M}(y, v)$ acts naturally on $E_v$.

Let $\overline{M}(y, v)^\wedge$ be the set of irreducible representations of $\overline{M}(y, v)$ up to isomorphisms. For each $\rho \in \overline{M}(y, v)^\wedge$, put $E_{v, \rho} = (\rho^* \otimes E_v)|_{\overline{M}(y, v)}$, where $\rho^*$ is the dual representation of $\rho$. Then $E_{v, \rho}$ is an $H$-module, and $E_v$ is decomposed as

$$E_v = \bigoplus_{\rho \in \overline{M}(y, v)^\wedge} \rho \otimes E_{v, \rho}.$$
by [L7, 8.10] that \( E_{v,\rho} \neq 0 \) if and only if \( \rho \) occurs in the restriction of \( \overline{M}(y) \)-module \( H_*^{(e)}(P_y, \hat{L}) \) to \( \overline{M}(y, v) \). The \( \mathbf{H} \)-modules \( E_{v,\rho} \) are called standard modules.

**Remarks 2.14.** (i) Standard modules \( E_{v,\rho} \) are parametrized in [L7] (i.e., in the setting that \( G \) and \( \mathfrak{g} \) are defined over \( \mathbb{C} \)) as \( E_{h, r_0, \rho} \) in terms of the pair \((h, r_0) \in \mathfrak{g} \oplus \mathbb{C} \) such that \([h, y] = 2r_0y \) with \( h \) semisimple. This is also possible in our situation, though we cannot use the Lie algebra \( \mathfrak{g} \) over \( k \). Since \( p \) is good, we have corresponding objects \( G_\mathbb{C}, \mathfrak{g}_\mathbb{C} \), and the parametrization of nilpotent orbits and the structure of \( \overline{M}(y) \) are the same for \( \mathfrak{g}_\mathbb{C} \) also. If we consider the maximal torus \( T(y)_\mathbb{C} \) in \( M(y)_\mathbb{C} \) corresponding to \( T(y) \) in \( M(y) \), the space \( V^* \) may be identified (under a choice of an isomorphism \( \hat{Q}_l \simeq \mathbb{C} \)) with the dual of the Cartan subalgebra \( \mathfrak{h}(y)_\mathbb{C} \) of a maximal reductive subalgebra \( \mathfrak{m}(y)_\mathbb{C} \) of \( \mathfrak{m}(y)_\mathbb{C} \). Then the action of \( \overline{M}(y) \) on \( S(V^*)^{W(y)} \) coincides with the action of \( \overline{M}(y) \) on \( S(\mathfrak{h}(y)_\mathbb{C})^{W(y)} \simeq S(\mathfrak{m}(y)_\mathbb{C})^{M(0)(y)_\mathbb{C}} \). Here

\[
\mathfrak{m}(y)_\mathbb{C} = \text{Lie } M^0(y) = \{(x, r_0) \in \mathfrak{g}_\mathbb{C} \oplus \mathbb{C} \mid [x, y] = 2r_0y \}.
\]

Moreover, the action of \( \overline{M}(y) \) on \( S(\mathfrak{m}(y)_\mathbb{C})^{M(0)(y)_\mathbb{C}} \) is induced from the action of \( M(y)_\mathbb{C}, (g_1, t) : (x, r_0) \mapsto (t^{-2} \text{Ad}(g_1)x, t^{-2}r_0) \). Hence \( V_1 \) is identified with the set of semisimple \( M^0(0)(y)_\mathbb{C} \)-orbits on \( \mathfrak{m}(y)_\mathbb{C} \). This implies, in our case, that \( E_{v,\rho} \) may be expressed as \( E_{h, r_0, \rho} \), and \( \overline{M}(y, v) \) as \( \overline{M}(y, h, r_0) \), if \( (h, r_0) \) is a semisimple orbit in \( \mathfrak{g}_\mathbb{C} \) corresponding to \( v \in V_1 \).

(ii) Standard modules play a crucial role in the representation theory of \( \mathbf{H} \). The structure of \( \mathbf{H} \)-module \( E_{v,\rho} \) was studied throughly in [L8], [L9]. However, the result in [L7] is enough for our purpose.

In view of the above remarks, the following result of Lusztig can be applied to our setting.

**Theorem 2.15.** ([L7, Theorem 8.17]) Let \( (h, r_0) \in \mathfrak{g}_\mathbb{C} \oplus \mathbb{C} \) be a semisimple element such that \( r_0 \neq 0 \). Then

(i) Let \( Y_{(h, r_0)} = \{x \in \mathfrak{g}_\mathbb{C} \mid [h, x] = 2r_0x \} \). Then \( Y_{(h, r_0)} \) consists of nilpotent elements, and \( Z_{\mathfrak{g}_\mathbb{C}}(h) \) acts (by the adjoint action) on \( Y_{(h, r_0)} \) with finitely many orbits.

(ii) Let \( y \) be an element in the unique open dense orbit in \( Y_{(h, r_0)} \). Then \( (h, r_0) \in \mathfrak{m}(y)_\mathbb{C} \). Let \( \rho \in \overline{M}(y, h, r_0)^\wedge \) be such that \( E_{h, r_0, \rho} \neq \{0\} \). Then \( E_{h, r_0, \rho} \) is a simple \( \mathbf{H} \)-module.
2.16. Here we summarize the properties connecting the equivariant homology with the ordinary cohomology. Let $M$ be a connected algebraic group, $X$ an $M$-variety and $E$ an $M$-equivariant local system on $X$. We consider $H^*_M(X,E)$. For each $i$, we define $F^i$ as the $H^*_M$-submodule of $H^*_M(X,E)$ generated by $\bigoplus_{j\leq i} H^j_M(X,E)$. Then $F^i$ gives a filtration $F^0 \subseteq F^1 \subseteq \cdots$ and $F^i = 0$ for $i < 0$. Put $\Pi_i = H^i_M(X,E)/H^i_M(X,E) \cap F^{i-1}$. We have a natural injection $\Pi_i \to F^i/F^{i-1}$ as $Q_l$-spaces. Since $F^i/F^{i-1}$ is an $H^*_M$-module, this is extended to an $H^*_M$-linear map

\[ H^*_M \otimes Q_l \to F^i/F^{i-1}. \]  

The natural homomorphism $H^i_M(X,E) \to H^i_e(X,E)$ is zero on $H^i_M(X,E) \cap F^{i-1}$, and it factors through a $Q_l$-linear map

\[ \Pi_i \to H^i_e(X,E). \]  

Lusztig showed in [L7, 7.2] that the maps (2.16.1) and (2.16.2) are isomorphisms whenever $H^e_{\text{odd}}(X,E) = 0$, and in that case we obtain an isomorphism

\[ H^*_M \otimes Q_l H^i_e(X,E) \cong F^i/F^{i-1}. \]  

We now consider the case where $X = P_y$, $E = \hat{\mathcal{L}}$ and $M = M^0(y)$. It is known that $H^e_{\text{odd}}(P_y,\hat{\mathcal{L}}) = 0$ by [L3, V, 24.8], and so the previous argument can be applied. We consider $E_v$ as in (2.13.1) and $H^*_M(y)$-module $(Q_l)^v$. We define an $Q_l$-space $F^i_v$ by $F^i_v = (Q_l)^v \otimes H^*_M(y) F^i$. Then $F^i_v$ is naturally identified with a quotient of $\bigoplus_{j\leq i} H^j_M(y)(P_y,\hat{\mathcal{L}})$. We denote by $f_i : F^{i-1}_v \to F^i_v$ the natural map induced from $F^{i-1} \hookrightarrow F^i$. It follows from (2.16.3) we have an exact sequence of $Q_l$-spaces

\[ F^{i-1}_v \xrightarrow{f_i} F^i_v \longrightarrow H^i_e(P_y,\hat{\mathcal{L}}) \longrightarrow 0. \]  

In particular, we have

\[ F^0_v \simeq H^0_e(P_y,\hat{\mathcal{L}}). \]  

2.17. We consider the $F_q$-structure on the equivariant homology. Assume that $G$ and $X$ are defined over $F_q$ with Frobenius map $F$, and $G$ acts on $X$ over $F_q$. Let $E$ be an $G$-equivariant local system on $X$ such
that $F^*\mathcal{E} \cong \mathcal{E}$. We fix an isomorphism $\varphi : F^*\mathcal{E} \cong \mathcal{E}$. Then $\varphi$ induces natural linear isomorphisms on $H^*_G(X, \mathcal{E})$, $H^*_G(X, \mathcal{E})$, etc. In fact, one can choose a $G$-variety $\Gamma$ so that $\Gamma$ is defined over $\mathbb{F}_q$. (We may assume that $G$ is an $F$-stable closed subgroup of some $GL_r$.) The case where $GL_r$ has a split $\mathbb{F}_q$-structure, the construction of $\Gamma$ in 2.2 works well. If $GL_r$ is of non-split type, we choose $F = \sigma_0F_0$, where $F_0$ is a split Frobenius, and $\sigma_0$ is an automorphism of $GL_r$ defined by $\sigma_0(g) = g^{-1}$. By choosing similar Frobenius maps for $GL_{r'}$ and $GL_{r'+r'}$, the inclusions in (2.2.1) are $F$-equivariant. Hence $\Gamma = \{e\} \times GL_{r'}\backslash GL_{r+r'}$ is defined over $\mathbb{F}_q$.

Then the maps $\pi : \Gamma \times X \rightarrow rX$, $p : \Gamma \times X \rightarrow X$ are defined over $\mathbb{F}_q$. Hence $r\mathcal{E}$ inherits an $\mathbb{F}_q$-structure of $\mathcal{E}$, which induces a linear map on $H^*_G(X, \mathcal{E}) = H^3(\Gamma X, r\mathcal{E})$. The thus obtained linear map is independent of the choice of $\Gamma$. In fact, if $\Gamma'$ is another choice, we have an isomorphism $H^3(\Gamma X, r\mathcal{E}) \cong H^3(\Gamma' X, r\mathcal{E})$, etc. as in [L7, 1.1], which are compatible with the induced $F$-actions on them.

§3. $G = SL_n$ with $F$ of split type

3.1. In this section, we assume that $p$ is arbitrary, and consider $G = SL_n$ with the standard Frobenius map $F$ on $G$, i.e., for $g = (g_{ij}) \in G$, $F(g) = (g_{ij}^q)$. Let $V = k^n$ with the standard basis $e_1, \ldots, e_n$ and we identify $SL_n$ with $SL(V)$.

Let $\mathfrak{g} = \mathfrak{sl}_n$ be the Lie algebra of $G$, and we denote by $F$ the corresponding Frobenius map on $\mathfrak{g}$. The unipotent classes in $G$ and nilpotent orbits in $\mathfrak{g}$ are parametrized by partitions of $n$, via Jordan normal form. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition of $n$, and let $C_\lambda$ (resp. $C_\lambda$) be the corresponding unipotent class in $G$ (resp. nilpotent orbit in $\mathfrak{g}$). Each $C_\lambda$ is $F$-stable, and we construct a specific nilpotent transformation $y = y_\lambda \in C^F_\lambda$ by defining a basis $\{y^a f_j \mid 1 \leq j \leq r, 0 \leq a < \lambda_j\}$ of $V$ obtained from the standard basis as follows:

$$(3.1.1) \quad y^a f_j = e_i \quad \text{with} \quad i = \lambda_1 + \cdots + \lambda_{j-1} + a.$$ 

Then $u_\lambda = y_\lambda + 1 \in C^F_\lambda$. The element $y_\lambda \in C^F_\lambda$ (resp. $u_\lambda \in C^F_\lambda$) is called the split element corresponding to $\lambda$.

3.2. By [L2], [LS], the generalized Springer correspondence for the case where $G = SL_n$ is described as follows. Let $n'$ be the largest divisor of $n$ which is prime to $p$. Then the center $Z_G$ is a cyclic group of order $n'$. For a divisor $d$ of $n'$, consider a Levi subgroup $L$ of $P$ of the type
\( A_{d-1} \times \cdots \times A_{d-1} \) \((n/d\)-factors). Let \( C \) be the regular unipotent class in \( L \). Then for \( v \in C \), \( A_L(v) = Z_L/Z_L^0 \cong \mathbb{Z}/d\mathbb{Z} \). Let \( E \) be an \( L \)-equivariant local system on \( C \) corresponding to a character \( \rho_0 \) of \( A_L(v) \) of order \( d \). Then \((C, E)\) is a cuspidal pair on \( L \), and any cuspidal pair on a Levi subgroup of a parabolic subgroup of \( G \) is obtained in this way. Hence for a Levi subgroup \( L \) determined by \( d \), there exist exactly \( \varphi(d) \) cuspidal pairs in \( L \), where \( \varphi \) is the Euler function.

Let \( K \) be as in (1.2.2) with respect to the cuspidal pair \((C, E)\) on \( L \). Let \( C' \) be a unipotent class in \( G \) corresponding to a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \). Then for \( u \in C' \), \( A_G(u) \) is a cyclic group of order \( n'_\lambda \), where \( n'_\lambda \) is the greatest common divisor of \( n', \lambda_1, \lambda_2, \ldots, \lambda_r \). Let \( E' \) be the local system on \( C' \) corresponding to \( \rho = A_G(u)^\wedge \). The condition for \( C' \) such that \( I(C', E') \) is a component of \( K \) (up to shift) is that each \( \lambda_i \) is divisible by \( d \). In this case \( n'_\lambda \) is divisible by \( d \), and we have a surjective homomorphism \( A_G(u) \to A_L(v) \) which factors through the natural maps \( Z_G \to A_G(u) \) and \( Z_G \to A_L(v) \). Let \( \rho \in A_G(u)^\wedge \) be the character obtained as the pull back of \( \rho_0 \in A_L(v)^\wedge \). Then \( I(C', E') \) is the unique component in \( K \) whose support is \( \overline{C'} \).

Now \( \mathcal{W} = N_G(L)/L \) is isomorphic to the symmetric group \( S_{n/d} \). The irreducible character \( E = E_\mu \in S_{n/d}^\wedge \) corresponding to \((C', E')\) under the generalized Springer correspondence is given by \( \mu = (\lambda_1/d, \lambda_2/d, \ldots) \).

3.3. We fix an \( F \)-stable Borel subgroup \( B \) of \( G \) and an \( F \)-stable maximal torus contained in \( B \), where \( B \) (resp. \( T \)) is the subgroup of \( G \) consisting of upper triangular matrices (resp. diagonal matrices). We fix \( d \) as in 3.2, and put \( t = n/d \). Let \( P = LU_P \) be the parabolic subgroup of \( G \) containing \( B \), where \( L \) is the Levi subgroup of \( P \) containing \( T \) of type \( A_{d-1} \times \cdots \times A_{d-1} \), \((t\)-times). Hence \( P \), \( L \) and \( U_P \) are all \( F \)-stable. Let \((C, E)\) be the cuspidal pair in \( L \) corresponding to \( \rho_0 \in A_L(v)^\wedge \) as in 3.2, and \((C, \mathcal{L})\) the corresponding objects in \( I \). The unipotent class \( C \) in \( L \) can be identified with \( C_1 \times \cdots \times C_t \) in \( SL_d \times \cdots \times SL_d \) with \( C_i \) regular unipotent in \( SL_d \). We choose \( v = v_0 \in C^F \) so that \( v_0 \) is a product of split elements in \( C_i^F \), and let \( y_0 = v_0 - 1 \) the corresponding element in \( C^F \). Let \( \overline{A_L(v_0)} \) be as in 1.3. Since \( A_L(v_0) \) is abelian, \( \rho_0 \in A_L(v)^\wedge \) is linear. We choose an extension \( \overline{\rho}_0 \) so that \( \overline{\rho}_0(\sigma) = 1 \). This corresponds to an isomorphism \( \varphi_0 : F^*E \cong E \) which induces the identity map on the stalk \( \mathcal{E}_{v_0} \).

Let \( \lambda = (\lambda_1, \ldots, \lambda_r) \) be a partition of \( n \) such that all the \( \lambda_i \) are divisible by \( d \), and \( u = u_\lambda \) the split unipotent element in \( G^F \). As in the case of \((C, E)\),
we choose an extension $\tilde{\rho}$ of $\rho \in A_G(u)^\wedge$ corresponding to $\mathcal{E}'$ by the condition that $\tilde{\rho}(\sigma) = 1$, and consider $\gamma = \gamma(v, \tilde{\rho}_0, u, \tilde{\rho})$ as in 1.3. Passing to the Lie algebra situation, we consider $y = y_\lambda \in \mathfrak{g}^F$ and $y_0 \in \mathfrak{c}^F$. Under this setting, we write $\gamma$ as $\gamma = \gamma(y_0, \tilde{\rho}_0, y, \tilde{\rho})$. We consider the subvariety $\mathcal{P}_y$ of $G/P$ as given in (1.6.1). As in 1.6, the map $\varphi_0$ induces a linear isomorphism $\Phi$ on $H^{a_0+r}_c(\mathcal{P}_y, \mathcal{L})$. We have

**Theorem 3.4.** Assume that $p$ is arbitrary, and let $G = SL_n$ with the standard Frobenius map $F$. Then $\Phi$ acts on $H^{a_0+r}_c(\mathcal{P}_y, \mathcal{L}) = H^{a_0+r}_c(\mathcal{P}_y, \mathcal{L})_{\rho}$ as $q^{(a_0+r)/2}$ times identity. In particular, we have $\gamma(y_0, \tilde{\rho}_0, y, \tilde{\rho}) = 1$.

**3.5.** The remainder of this section is devoted to the proof of Theorem 3.4. Since the second statement easily follows from the first one, we concentrate to the proof of the first statement. First we note that $\mathcal{P}_y$ may be identified with the set $\mathcal{F}_y$ of partial flags

$$D = (V_d \subset V_{2d} \subset \cdots \subset V_{(t-1)d}),$$

such that $D$ is $y$-stable and that $y$ induces a regular nilpotent transformation on $V_{id}/V_{(i-1)d}$ for each $i \geq 1$. (Here $V_j$ denotes a subspace of $V$ with dim $V_j = j$.)

Let $\mathcal{G}_y$ be the set of $d$-dimensional subspaces $V_d$ of $V$ such that $V_d$ is $y$-stable and that $y$ acts as a regular nilpotent transformation on $V_d$. We have a natural surjective map $p : \mathcal{F}_y \to \mathcal{G}_y$ by $p(D) = V_d$. Then $\mathcal{G}_y$ is identified with the variety $\mathbf{P}(\text{Ker } y^d) - \mathbf{P}(\text{Ker } y^{d-1})$; for each $v \in \text{Ker } y^d - \text{Ker } y^{d-1}$, the space spanned by $v, yv, \ldots, y^{d-1}v$ gives an element in $\mathcal{G}_y$. We have a filtration of $\mathcal{G}_y$

$$\mathcal{G}_y = \mathcal{G}_0 \supset \mathcal{G}_1 \supset \cdots,$$

where $\mathcal{G}_i - \mathcal{G}_{i+1} \simeq A^{s-i}$ with dim $\mathcal{G}_y = s = d(\text{dim Ker } y) - 1$. Here $\mathcal{G}_i$ is defined by $\mathbf{P}(U_i) - \mathbf{P}(\text{Ker } y^{d-1})$ for a certain subspace $U_i$ of $\text{Ker } y^d$ containing $\text{Ker } y^{d-1}$ such that $\text{Ker } y^d = U_0 \supset U_1 \supset \cdots$. Let us choose a non-zero vector $w_i \in U_i - U_{i+1}$ for each $i$. We can choose some $e_j$ as $w_i$. As in the case of $\mathcal{B}_u$ for $GL_n$, one can define a map $f^{(i)} : A^{s-i} \to Z_{\tilde{G}}(y)$,

$v \mapsto f^{(i)}_v$ such that $f^{(i)}_v \cdot w_i = v$ for $v \in U_i - U_{i-1}$, under the identification $\mathbf{P}(U_i) - \mathbf{P}(U_{i-1}) \simeq A^{s-i}$. (Here $\tilde{G}$ denotes $GL_n$.) Let $V_d^{(i)}$ be the element in $\mathcal{G}_y$ corresponding to $w_i$. Then $y$ induces a nilpotent transformation $\overline{y}$ on $\overline{V} = V/V_d^{(i)}$, which corresponds to a partition $\lambda'$ of $n - d$ obtained from $\lambda$.
by replacing some $\lambda_j$ by $\lambda_j - d$. Moreover, $p^{-1}(V_d^{(i)})$ is isomorphic to $\mathcal{F}_{\overline{G}}$, the corresponding variety for $SL(\overline{V})$, under the correspondence

$$D = (V_d^{(i)} \subset V_{2d} \subset \cdots \subset V_{(t-1)d}) \longleftrightarrow \overline{D} = (\overline{V}_{2d} \subset \cdots \subset \overline{V}_{(t-1)d})$$

with $\overline{V}_{jd} = V_{jd}/V_d^{(i)}$. As in the case of $GL_n$, by using the map $f^{(i)} : \mathbb{A}^{n-i} \to Z_G(y)$, we have an isomorphism

$$p^{-1}(V_d^{(i)}) \times (G_i - G_{i+1}) \simeq p^{-1}(G_i - G_{i+1}), \quad (D, v) \mapsto f_v^{(i)} \cdot D.$$  

Note that $\mathcal{F}_y$ and $G_y$ have natural $\mathbb{F}_q$-structures inherited from $G/P$. Then $G_i$ are all $F$-stable, and the isomorphism in (3.5.1) is $F$-equivariant.

3.6. Let $Q$ be the maximal parabolic subgroup of $G$ containing $P$ of type $A_{n-d-1} \times A_{d-1}$. Let $\mathcal{G}$ be the set of subspaces of dimension $d$ in $V$. Then $\mathcal{G}$ may be identified with $G/Q$ and $\mathcal{G}_y$ is a locally closed subvariety of $\mathcal{G}$. The map $p : \mathcal{F}_y \to \mathcal{G}_y$ is obtained from the map $G/P \to G/Q$ by restricting it to $\mathcal{P}_y$, which we also denote by $p$. Now, $V^{(i)}_d \in \mathcal{G}_y$ corresponds to $gQ \in G/Q$ for some $g = g_i \in G$ and $p^{-1}(V^{(i)}_d)$ may be identified with $p^{-1}(gQ)$, where

$$p^{-1}(gQ) = \{ xP \in gQ/P \mid \text{Ad}(x)^{-1}y \in \mathcal{C} + n_P \}.$$  

We may choose $g$ so that $gP \in \mathcal{P}_y$.

We note that $Q/P$ is isomorphic to $M/P_M$, where $M$ is the subgroup of $G$ isomorphic to $SL_{n-d}$, and is isogeneous to a component of the Levi subgroup of $Q$ containing $T$. Then $P_M = P \cap M$ is the parabolic subgroup of $M$ of type $A_{d-1} \times \cdots \times A_{d-1}$, ($t-1$ factors), and $L_M = L \cap M$ is the Levi subgroup of $P_M$. The regular nilpotent orbit $\mathcal{C}$ in $l$ can be written as $\mathcal{C} = \mathcal{C}_M \times \mathcal{C}_t$, where $\mathcal{C}_M$ is the regular nilpotent orbit in $\text{Lie} L_M = l_M$ and $\mathcal{C}_t$ is the regular nilpotent orbit in the $t$-th component of $l$. Since $\text{Ad}(g)^{-1}y \in \mathcal{C} + n_P$, one can write $\text{Ad}(g)^{-1}y = y' + z'$ with $y' \in \mathcal{m}$ and $z' \in \mathcal{C}_t + n_Q$. (Here $\mathcal{m} = \text{Lie} M$ and $n_Q = \text{Lie} U_Q$.) Set

$$\mathcal{P}^M_y = \{ xP_M \in M/P_M \mid \text{Ad}(x)^{-1}y' \in \mathcal{C}_M + n_{P_M} \},$$

$$\mathcal{P}^M_{y'} = \{ x \in M \mid \text{Ad}(x)^{-1}y' \in \mathcal{C}_M + n_{P_M} \}.$$  

We note that

$$\mathcal{P}^M_{y'} \simeq p^{-1}(gQ).$$

The map $xP_M \mapsto gxP$ gives an isomorphism $\mathcal{P}^M_{y'} \simeq p^{-1}(gQ)$. 


In fact, since $M$ normalizes $C_t U_Q$, we have $\text{Ad}(x)^{-1} z' \in C_t + n_Q$. Then the condition $\text{Ad}(gx)^{-1} y \in C + n_P$ is equivalent to the condition $\text{Ad}(x)^{-1} y' \in C_M + n_{PM}$. (3.6.1) follows from this.

By (3.6.1), one can define an injective map $\iota : P^M_y \to P_y$. Similarly, $\hat{P}^M_y$ is isomorphic to the set $\{x' \in gM \mid \text{Ad}(x')^{-1} y \in C + n_P\}$ which is a subset of $\hat{P}_y$. Hence we have an injective map $\hat{i} : \hat{P}^M_y \to \hat{P}_y$. Now it is easy to see that the following diagram commutes.

(3.6.2)

Here the left vertical map is an injection $\iota' : C_M \to C$, $x \mapsto (x, y''$), where $y''$ is the projection of $z' \in C_t + n_Q$ to $C_t$, i.e., the projection of $\text{Ad}(g)^{-1} y \in C + n_P$ on $C_t$. The horizontal maps $\alpha'$, $\beta'$ are defined in a similar way as $\alpha$ and $\beta$ by replacing $G$ by $M$.

Let $\mathcal{L}$ and $\hat{\mathcal{L}}$ be local systems on $C$ and $P_y$, respectively, as in 1.6. We denote by $\mathcal{L}_M$ and $\hat{\mathcal{L}}_M$ similar objects for $C_M$ and $P^M_y$ as $\mathcal{L}$, $\hat{\mathcal{L}}$ for $C$ and $P_y$. Then $\mathcal{L}_M$ coincides with $(\iota')^* \mathcal{L}$. This implies, by (3.6.2), that

(3.6.3)

$$\iota^* \hat{\mathcal{L}} = \hat{\mathcal{L}}_M.$$ 

Put

$$Y_y = \{x Q \in G/Q \mid \text{Ad}(x)^{-1} y \in m + C_t + n_Q\},$$

$$\hat{Y}_y = \{x \in G \mid \text{Ad}(x)^{-1} y \in m + C_t + n_Q\}.$$ 

Then $Y_y$ is isomorphic to $G_y$. We consider the subset $Y_i$ of $Y_y$ corresponding to $G_i$. Since $G_i - G_{i+1}$ coincides with the set $\{f^{(i)}_v \cdot w_i \mid v \in A^{s-i}\}$, one can write as $Y_i - Y_{i+1} = \{f^{(i)}_v g_i Q \mid v \in A^{s-i}\}$. Then we have the following commutative diagram

(3.6.4)
Here \( y''_i = y'' \) is as in (3.6.2), and \( \hat{\alpha}(x) \) is the \( C_t \)-component of \( \text{Ad}(x)^{-1}y \in m + C_t + n_Q \), \( \hat{\beta}(x) = xQ \). All the vertical maps are natural inclusions and the lower horizontal arrows are the restrictions of upper ones. Note that the right lower horizontal map is an isomorphism since \( Y_i - Y_{i+1} \approx \mathbb{A}^{s-i} \).

Let \( \mathcal{L}_t \) be the cuspidal local system on \( C_t \). Then we have a local system \( \hat{\mathcal{L}}_t \) on \( Y_y \) by the condition that \( \hat{\alpha}^*\mathcal{L}_t = \hat{\beta}^*\hat{\mathcal{L}}_t \). Since \( \mathcal{L}_t \) is a local system of rank 1, it follows from (3.6.4) that

\[
\text{(3.6.5) The restriction of } \hat{\mathcal{L}}_t \text{ on } Y_i - Y_{i+1} \text{ is the constant sheaf } \mathbb{Q}_t.
\]

We now consider the commutative diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xleftarrow{\hat{\alpha}} & \hat{\mathcal{P}}_y \\
\uparrow & & \uparrow \\
C_M \times \{y''_i\} & \xleftarrow{\alpha''} & \beta^{-1}(p^{-1}(Y_i - Y_{i+1})) & \xrightarrow{\beta''} & p^{-1}(Y_i - Y_{i+1}).
\end{array}
\]

Here all the vertical maps are natural inclusions, and the horizontal maps \( \alpha'' \) and \( \beta'' \) are the restrictions of \( \alpha \) and \( \beta \). By (3.5.1), we have

\[
\text{(3.6.7)} \quad p^{-1}(Y_i - Y_{i+1}) \simeq \mathcal{P}^M_{y'} \times (Y_i - Y_{i+1}),
\]

\[
\beta^{-1}(p^{-1}(Y_i - Y_{i+1})) \simeq \hat{\mathcal{P}}^M_{y'} \times f^{(i)}(\mathbb{A}^{s-i})g_i,
\]

and under the above isomorphisms, the maps \( \alpha'' \), \( \beta'' \) are given as

\[
\alpha''(x, f^{(i)}g_i) = (\alpha'(x), y''_i), \quad \beta''(x, f^{(i)}g_i) = (\beta'(x), v)
\]

for \( x \in \mathcal{P}^M_{y'}, \ v \in Y_i - Y_{i+1} \approx \mathbb{A}^{s-i} \).

Now the restriction of \( \mathcal{L} \) to \( C_M \times \{y''_i\} \) is a local system \( \mathcal{L}_M \boxtimes \mathbb{Q}_t \). Hence by making use of (3.6.6) and (3.6.7), we have

\[
\text{(3.6.8) Under the isomorphism } p^{-1}(Y_i - Y_{i+1}) \simeq \mathcal{P}^M_{y'} \times (Y_i - Y_{i+1}), \text{ the restriction of } \hat{\mathcal{L}} \text{ on } p^{-1}(Y_i - Y_{i+1}) \text{ coincides with } \hat{\mathcal{L}}_M \boxtimes \mathbb{Q}_t.
\]

It follows from (3.6.8) that we have an isomorphism

\[
\text{(3.6.9) } H^k_c(p^{-1}(Y_i - Y_{i+1}), \hat{\mathcal{L}}) \simeq H^{k'}_c(\mathcal{P}^M_{y'}, \hat{\mathcal{L}}_M),
\]

where \( k \equiv k' \mod 2 \). Then using the locally trivial filtration of \( \mathcal{P}_y = p^{-1}(Y_0) \supset p^{-1}(Y_1) \supset \cdots \), and by induction on the rank of \( G \), we see that

\[
\text{(3.6.10) } H^{\text{odd}}_c(p^{-1}(Y_i), \hat{\mathcal{L}}) = 0
\]

for any \( i \geq 0 \).
3.7. We are now ready to prove Theorem 3.4. Put $m = a_0 + r$. First we note the following.

\[(3.7.1) \quad H^m_c (P_y, \hat{\mathcal{L}}) = H^m_c (P_y, \hat{\mathcal{L}})_{\rho}, \text{ and the map } \Phi \text{ acts on } H^m_c (P_y, \hat{\mathcal{L}}) \text{ as a scalar multiplication.}\]

In fact, it follows from Section 2 that $H^m_c (P_y, \hat{\mathcal{L}})$ has a natural structure of $W \times A_G (y)$-module, which is compatible with the isomorphisms (1.4.3) and (1.6.2). Hence by the generalized Springer correspondence, it is decomposed as

$$H^m_c (P_y, \hat{\mathcal{L}}) \simeq \bigoplus_{\rho' \in A_G (y)} V_{y, \rho} \otimes \rho',$$

where $V_{y, \rho'}$ is an irreducible $W$-module whenever it is non-zero. Now the explicit description of the generalized Springer correspondence in the case of $SL_n$ (see 3.2) shows that $\rho$ is the unique character such that $V_{y, \rho} \neq 0$. Hence $H^m_c (P_y, \hat{\mathcal{L}}) = H^m_c (P_y, \hat{\mathcal{L}})_{\rho}$. Since $A_G (y)$ is abelian, $H^m_c (P_y, \hat{\mathcal{L}})$ is an irreducible $W$-module. It is easy to see that the map $\Phi$ on $H^m_c (P_y, \hat{\mathcal{L}})$ commutes with the action of $W$. Hence $\Phi$ is a scalar multiplication, and so (3.7.1) holds.

Note that in the discussion of 3.5 and 3.6, $G_y$, $Y_y$, etc. have natural $F_q$-structures. We may choose the filtration of $G_y$ and $Y_y$ compatible with the $F_q$-structure, i.e., all the $Y_i$ and $\hat{Y}_y$ are $F$-stable. Then all the diagrams and formulas there hold with $F_q$-structure. We consider the top piece $Y_0 - Y_1$ of the filtration of $Y_y$. In this case, we may choose $g = g_0 = 1$ in the discussion in 3.6, and so $y$ is decomposed as $y = y' + z'$ with $y' \in m$ and $z' \in C_l + n_Q$. Hence $y'$ (resp. $y''$) is the projection of $y$ on $m$ (resp. on $C_l$). Since $y$ is a split element, $y'$, $y''$ are also split. Let

$$m' = (\dim M - \dim C_{y'}) - (\dim L_M - \dim C_M),$$

where $C_{y'}$ is the nilpotent orbit in $m$ containing $y'$. Since $C$ is the regular nilpotent orbit in $l$, we see easily that $m = 2 \dim B_y$, where $B_y$ is the variety of Borel subgroups whose Lie algebra contains $y$. Similarly, we have $m' = 2 \dim B_{y'}^M$. Then by using the locally trivial filtration of $B_y$, we see that

\[(3.7.2) \quad m - m' = 2d \dim \ker y = 2s.\]

In fact, assume that $y = y_\lambda$ with $\lambda = (\lambda_1, \ldots, \lambda_k)$. By using the locally trivial filtration arising from the maximal parabolic subgroup $P_1$ of $G$ with Levi
subgroup $L_1$ of type $A_{n-2}$, one obtains that $\dim B_y - \dim B_{y_1}^{L_1} = \dim \ker y$, where $y_1$ is a nilpotent element in $\text{Lie } L_1$ of type $\lambda' = (\lambda_1, \ldots, \lambda_k - 1)$. In the same way, one can find a nilpotent element $y_2 \in \text{Lie } L_2$ with type $(\lambda_1, \ldots, \lambda_k - 2)$ such that $\dim B_{y_1}^{L_1} - \dim B_{y_2}^{L_2} = \dim \ker y$, where $L_2$ is a Levi subgroup of the maximal parabolic subgroup $P_2$ of $L_1$. Repeating this procedure, one can find similar formulas for $L_1 \supset L_2 \supset \cdots \supset L_d$ with $B_{y_d}^{L_d} = B_{y_1}^{L_1}$. (3.7.2) follows from this.

Since $Y_0 - Y_1 \simeq A^s$, we have an isomorphism with $F_q$-structures

$$H_c^m(p^{-1}(Y_0 - Y_1), \hat{\mathcal{L}}) \simeq H_{c'}^m(P_{y'}^M, \hat{\mathcal{L}}_M)[s]$$

(3.7.3)

as a special case of (3.6.9), where $[s]$ is the Tate twist. (The compatibility of the Frobenius actions comes from the discussion in 3.6 by noticing that $y''$ is a split element in $C_t$.) Let $\Phi_M$ be the map on $H_{c'}^m(P_{y'}^M, \hat{\mathcal{L}}_M)$ defined in a similar way as $\Phi$. By induction on the rank of $G$, we may assume that $\Phi_M$ acts on $H_{c'}^m(P_{y'}^M, \hat{\mathcal{L}}_M)$ as a scalar multiplication by $q^{m'/2}$. Then by (3.7.3), $\Phi$ acts on $H_c^m(p^{-1}(Y_0 - Y_1), \hat{\mathcal{L}})$ as a scalar multiplication by $q^{m'/2}$. Now by using the cohomology long exact sequence with respect to the closed immersion $p^{-1}(Y_1) \subset p^{-1}(Y_0) = \mathcal{P}_u$, together with (3.6.10), we see that the natural map

$$H_c^m(p^{-1}(Y_0 - Y_1), \hat{\mathcal{L}}) \longrightarrow H_c^m(P_{y}, \hat{\mathcal{L}})$$

is injective. This proves the theorem since $\Phi$ acts on $H_c^m(P_{y}, \hat{\mathcal{L}})$ by a scalar multiplication by (3.7.1).

§4. $G = SL_n$ with $F$ of non-split type

4.1. In this section, we assume that $G = SL_n$ as in Section 3, and that $p$ is large enough so that the argument in Section 2 can be applied (e.g., $p > 3(n - 1)$). Let $F = \sigma F_0$ be the twisted Frobenius map on $G$, where $F_0$ is the standard Frobenius map over $F_q$ as in 3.1, and $\sigma$ is the graph automorphism on $G$ of order 2. Here we take $\sigma : G \rightarrow G$ defined by $\sigma(g) = w_0^{-1}g^{-1}w_0$ for $g \in G$ (where $w_0$ is the permutation matrix in $GL_n$ corresponding to the longest element in $S_n$, and $^tg$ means the transpose of the matrix $g = (g_{ij})$). Then $B$ and $T$ in 3.3 are $F$ and $F_0$-stable.

Unipotent classes in $G$ are all $F$-stable. In order to describe elements in $C^F$ for each unipotent class $C$, we introduce a sesqui-linear form as follows. Let $V \simeq k^n$ be as in 3.1, and $V_0$ the $F_q$-subspace of $V$ generated by $\{e_i\}$. We define a sesqui-linear form $\langle , \rangle$ on $V_0$ by $\langle \sum_i a_i e_i, \sum_j b_j e_j \rangle = \sum_i a_i b_{n-i}^j$. Then it is easy to see that for $g \in G^{F_0^2}$, $g \in G^F$ if and only if $\langle gv, gw \rangle = \langle v, w \rangle$. The Frobenius actions comes from the discussion in 3.6 by noticing that $\Phi_M$ acts on $H_{c'}^m(P_{y'}^M, \hat{\mathcal{L}}_M)$ as a scalar multiplication by $q^{m'/2}$.
\[ \langle v, w \rangle \] for any \( v, w \in V_0 \). Let \( \mathfrak{g} \) be the Lie algebra of \( G \), on which \( F \) acts naturally. Then for \( x \in \mathfrak{g}^{F^2}, x \in \mathfrak{g}^F \) if and only if \( \langle xv, w \rangle + \langle v, xw \rangle = 0 \) for any \( v, w \in V_0 \).

**4.2.** For a partition \( \lambda = (\lambda_1, \ldots, \lambda_r) \) of \( n \), we shall construct a nilpotent element \( y_\lambda \in \mathfrak{g}^{F^2} \). First we note that there exist basis vectors \( f_j^{(i)} \) \( (1 \leq i \leq r, 1 \leq j \leq \lambda_i) \) of \( V_0 \) satisfying the property that

\[
\langle f_j^{(i)}, f_k^{(i')} \rangle = \begin{cases} 
1 & \text{if } i = i', j + k = \lambda_i + 1 \text{ and } j \neq k, \\
\pm 1 & \text{if } i = i', j + k = \lambda_i + 1 \text{ and } j = k, \\
0 & \text{otherwise.}
\end{cases}
\]

In fact, we can choose \( f_j^{(i)} = e_k \) for some \( k \) if \( \lambda_i \) is even. If \( \lambda_i \) is odd, put \( \lambda_i = 2t_i + 1 \). Then \( f_j^{(i)} \) is of the form \( e_k \) if \( j \neq t_i + 1 \), and we can choose \( f_{t_i+1}^{(i)} \) from one of the vectors \( e_k \pm \frac{1}{2} e_{n-k+1} \) with \( 2k \neq n + 1 \) (note that \( p > 2 \)), or \( e_t \) with \( n = 2t + 1 \).

Put \( t_i = [\lambda_i/2] \) \( ([ ] \) is the Gauss symbol) for each \( \lambda_i \). We now define a nilpotent transformation \( y_\lambda \in \mathfrak{g}^{F^2} \) on \( V_0 \) by

\[
y_\lambda f_j^{(i)} = \begin{cases} 
 f_{j+1}^{(i)} & \text{if } 1 \leq j \leq t_i - 1, \\
 \varepsilon_i f_{j+1}^{(i)} & \text{if } j = t_i, \\
 -f_{j+1}^{(i)} & \text{if } t_i + 1 \leq j \leq \lambda_i - 1, \\
 0 & \text{if } j = \lambda_i,
\end{cases}
\]

where \( \varepsilon_i = 1 \) if \( \lambda_i \) is even, and \( \varepsilon_i = \langle f_{j+1}^{(i)}, f_{j+1}^{(i')} \rangle \) if \( \lambda_i \) is odd. Then

\[
(4.2.1) \quad \{ y_\lambda^j f_1^{(i)} \mid 1 \leq i \leq r, 0 \leq j \leq \lambda_i - 1 \}
\]

gives a basis of \( V_0 \) satisfying the relation

\[
(4.2.2) \quad \langle y_\lambda^j f_1^{(i)}, y_\lambda^{\lambda_i-j+1} f_1^{(i')} \rangle = (-1)^{2j} a_i, \quad (a_i = \pm 1)
\]

and \( \langle y_\lambda^j f_1^{(i)}, y_\lambda^k f_1^{(i')} \rangle = 0 \) for all other pairs. It follows from this that \( y_\lambda \in \mathfrak{g}^{F^2} \).

Let \( d \) be as in 3.2, and assume that \( d \geq 2 \). We assume that the partition \( \lambda \) satisfies the condition that all the parts \( \lambda_i \) are divisible by \( d \). We shall
construct a nilpotent element $y_1 \in \mathfrak{g}^F$ of type $\nu = (d, \ldots, d)$ associated to $y_\lambda$. We define a map $y_1$ on $V_0$ by

$$(4.2.3) \quad y_1 f_j^{(i)} = \begin{cases} 0 & \text{if } j \equiv 0 \pmod{d}, \\ y_\lambda f_j^{(i)} & \text{otherwise.} \end{cases}$$

Then in view of (4.2.2), it is easy to check that $y_1$ leaves the form $\langle \cdot, \cdot \rangle$ invariant, and we have $y_1 \in \mathfrak{g}^F$.

4.3. Let $L$ be a Levi subgroup of the standard parabolic subgroup $P$ of $G$ of type $A_{d-1} \times \cdots \times A_{d-1}$ ($t = n/d$-factors). (Here $P$ and $L$ are as in 3.1 with respect to $F_0$, $P$ is $\sigma$-stable, $\sigma$ permutes the $i$-th factor and $(t-i+1)$-th factor, etc.) Thus $P$ and $L$ are $F$-stable. Let $\mathcal{C}$ be the regular nilpotent orbit in $t$. We choose a representative $y_0 \in C^F$ in the following way; we define a basis $\{e_j^{(i)} \mid 1 \leq i \leq t, 1 \leq j \leq d\}$ of $V_0$ by $e_j^{(i)} = e_{(i-1)d+j}$. Then in the case where $t$ is even, or $t$ is odd and $i \neq (t + 1)/2$, we define

$$y_0 e_j^{(i)} = \begin{cases} e_{j+1}^{(i)} & \text{if } 1 \leq i \leq [t/2], j \neq d, \\ -e_{j+1}^{(i)} & \text{if } t - [t/2] + 1 \leq i \leq t, j \neq d, \\ 0 & \text{if } j = d. \end{cases}$$

If $t$ is odd and $i = (t + 1)/2$, let $V_1$ be the subspace of $V$ spanned by $e_j^{(i)}$ with $1 \leq j \leq d$. We define $y_0|_{V_1}$ as a regular nilpotent element $y_\lambda \in \mathfrak{sl}_d^F$ as in 4.2.

Let $(\mathcal{C}, \mathcal{L})$ be the cuspidal pair in $t$ corresponding to an $F$-stable character $\rho_0$ of $A_L(y_0)$. We have a natural homomorphism $A_L(y_0) \to A_G(y_0)$. Since $A_G(y_0)$ is a cyclic group of order $d$, this gives an isomorphism compatible with $F$-action. Thus $\rho_0$ is regarded as an $F$-stable character of $A_G(y_0)$. Since $y_0$ and $y_1$ are conjugate under $G$, there exists $c_1 \in A_G(y_0)$ (up to $F$-conjugacy) such that $y_1$ is obtained from $y_0$ by twisting by $c_1$. Since $\rho_0$ is $F$-stable, the value $\rho_0(c_1)$ is well-defined. This value is determined by $y_1$, hence by $y_\lambda$, which we denote by $\eta_\lambda$. Let $\gamma(y_0, \tilde{\rho}_0, y_\lambda, \tilde{\rho})$ be the scalar defined by choosing the extensions $\tilde{\rho}_0, \tilde{\rho}$ in a similar way as the case of split $F$ (cf. 3.3). Put $m = a_0 + r$ as before. We have the following theorem.

**Theorem 4.4.** Assume that $p$ is large enough so that Dynkin-Kostant theory can be applied. Let $w_0$ be the longest element in $W$. Then $\Phi w_0$ acts on $H_c^m(\mathcal{P}_{y_\lambda}, \mathcal{L}) = H_c^m(\mathcal{P}_{y_\lambda}, \mathcal{L})_{\tilde{\rho}}$ as a scalar multiplication by $\eta_\lambda(-q)^{m/2}$. In particular, $\gamma(y_0, \tilde{\rho}_0, y_\lambda, \tilde{\rho}) = \eta_\lambda(-1)^{m/2}$. 


4.5. The remainder of this section is devoted to the proof of the theorem. If we notice that the preferred extension \( \widetilde{V}_E \) of \( V_E \) is given by defining the action of \( \sigma \in \mathcal{W} \) by the action of \( w_0 \in \mathcal{W} \), the second statement follows easily from the first one. So we concentrate on the proof of the first statement. For \( y_1 \) of type \((d, \ldots, d)\), we construct a \( \mathfrak{sl}_2 \)-triple \( \{y_1, y_1^- , h_1\} \) as follows. On each Jordan block, \( y_1 \) can be expressed as a matrix of degree \( d \) with respect to the basis in (4.2.1) as

\[
Y = \begin{pmatrix}
0 & & \\
1 & \ddots & \\
& \ddots & 0 \\
& & 1 & 0
\end{pmatrix}.
\]

We define matrices \( Y^- , H \) of degree \( d \) by

\[
Y^- = \begin{pmatrix}
0 & 1 \cdot (d - 1) & \cdots & 2(d - 2) \\
0 & 2(d - 2) & \cdots & 0 \\
& \ddots & \ddots & \ddots \\
& & \ddots & (d - 1) \cdot 1
\end{pmatrix},
\]

\[
H = \begin{pmatrix}
1 - d & 3 - d & \cdots & d - 1 \\
3 - d & \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
& & \ddots & d - 1
\end{pmatrix}.
\]

Then \([H, Y] = 2Y, [H, Y^-] = -2Y, [Y, Y^-] = H\). Thus by combining these matrices for all the Jordan blocks, one obtains \( y_1^-, h_1 \in \mathfrak{g} \) satisfying the property that \([h_1, y_1] = 2y_1, [h_1, y_1^-] = -2y_1^-, [y_1, y_1^-] = h_1\) as asserted. It follows easily from the construction that \( y_1^-, h_1 \in \mathfrak{g}^F \).

We define a transversal slice \( \Sigma \) with respect to the orbit through \( y_1 \) in \( \mathfrak{g} \) by \( \Sigma = y_1 + Z_\mathfrak{g}(y_1^-) \). Hence \( \Sigma \) is \( F \)-stable. We have the following lemma.

**Lemma 4.6.** Let \( y_\lambda \) be as in 4.2. Then we have \( y_\lambda \in \Sigma^F \).

**Proof.** We write \( y_\lambda \) as \( y_\lambda = y_1 + y \). It is enough to show that \( y \in Z_\mathfrak{g}(y_1^-) \). Now \( y \) is a nilpotent transformation on \( V_0 \) determined by the condition that

\[
y : y_\lambda^j f_1^{(i)} \mapsto y_\lambda^{j+1} f_1^{(i)}
\]
for \( j \equiv 0 \pmod{d} \), and it maps all other \( y^i_jf^{(i)}_1 \) to 0. Since \( y^-_i \) maps \( y^i_jf^{(i)}_1 \) to \( y^{i-1}_jf^{(i)}_1 \) up to scalar if \( j \not\equiv 1 \pmod{d} \), and to 0 if \( j \equiv 1 \pmod{d} \). we see easily check that \( y^-_i \circ y = y \circ y^-_i = 0 \) on \( V_0 \). Hence \( y \in Z_g(y^-_i) \).

4.7. By using the \( \mathfrak{sl}_2 \)-triple \( \{y_1, y^-_1, h_1\} \), one can define a Lie algebra homomorphism \( \phi_1 : \mathfrak{sl}_2 \to \mathfrak{g} \) as in 2.7. The construction of \( \mathfrak{sl}_2 \)-triple given in 4.5 works well for \( y_\lambda \) in general, and one gets the \( \mathfrak{sl}_2 \)-triple containing \( y_\lambda \). We denote by \( \phi_\lambda \) the homomorphism \( \mathfrak{sl}_2 \to \mathfrak{g} \) obtained from it. Thus \( Z_G(\phi), M_G(\phi), \) are defined as in 2.7 for \( \phi = \phi_1, \phi_\lambda \).

Let \( \pi : \mathfrak{g} \to \mathfrak{g} \) be as in 2.8. Then \( \mathcal{P}_y \subset \mathfrak{g} \), and the local system \( \hat{\mathcal{L}} \) on \( \mathcal{P}_y \) can be extended to a local system on \( \hat{\mathfrak{g}} \) (cf. 2.8), which we denote also by \( \hat{\mathcal{L}} \) as in 2.8. Put \( K_1 = \pi_1\hat{\mathcal{L}} \). \( K_1 \) is essentially the same as \( K = \pi_1\hat{\mathcal{L}}^* \) in 2.8, and so \( K_1[\delta] \) is a perverse sheaf on \( \mathfrak{g} \) with a canonical \( \mathcal{W} \)-action, where \( \delta \) is as in 2.8. By making use of the transversal slice \( \Sigma \), we show the following proposition.

**Proposition 4.8.** There exist natural maps of \( \mathcal{W} \)-modules, which make the following diagram commutative.

\[
\begin{array}{ccc}
\mathbb{H}^i(\mathfrak{g}, K_1) & \xrightarrow{\pi_1} & H^i_c(\mathcal{P}_y_1, \hat{\mathcal{L}}) \\
\pi_\lambda \downarrow & & \downarrow \xi_\lambda \\
H^i_c(\mathcal{P}_y_\lambda, \hat{\mathcal{L}}) & & \\
\end{array}
\]

Moreover, the map \( \xi_\lambda \) is equivariant with respect to the actions of \( \Phi \) on both cohomologies.

**Proof.** By the inclusion \( \{y_\lambda\} \subset \Sigma \subset \mathfrak{g} \), we have the canonical maps

\[
\begin{array}{ccc}
\mathbb{H}^i(\mathfrak{g}, K_1) & \xrightarrow{\pi_1} & \mathbb{H}^i(\Sigma, K_1) \\
\downarrow & & \downarrow \\
\mathcal{H}^i_{y_\lambda}(K_1) & & \mathcal{H}^i_{y_\lambda}(K_1) \\
\end{array}
\]

(4.8.1)

Since \( K_1 \) is a \( \mathcal{W} \)-complex with respect to the trivial action of \( \mathcal{W} \) on \( \mathfrak{g} \), the above maps are \( \mathcal{W} \)-equivariant. Since \( K_1 = \pi_1\hat{\mathcal{L}} \), we have

\[
\mathcal{H}^i_{y_\lambda}(K_1) \simeq H^i_c(\mathcal{P}_y_\lambda, \hat{\mathcal{L}})
\]
by the proper base change theorem. On the other hand, \( K_1[\delta] \) is a perverse sheaf on \( g \). Since the morphism \( G \times \Sigma \to g \) is smooth with all fibres of pure dimension equal to \( \dim Z_G(y_1) \), by a similar argument as in [L6, 3.2], \( K_1[\dim \Sigma] \mid_{\Sigma} \) is a perverse sheaf on \( \Sigma \). \( \Sigma \) is stable under the \( G_m \)-action (\( t : x \mapsto t^{i-2}x \) for each \( x \in g_i \) with respect to the grading \( g = \bigoplus g_i \) associated to \( \phi_1 : sl_2 \to g \)), and contracts to \( y_1 \in \Sigma \). Since \( K_1 \) is \( G_m \)-equivariant, the canonical map \( H^i(\Sigma, K_1) \to \mathcal{H}^i_{y_1}(K_1) \) gives rise to an isomorphism
\[
\mathbb{H}^i(\Sigma, K_1) \simeq \mathcal{H}^i_{y_1}(K_1) \simeq H^i_c(\mathcal{P}_{y_1}, \mathcal{L}).
\]
The proposition follows from this. \( \square \)

For the special case where \( i = 0 \), we have the following more precise result.

**Lemma 4.9.** The maps \( \pi_1, \pi_\chi \) in Proposition 4.8 give isomorphisms
\[
\mathbb{H}^0(\mathfrak{g}, K_1) \simeq H^0_c(\mathcal{P}_{y_1}, \mathcal{L}) \simeq \Gamma(\mathcal{C}, \mathcal{L})
\]
for \( y = y_1, y_\chi \). In particular, \( \xi_\chi : H^0_c(\mathcal{P}_{y_1}, \mathcal{L}) \to H^0_c(\mathcal{P}_{y_\chi}, \mathcal{L}) \) is an isomorphism.

**Proof.** We consider the following commutative diagram
\[
\begin{array}{ccccccccc}
\mathcal{C} & \xleftarrow{\alpha} & \mathcal{g} & \xrightarrow{\beta} & \mathcal{g} & \xrightarrow{\pi} & \mathfrak{g} \\
\uparrow{\hat{j}} & & \uparrow{\hat{j}} & & \uparrow{j} & & \uparrow{id} \\
\mathcal{C} & \xleftarrow{\alpha} & \hat{\mathcal{g}} & \xrightarrow{\beta} & \mathcal{g} & \xrightarrow{\pi} & \mathfrak{g}
\end{array}
\]
(4.9.1)
where the lower horizontal maps are as in 2.8, and
\[
\hat{\mathcal{g}}' = \{ (x, gP) \in \mathfrak{g} \times G/P \mid \text{Ad}(g^{-1})x \in \mathcal{C} + \mathfrak{j} + nP \},
\hat{\mathcal{g}} = \{ (x, g) \in \mathfrak{g} \times G \mid \text{Ad}(g^{-1})x \in \mathcal{C} + \mathfrak{j} + nP \}
\]
and \( \alpha, \beta \) and \( \pi \) are maps defined in a similar way as \( \alpha, \beta \) and \( \pi \). \( \hat{j}, j \) are open immersions, and \( \pi \) is proper. Now the local system \( \mathcal{L} \) on \( \hat{\mathfrak{g}} \) is determined by the condition that \( \alpha^* \mathcal{L} = \beta^* \mathcal{L} \). Since the square in the left hand side in (4.9.1) is cartesian, \( \hat{j}_!(\alpha^* \mathcal{L}) \simeq \alpha_!(\hat{j}_! \mathcal{L}) \). The middle square is also cartesian, and we have
\[
\overline{\beta} \circ (\hat{j}_! \mathcal{L}) \simeq \hat{j}_!(\beta^* \mathcal{L}) \simeq \hat{j}_!(\alpha^* \mathcal{L}) \simeq \overline{\alpha} \circ (\hat{j}_! \mathcal{L}).
\]
By the definition of the direct image with compact support, we have \( \pi_! \mathcal{L} = \pi_*(j_! \mathcal{L}) \). Then
\[
\mathbb{H}^0(\mathfrak{g}, K_1) = \mathbb{H}^0(\mathfrak{g}, \pi_! \mathcal{L}) = \mathbb{H}^0(\mathfrak{g}, \pi_*(j_! \mathcal{L})) = H^0(\mathfrak{g}', j_! \mathcal{L}).
\]
Similarly, by using the open immersion
\[ j : \mathcal{P}_y \hookrightarrow \mathcal{P}_y = \{gP \in G/P \mid \text{Ad}(g^{-1})y \in \mathcal{C} + \mathfrak{n}_P\}, \]
we see that \( H^0_c(\mathcal{P}_y, \mathcal{L}) \simeq H^0(\mathcal{P}_y, j_! \mathcal{L}) \) for \( y = y_1, y_\lambda \). It follows that the maps \( \pi_1, \pi_\lambda \) in Proposition 4.8 for \( i = 0 \) are nothing but the restriction \( \Gamma(\mathfrak{g}', j_! \mathcal{L}) \to \Gamma(\mathcal{P}_y, j_! \mathcal{L}) \) of the global section of the sheaf \( j_! \mathcal{L} \) on \( \mathfrak{g}' \) for \( y = y_1, y_\lambda \). But since \( \mathcal{P}_y = \mathcal{P}_{\mathfrak{n}_P} \), we have
\[
\Gamma(\mathfrak{g}', j_! \mathcal{L}) \simeq \Gamma(\mathcal{P}_y, j_! \mathcal{L}) \simeq \Gamma(\mathcal{P}_y, j_! \mathcal{L}).
\]
Finally, we note that \( j_! \mathcal{L} \simeq j_* \mathcal{L} \) since \( \mathcal{L} \) is the cuspidal local system and so is clean ([L7, 2.2]). Hence
\[
\Gamma(\mathcal{C}, j_! \mathcal{L}) \simeq \Gamma(\mathcal{C}, j_* \mathcal{L}) \simeq \Gamma(\mathcal{C}, \mathcal{L})
\]
as asserted.

4.10. Let \( \phi_0 : \mathfrak{sl}_2 \to \mathfrak{I} \subset \mathfrak{g} \) be the Lie algebra homomorphism such that \( \phi_0 \left( \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right) = y_0 \) constructed as in 4.3. Thus \( \phi_0 \) is \( F \)-equivariant with respect to the twisted Frobenius action on \( \mathfrak{sl}_2 \). Put \( G_0 = Z_G^0(\phi_0) \) and \( T_0 = Z_L^0(\phi_0) \). Then \( G_0 \) and \( T_0 \) are \( F \)-stable. It is checked that \( G_0 \) is isomorphic to \( SL_t \), and \( F \) acts as a twisted Frobenius endomorphism on \( SL_t \). By (2.9.1) we have \( T_0 \simeq Z_L^0 \), and under the identification \( G_0 \simeq SL_t \), \( T_0 \) coincides with a maximally split maximal torus of \( SL_t \), and \( \mathcal{W} = N_G(Z_L^0)/L \) is naturally isomorphic to the Weyl group of \( G_0 \) with respect to \( T_0 \).

\( F \) acts naturally on \( \mathcal{W} \simeq S_t \), as a conjugation by \( w_0 \in \mathcal{W} \), where \( w_0 \) is the longest element in \( \mathcal{W} \). Thus \( Fw_0 \) acts trivially on \( \mathcal{W} \). By 2.17, \( F \) acts naturally on \( H^*_{T_0} = \bigoplus_i H^*_{T_0} \simeq S(\mathfrak{h}^*) \), where \( \mathfrak{h}^* = Q_l \otimes \mathbb{Z} X(T_0) \). \( \mathcal{W} \) also acts on \( H^*_{T_0} \), which coincides with the action of \( \mathcal{W} \) on \( S(\mathfrak{h}^*) \) induced from the action of \( \mathcal{W} \) on \( X(T_0) \) (cf. (2.3.2)). We have the following lemma.

**Lemma 4.11.** \( Fw_0 \) acts on \( H^*_{T_0} \) as a scalar multiplication by \( (-q)^i \).
Proof. $Fw_0$ commutes with the graded algebra structure of $H^*_{T_0}$. Since $H^*_{T_0}$ is generated by $H^2_{T_0}$, it is enough to show that $Fw_0$ acts on $H^2_{T_0}$ as a scalar multiplication by $-q$. We show this by modifying the arguments used in the proof of Lemma 2.4. Let $\Gamma$ be as in the proof of Lemma 2.4 (with respect to $G_0$). We consider the locally trivial fibration $f : T_0 \setminus \Gamma \to G_0 \setminus \Gamma$. We may assume that $\Gamma$ is defined over $F_q$, and $f$ is $F$-equivariant. We consider the spectral sequence

\begin{equation}
H^p(G_0 \setminus \Gamma, R^q f_* \overline{Q}_l) \implies H^{p+q}(T_0 \setminus \Gamma),
\end{equation}

which have natural actions of $\mathcal{W}$ (cf. 2.4) and of $F$. Let $\theta$ be the reflection representation of $\mathcal{W}$. Then (4.11.1) implies a spectral sequence

\[H^p(G_0 \setminus \Gamma, R^q f_* \overline{Q}_l)_{\theta} \implies H^{p+q}(T_0 \setminus \Gamma)_{\theta},\]

where $X_{\theta}$ denotes the $\theta$-isotypic subspace for a $\mathcal{W}$-module $X$. As in 2.4, we have

\[H^p(G_0 \setminus \Gamma, R^q f_* \overline{Q}_l) \simeq H^p(G_0 \setminus \Gamma) \otimes H^q(T_0 \setminus G_0),\]

and so

\[H^p(G_0 \setminus \Gamma, R^q f_* \overline{Q}_l)_{\theta} \simeq H^p(G_0 \setminus \Gamma) \otimes H^q(T_0 \setminus G_0)_{\theta}\]

since $\mathcal{W}$ acts trivially on $H^p(G_0 \setminus \Gamma)$. Now it is known that $\bigoplus_i H^{2i}(T_0 \setminus G_0)$ is a graded regular representation of $\mathcal{W}$, and that

\[H^q(T_0 \setminus G_0)_{\theta} = \begin{cases} H^q(T_0 \setminus G_0) & \text{if } q = 2, \\ 0 & \text{if } q < 2. \end{cases}\]

Since $H^*(T_0 \setminus \Gamma) = H^*_{T_0} \simeq S(\mathfrak{h}^*)$, we have $H^2(T_0 \setminus G_0)_{\theta} = H^2(T_0 \setminus \Gamma)$. Moreover, $H^0(G_0 \setminus \Gamma) = H^0_G = \overline{Q}_l$ be Lemma 2.4. It follows that

\[H^2(T_0 \setminus \Gamma) \simeq H^2(T_0 \setminus G_0).\]

This isomorphism is compatible with the actions of $F$ and $\mathcal{W}$. It is well-known that $Fw_0$ acts as a scalar multiplication $-q$ on $H^2(T_0 \setminus G_0) = H^2(B_0 \setminus G_0)$, where $B_0$ is the $F$-stable Borel subgroup of $G_0$ containing $T_0$. Hence $Fw_0$ acts similarly on $H^2_{T_0} = H^2(T_0 \setminus \Gamma)$. This proves the lemma. □

4.12. We consider the equivariant homology $H^{M_0(y_\lambda)}_*(\mathcal{P}_{y_\lambda}, \mathcal{L}^*)$, where $M_0(y_\lambda) = M_G(y_\lambda)$. By results in Section 2, the graded Hecke algebra $H = S \otimes \overline{Q}_l[\mathcal{W}]$ acts on $H^{M_0(y_\lambda)}_*(\mathcal{P}_{y_\lambda}, \mathcal{L}^*)$, where $S = S(\mathfrak{h}^*) \otimes \overline{Q}_l[\mathfrak{r}]$ as
defined in 2.9 with $S(\mathfrak{h}^*)$ in 4.10. We shall construct a standard $\mathbf{H}$-module $E_{v,\rho'}$ obtained from $H^*_{M^0}(y_\lambda)(\mathcal{P}_{y_\lambda},\hat{\mathcal{L}}^*)$ for a certain pair $(v,\rho')$. Let $y$ be the nilpotent element in $\mathfrak{g}_C$ corresponding to $y_\lambda \in \mathfrak{g}$. We choose $y^-,h_0 \in \mathfrak{g}_C$ such that $\{y,y^-,h_0\}$ forms an $\mathfrak{sl}_2$-triple. Put $h = h_0$, $r_0 = 1$. Then $(h,r_0) \in \mathfrak{m}(y)C$ with $h$ semisimple. We denote by $v$ an element in $H^*_{M^0}(y_\lambda)$ corresponding to the $M^0(y)$-orbit of $(h,r_0)$. Let $\rho$ be the irreducible character of $\text{Ad}(y_\lambda)$ as in 4.3. Since $A_G(y_\lambda) \simeq M(y_\lambda) \simeq \overline{M}(y)$, one can regard $\rho$ as a character of $\overline{M}(y)$. Let $\rho^*$ be the dual representation of $\rho$.

Under the notation in Remark 2.14 and Theorem 2.15, we note that

(4.12.1) Let $v$ be as above. Then $E_{v,\rho'}$ is a simple $\mathbf{H}$-module, where $\rho'$ is the restriction of $\rho^*$ on $\overline{M}(y,v)$.

It is enough to show that $(h,r_0)$ satisfies the property in Theorem 2.15. By Dynkin-Kostant theory, $y$ is contained in the open dense orbit in $Y(h,r_0) = \mathfrak{g}_2$ (the graded space with respect to $h$) under the action of $Z_{G_C}(h)$. It remains to show that $\rho'$ occurs in $H^*_{\{e\}}(\mathcal{P}_{y_\lambda},\hat{\mathcal{L}}^*)$. But this is clear since $H^c_{\{e\}}(\mathcal{P}_{y_\lambda},\hat{\mathcal{L}}) = H^c_{\{e\}}(\mathcal{P}_{y_\lambda},\hat{\mathcal{L}})_\rho$. Thus (4.12.1) holds.

4.13. The $\mathbb{F}_q$-structure $\varphi_0 : F^*\mathcal{L} \simeq \mathcal{L}$ induces a linear isomorphism $\Phi$ on $H^*_{\{e\}}(\mathcal{P}_{y_\lambda},\hat{\mathcal{L}})$. $\varphi_0$ also induces a linear isomorphism $\Psi$ on $H^*_{M^0}(y_\lambda)(\mathcal{P}_{y_\lambda},\hat{\mathcal{L}}^*)$ satisfying the following property; by [L6, 7.2, (d)], there exists a $\bar{Q}_l$-linear isomorphism

(4.13.1) $\bar{Q}_l \otimes_{H^*_{M^0}(y_\lambda)} H^*_{M^0}(y_\lambda)(\mathcal{P}_{y_\lambda},\hat{\mathcal{L}}^*)$ $\longrightarrow$ $H^*_{\{e\}}(\mathcal{P}_{y_\lambda},\hat{\mathcal{L}}^*)$,

where $\bar{Q}_l$ is regarded as an $H^*_{M^0}(y_\lambda)$-module via the canonical map $H^*_{M^0}(y_\lambda) \to H^*_{\{e\}} = \bar{Q}_l$. $F$ acts naturally on $H^*_{M^0}(y_\lambda)$ and on $H^*_{\{e\}}$, and the last map is $F$-equivariant with respect to the trivial action on $\bar{Q}_l$. Thus $\Psi$ induces a linear map $\overline{\Psi}$ on the left hand side of (4.13.1). The $\bar{Q}_l$-linear map in (4.13.1) is compatible with $\overline{\Psi}$ and the map $\Phi^*$ on $H^*_{\{e\}}(\mathcal{P}_{y_\lambda},\hat{\mathcal{L}}^*) = H^c_{\{e\}}(\mathcal{P}_{y_\lambda},\hat{\mathcal{L}})^*$, where $\Phi^*$ is the transposed inverse of $\Phi$.

Note that $H^c_{\{e\}}(\mathcal{P}_{y_\lambda},\hat{\mathcal{L}})$ is an irreducible $\mathcal{W}$-module. Since $\Phi w_0$ commutes with all the elements in $\mathcal{W}$, we see that $\Phi w_0$ acts on $H^c_{\{e\}}(\mathcal{P}_{y_\lambda},\hat{\mathcal{L}})$ as a scalar multiplication. Then we have the following lemma.

**Lemma 4.14.** Assume that $\Phi w_0$ acts on $H^c_{\{e\}}(\mathcal{P}_{y_\lambda},\hat{\mathcal{L}})$ by a scalar multiplication by $\zeta$. Then $\Phi w_0$ acts on $H^0_{\{e\}}(\mathcal{P}_{y_\lambda},\hat{\mathcal{L}})$ by a scalar multiplication by $\zeta(-q^{-1})^{m/2}$. 

---

*Note: The IP address 54.70.40.11, on 04 Jun 2019 at 17:33:05, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms.*
Proof. Let \( v = (h, r_0) \) be as in (4.12.1). Let \( \gamma_v : H^*_{M^0(\lambda)} \to \tilde{Q}_l \) be the algebra homomorphism corresponding to \( v \) (cf. 2.13). Since \( M^0(\lambda) \) is \( F \)-stable, \( F \) acts naturally on \( H^*_{M^0(\lambda)} \) such that \( \Psi(mx) = F(m)\Psi(x) \) for \( m \in H^*_{M^0(\lambda)} \) and \( x \in H^*_{M^0(\lambda)}(\mathcal{P}_{y\lambda}, \mathcal{L}^*) \). Since \( H^*_G(\lambda) \simeq Z^*_G(\lambda) \times G_m \), we have \( H^*_{M^0(\lambda)} \simeq S(h_1^*)W_1 \otimes \tilde{Q}_l[r] \), where \( W_1 \) is the Weyl group of a reductive group \( Z^*_G(\phi_\lambda) \) and \( h_1^* = \tilde{Q}_l \otimes \mathbb{Z}X(T_1) \) with a maximally split maximal torus \( T_1 \) of \( Z^*_G(\phi_\lambda) \). We note that

(4.14.1) The maximal ideal \( \text{Ker} \gamma_v \) in \( H^*_{M^0(\lambda)} \) is generated by homogeneous polynomials.

In fact, by the previous argument, we may replace \( H^*_{M^0(\lambda)} \) by \( S(m(y)C, r)^{M(\lambda)} \), and \( v \) by \( (h, r_0) \in m(y)C, r \). It is enough to show that if a polynomial function \( f \) on \( m(y)C, r \) which is invariant under the action of \( M^0(\lambda) \) vanishes on \( (h, r_0) \), then its homogeneous parts also vanish at \( (h, r_0) \). But the \( G_m \)-action on \( m(y)C \) implies that \( t : (h, r_0) \mapsto (t^{-2}h, t^{-2}r_0) \). Since \( f \) is invariant under \( M^0(\lambda) \), we see that \( f \) vanishes also on \( (t^{-2}h, t^{-2}r_0) \) for any \( t \in C^* \). It follows that each homogeneous part of \( f \) also vanishes at \( (h, r_0) \) as asserted.

Next we note that

(4.14.2) The maximal ideal \( \text{Ker} \gamma_v \) is \( F \)-stable.

Let \( w_1 \) be the longest element in \( W_1 \). As in Lemma 4.11, \( Fw_1 \) acts on \( S(h_1^*)_i \); (the \( i \)-th homogeneous part) as a scalar multiplication by \( (-q)^i \). Hence \( F \) acts on \( S(h_1^*)_i \) by a scalar multiplication by \( (-q)^i \). Also, \( F \) acts on \( \tilde{Q}_l[r]_i \) as a scalar multiplication by \( q^i \). Since \( \text{Ker} \gamma_v \) is a homogeneous ideal, \( F \) stabilizes \( \text{Ker} \gamma_v \). Hence (4.14.2) holds.

Now \( E_{v, \rho'} \) is obtained as the quotient of \( H^*_s(M^0(\lambda))(\mathcal{P}_{y\lambda}, \mathcal{L}^*)_{\rho'} \) by the \( \mathbf{H} \)-submodule \( I_v = \text{Ker} \gamma_v \cdot H^*_s(M^0(\lambda))(\mathcal{P}_{y\lambda}, \mathcal{L}^*)_{\rho'} \). Since \( \text{Ker} \gamma_v \) is \( F \)-stable, we see that \( I_v \) is \( \Psi \)-stable. Thus \( \Psi \) induces a linear map on \( E_{v, \rho'} \). We consider the filtration \( F^0 \subseteq F^1 \subseteq \cdots \) of \( H^*_s(M^0(\lambda))(\mathcal{P}_{y\lambda}, \mathcal{L}^*) \) as in 2.16. Then each \( F^i \), as well as its \( \rho' \)-isotypic part \( F^i_{\rho'} \), is \( \Psi \)-stable. Then \( (F^i_{\rho'})_v \) is also \( \Psi \)-stable since it is the quotient of \( F^i_{\rho'} \) by \( F^i_{\rho'} \cap I_v \). By (2.16.5) and by our assumption, \( \Psi w_0 \) acts on the non-zero space \( F^0_v = (F^0_{\rho'})_v \) as a scalar multiplication by \( \zeta^{-1} \). This implies that \( \Psi w_0 \) acts on \( H^*_s(M^0(\lambda))(\mathcal{P}_{y\lambda}, \mathcal{L}^*) \) modulo \( I_v \) by \( \zeta^{-1} \). On the other hand, since \( E_{v, \rho'} \) is a simple \( \mathbf{H} \)-module, \( H^*_s(M^0(\lambda))(\mathcal{P}_{y\lambda}, \mathcal{L}^*)_{\rho'} \) is generated by \( H^*_s(M^0(\lambda))(\mathcal{P}_{y\lambda}, \mathcal{L}^*)_{\rho'} \) mod \( I_v \) as an \( \mathbf{H} \)-module.
module. Since \( r \) acts as a scalar multiplication by \( r_0 \) on \( E_{v, \rho'} \), the action of \( H \) on \( E_{v, \rho'} \) is given by the action of \( S(\mathfrak{h}^*) = H^*_{T_0} \) and of \( \mathcal{W} \). Note that 
\[
\Psi w_0(\xi x) = F w_0(\xi) \Psi w_0(x) \quad \text{for} \quad \xi \in S, \quad x \in H^{M_0(y_\lambda)}_*(\mathcal{P}_{y_\lambda}, \hat{\mathcal{L}}^*) .
\]
The action of \( \mathcal{W} \) preserves the grading of \( H^{M_0(y_\lambda)}_*(\mathcal{P}_{y_\lambda}, \hat{\mathcal{L}}^*) \), and \( \Psi w_0 \) commutes with \( \mathcal{W} \). It follows, by Lemma 4.11 that \( \Psi w_0 \) acts on \( H^{M_0(y_\lambda)}_*(\mathcal{P}_{y_\lambda}, \hat{\mathcal{L}}^*)_{\rho'} \) modulo \( I_v \) as a scalar multiplication by \( \zeta^{-1}(-q)^{m/2} \). Let \( f_m \) be the map \( F^{m-1}_v \to F^m_v \) as in (2.16), which is \( \overline{M}(y_\lambda, v) \)-equivariant. Since \( (F^m_v)_{\rho'}/(\text{Im } f_m)_{\rho'} \) is regarded as a natural quotient of \( H^{M_0(y_\lambda)}_*(\mathcal{P}_{y_\lambda}, \hat{\mathcal{L}}^*)_{\rho'} \) modulo \( I_v \), \( \Psi w_0 \) acts on \( (F^m_v)_{\rho'}/(\text{Im } f_m)_{\rho'} \) as \( \zeta^{-1}(-q)^{m/2} \). Since \( H^{\langle e \rangle}(\mathcal{P}_{y_\lambda}, \hat{\mathcal{L}}^*) \) is isomorphic to \( F^m_v/\text{Im } f_m \) by (2.16.4), we see that \( \Psi w_0 \) acts on \( H^{\langle e \rangle}(\mathcal{P}_{y_\lambda}, \hat{\mathcal{L}}^*)_{\rho'} \) by a scalar multiplication by \( \zeta^{-1}(-q)^{m/2} \), which coincides with the action of \( \Phi^* w_0 \) on it. We claim that \( H^{\langle e \rangle}(\mathcal{P}_{y_\lambda}, \hat{\mathcal{L}}^*) = H^{\langle e \rangle}(\mathcal{P}_{y_\lambda}, \hat{\mathcal{L}}^*)_{\rho'} \). In fact,
\[
H^{\langle e \rangle}(\mathcal{P}_{y_\lambda}, \hat{\mathcal{L}}^*) = H^0_c(\mathcal{P}_{y_\lambda}, \hat{\mathcal{L}})^* = \Gamma(\mathcal{L}, \mathcal{L})^*
\]
by Lemma 4.9. \( A_L(y_0) \) acts on \( \Gamma(\mathcal{L}, \mathcal{L}) \) by the character \( \rho_0 \). Since \( \rho \) is the pull back of \( \rho_0 \) under the map \( A_G(y_\lambda) \to A_L(y_0) \) (cf. 3.2), the action of \( \overline{M}(y_\lambda) = A_G(y_\lambda) \) on \( H^0_c(\mathcal{P}_{y_\lambda}, \hat{\mathcal{L}}) \) is via \( \rho_0 \). Hence \( H^0_c(\mathcal{P}_{y_\lambda}, \hat{\mathcal{L}}) = H^0_c(\mathcal{P}_{y_\lambda}, \hat{\mathcal{L}})_{\rho} \)
and the claim follows.

Thus \( \Phi w_0 \) acts on \( H^0_c(\mathcal{P}_{y_\lambda}, \hat{\mathcal{L}}) = H^{\langle e \rangle}(\mathcal{P}_{y_\lambda}, \hat{\mathcal{L}})^* \) by a scalar multiplication by \( \zeta(-q^{-1})^{m/2} \) as asserted.

**4.15.** We are now ready to prove Theorem 4.4. First we note that \( \mathcal{W} \) acts trivially on \( H^0_c(\mathcal{P}_{y_\lambda}, \hat{\mathcal{L}}) \) for any \( y = y_\nu \) such that all the parts of \( \nu \) are divisible by \( d \). In fact, if \( y_\nu \) is regular nilpotent, \( a_0 + r = 0 \) by (1.3.1) since \( \mathcal{L} \) is also a regular nilpotent class in \( L \). It follows, by the generalized Springer correspondence (see 3.2), that \( H^0_c(\mathcal{P}_{y_\nu}, \hat{\mathcal{L}}) \) is the irreducible \( \mathcal{W} \)-module corresponding to the unit representation. Thus by Lemma 4.9, \( H^0_c(\mathcal{P}_{y_\lambda}, \hat{\mathcal{L}}) \) is also a trivial \( \mathcal{W} \)-module for any \( y \).

Now assume that \( \Phi w_0 \) acts on \( H^m_e(\mathcal{P}_{y_\lambda}, \hat{\mathcal{L}}) \) by a scalar multiplication by \( \zeta \). Then by Lemma 4.14, \( \Phi w_0 \) acts on \( H^0_c(\mathcal{P}_{y_\lambda}, \hat{\mathcal{L}}) \) by a scalar multiplication by \( \zeta(-q^{-1})^{m/2} \). Since the map \( H^0_c(\mathcal{P}_{y_1}, \hat{\mathcal{L}}) \to H^0_c(\mathcal{P}_{y_\lambda}, \hat{\mathcal{L}}) \) is \( \Phi w_0 \)-equivariant isomorphism by Lemma 4.9 (and Proposition 4.8), we see that \( \Phi w_0 \) acts on \( H^0_c(\mathcal{P}_{y_1}, \hat{\mathcal{L}}) \) by a scalar multiplication by \( \zeta(-q^{-1})^{m/2} \). Since \( w_0 \) acts trivially on it, we see that

\[
(4.15.1) \quad \Phi \text{ acts on } H^0_c(\mathcal{P}_{y_1}, \hat{\mathcal{L}}) \text{ by } \zeta(-q^{-1})^{m/2}.
\]
On the other hand, by a similar argument as in the proof of Lemma 4.9, the natural map

\[ H_c^0(\mathcal{P}_{y_0}, \hat{\mathcal{L}}) \cong \Gamma(\mathcal{L}, \mathcal{L}) \rightarrow \mathcal{L}_{y_0} \]

gives an isomorphism. This isomorphism is compatible with the action of \( \Phi \) and of \( \varphi_0 \). It follows that \( \Phi \) acts on \( H_c^0(\mathcal{P}_{y_0}, \hat{\mathcal{L}}) \) as an identity map. Since \( y_1 \) is in the \( G \)-orbit of \( y_0 \), \( H_c^0(\mathcal{P}_{y_0}, \hat{\mathcal{L}}) \cong H_c^0(\mathcal{P}_{y_1}, \hat{\mathcal{L}}) \). As discussed in the proof of Lemma 4.14, \( \mathcal{A}_G(y_0) \) acts on \( H_c^0(\mathcal{P}_{y_0}, \hat{\mathcal{L}}) \) via \( \rho_0 \). We also note that \( \mathcal{A}_L(y_0) \cong \mathcal{A}_G(y_0) \). Since \( y_1 \) is \( G \)-conjugate to \( y_{c_1} \), an element twisted by \( c_1 \mathcal{A}_G(y_0) \) from \( y_0 \), we see that

\[ (4.15.2) \quad \Phi \text{ acts on } H_c^0(\mathcal{P}_{y_1}, \hat{\mathcal{L}}) \text{ by a scalar multiplication by } \rho_0(c_1) = \eta_\lambda. \]

Comparing (4.15.1) and (4.15.2), we see that \( \zeta = \eta_\lambda(-q)^{m/2} \). This proves the theorem.

4.16. In order to apply Theorem 4.4, we need to know \( c_1 \in \mathcal{A}_G(y_0) \) such that \( y_1 = (y_0)^{c_1} \). For a given \( y_0 \), we shall choose a specific \( y_1 \) and \( y_\lambda \), and determine \( c_1 \) explicitly. Put \( \lambda' = (\lambda_1', \ldots, \lambda_t') \) with \( \lambda_i' = \lambda_i/d \). Hence \( \lambda' \) is a partition of \( t \). Let \( \{e_j^{(i)}\} \) be the basis of \( V_0 \) as in 4.3. Put \( d' = \lfloor d/2 \rfloor \).

Let us define a subspace \( W_0 \) of \( V_0 \) and define \( y_0 \) by

\[ W_0 = \begin{cases} 
\{e_{d+1}^{(i)} \mid 1 \leq i \leq t\} & \text{if } d \text{ is odd}, \\
\{0\} & \text{if } d \text{ is even}.
\end{cases} \]

Also we define subspaces \( W_1, W_2 \) of \( V_0 \) by

\[ W_1 = \langle e_j^{(i)} \mid 1 \leq j \leq d', 1 \leq i \leq t \rangle, \]
\[ W_2 = \langle e_j^{(i)} \mid d - d' + 1 \leq j \leq d, 1 \leq i \leq t \rangle. \]

Clearly we have \( V_0 = W_1 \oplus W_0 \oplus W_2 \). We define a new basis \( \{h_j^{(i)} \mid 1 \leq j \leq d, 1 \leq i \leq t\} \) of \( V_0 \) satisfying the following conditions.

(i) \( h_j^{(i)} = e_j^{(i)} \) if \( e_j^{(i)} \in W_1 \).

(ii) The set \( \{h_j^{(i)} \mid d - d' + 1 \leq j \leq d, 1 \leq i \leq t\} \) coincides with the set of the basis \( \{e_j^{(i)}\} \) of \( W_2 \).
(iii) Let \( z \) be the number of \( i \) such that \( \lambda'_i \) is odd. Then

\[
\langle h^{(2t-1)}_j, h^{(2t)}_{d-j+1} \rangle = 1 \quad \text{for} \quad 1 \leq i \leq (t-z)/2, \ 1 \leq j \leq d,
\]

\[
\langle h^{(i)}_j, h^{(i)}_{d-j+1} \rangle = 1 \quad \text{for} \quad (t-z)/2 + 1 \leq i \leq t, \ 1 \leq j \leq d',
\]

\[
\langle h^{(i)}_{d+1}, h^{(i)}_{d+1} \rangle = \pm 1 \quad \text{for} \quad (t-z)/2 + 1 \leq i \leq t \text{ if } d: \text{ odd},
\]

\[
\langle h^{(i)}_j, h^{(j')}_{i} \rangle = 0 \quad \text{for all other cases.}
\]

(iv) \( \{h^{(i)}_{d+1}\} \) gives a basis of \( W_0 \) in the case where \( d \) is odd.

The conditions (i)–(iv) determines \( \{h^{(i)}_j\} \) uniquely except the vectors contained in \( W_0 \). We note that one can choose the basis \( \{h^{(i)}_j\} \) of \( W_0 \) so that the transition matrix between \( \{e^{(i)}_j\} \) and \( \{h^{(i)}_j\} \) has the determinant \( \pm 1 \) (see the construction of \( f^{(i)}_j \) in 4.2).

We define a nilpotent transformation \( y'_1 \in \mathfrak{g}^F \) as in 4.2 replacing \( f^{(i)}_j \) by \( h^{(i)}_j \). Then it is easy to construct \( y'_{\lambda} \in \mathfrak{g}^F \) such that \( y'_1 \) is obtained from \( y'_{\lambda} \) by a similar procedure as \( y_1 \) is obtained from \( y_{\lambda} \). Clearly, \( y'_1 \) (resp. \( y'_{\lambda} \)) is conjugate to \( y_1 \) (resp. \( y_{\lambda} \)). The argument in the proof of Theorem 4.4 works well for \( y'_1, y'_{\lambda} \). Thus we consider \( y'_1 \) and \( y'_{\lambda} \), and write them as \( y_1, y_{\lambda} \). We shall describe \( c_1 \in A_G(y_0) \) such that \( y_1 = (y_0)c_1 \). It follows from the construction that there exists \( g \in G = GL_n \) such that \( \text{Ad}(g)y_0 = y_1 \), where \( g \) stabilizes the subspaces \( W_0, W_1, W_2 \). More precisely, \( g \) acts trivially on \( W_1 \), and gives a permutation matrix with respect to the basis \( \{e^{(i)}_j\} \) up to \( \pm 1 \) on \( W_2 \). Thus by our choice of the basis \( \{h^{(i)}_j\} \), we have \( \det g = \pm 1 \). Let us take \( \alpha \in \overline{\mathbb{F}}_q \) such that \( \alpha^d = \pm 1 \) (if \( \det g = 1 \), we choose \( \alpha = 1 \)). We note that \( \text{Ker } y_0 \subset W_2 \), and that \( g \) stabilizes \( \text{Ker } y_0 \). We denote by \( g_0 \) the restriction of \( g \) on \( \text{Ker } y_0 \). Then we have

**Lemma 4.17.** Let the notations be as before. Then we have \( y_1 = (y_0)c_1 \), where \( c_1 \in A_G(y_0) \) is given, under the identification \( A_G(y_0) \cong \{x \in \overline{\mathbb{F}}_q \mid x^d = 1\} \), by

\[
c_1 = \alpha^{d(1-q)} \det g_0.
\]

**Proof.** Let \( \phi_0 : \mathfrak{sl}_2 \to \mathfrak{g} \) be as in 4.10, and we consider the group \( Z_G(\phi_0) \). Then \( A_G(y_0) \cong Z_G(\phi_0)/Z^0_G(\phi_0) \). We have \( Z_G(\phi_0) \cong \{x \in GL_t \mid \det x^d = 1\} \), where the element \( g_1 \in Z_G(\phi_0) \) corresponding to \( x \) is given as
follows; \( g_1 \) acts on the subspace \( V_j \) of \( V_0 \) spanned by \( \{ e_j^{(i)} | 1 \leq i \leq t \} \), for a fixed \( j \), as \( x \in GL_t \). Now if we can find \( g_1 \in G \) such that \( \text{Ad}(g_1)y_0 = y_1 \), then \( g_1^{-1}F(g_1) \in A_G(y_0) \), and it leaves \( \text{Ker} y_0 \) invariant. Moreover, the determinant of the restriction of \( g_1^{-1}F(g_1) \) gives rise to the corresponding element in \( A_G(y_0) \simeq \{ x \in \overline{F}_q | x^d = 1 \} \).

Now in our situation, if we put \( g_1 = \alpha^{-1}g \), we have \( g_1 \in G \) and \( \text{Ad}(g_1)y_0 = y_1 \). Then \( g_1^{-1}F(g_1) = \alpha^{-1}q^{-1}F(g) \). On the other hand, since \( F(g) = u_0(tg^{-1})w_0^{-1}, F(g) \) also stabilizes the subspaces \( W_0, W_1, W_2 \). Moreover, it acts on \( W_2 \) trivially, and on \( W_1 \) as a permutation of the basis \( \{ e_j^{(i)} \} \) up to sign. It follows that \( g^{-1}F(g) \) acts on the space \( \text{Ker} y_0 \) as \( g_0^{-1} \). Thus \( g_1^{-1}F(g_1) \) acts on \( \text{Ker} y_0 \) as a map \( \alpha^{1-q}g_0^{-1} \), and we have \( \det(\alpha^{1-q}g_0^{-1}) = \alpha^{k(1-q)} \det g_0 \) as asserted.

\[ \square \]

**References**


