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SEPARABLE DETERMINATION OF FRÉCHET DIFFERENTIABILITY OF CONVEX FUNCTIONS

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For a continuous convex function on an open convex subset of any Banach space a separability condition on its image under the subdifferential mapping is sufficient to guarantee the generic Fréchet differentiability of the function. This gives a direct insight into the characterisation of Asplund spaces.

It is well known that a continuous convex function ϕ on a separable Banach space X may be Fréchet differentiable on a dense G_δ subset of X but the image of ϕ under its subdifferential mapping be non-separable. For example, the James Tree space JT, [**6**], is separable and its dual JT^* is non-separable, but there exists an equivalent norm on JT which is Fréchet differentiable on a dense G_δ subset of JT, [**1**, p.246].

However in this note we show that for a continuous convex function on an open convex subset A of any Banach space X, if the subdifferential mapping produces a separable image on every closed separable subspace Y of X, then the function is Fréchet differentiable on a dense G_δ subset of A.

A Banach space X is said to be an *Asplund space* if every continuous convex function on an open convex subset A of X is Fréchet differentiable on a dense G_δ subset of A. A Banach space is Asplund if and only if every separable subspace has separable dual, [**9**, p.24]. Our general Propositions 1 and 2 give us direct insight into why such a structural property for a Banach space has implications for the Fréchet differentiability of all continuous convex functions defined on open convex subsets of the space.

A set-valued mapping Φ from a topological space A into non-empty subsets of a linear topological space X is *upper semi-continuous* at $t \in A$ if for every open set W where $\Phi(t) \subset W$ there exists an open neighbourhood U of t such that $\Phi(U) \subset W$. When Φ is upper semi-continuous on A and has convex compact images at each point we call Φ a *cusco* mapping. A cusco Φ is a *minimal cusco* if its graph does not contain the graph of any other cusco with the same domain. A minimal weak* cusco from a topological space A into subsets of the dual X^* of a Banach space X has the property

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that, for any open set U in A and any weak* closed convex set K in X^*, if $\Phi(t) \cap K \neq \emptyset$ for every $t \in U$ then $\Phi(U) \subset K$, [4, p.844]. Further, a set-valued mapping Φ from a topological space A into non-empty subsets of a linear topological space X is *lower semi-continuous* at $t_0 \in A$ if for every open subset W where $\Phi(t_0) \cap W \neq \emptyset$ there exists an open neighbourhood U of t_0 such that $\Phi(t) \cap W \neq \emptyset$ for each $t \in U$.

Proposition 1.

(i) *A weak* cusco Φ from a Baire space A into subsets of the dual X^* of a Banach space X where $\Phi(A)$ is separable, is norm lower semi-continuous at the points of a dense G_δ subset D of A.*

(ii) *If Φ is also a minimal weak* cusco then it is single-valued and norm upper semi-continuous at the points of D.*

Proof: (i) Since $\Phi(A)$ is separable there exists a countable base $\{B_n\}$ of open balls for the norm topology relative to $\Phi(A)$. If Φ is not norm lower semi-continuous at $t_0 \in A$ then there exists an $n_0 \in \mathbf{N}$ such that $\Phi(t_0) \cap B_{n_0} \neq \emptyset$ but in any neighbourhood U of t_0 there exists a $t \in U$ such that $\Phi(t) \cap B_{n_0} = \emptyset$. Now there exists $\overline{B_{n_1}} \subset B_{n_0}$ such that $\Phi(t_0) \cap B_{n_1} \neq \emptyset$. Then $\Phi(t) \cap B_{n_1} = \emptyset$. Now $\Phi(t) \subset C\left(\overline{B_{n_1}}\right)$ and since $C\left(\overline{B_{n_1}}\right)$ is weak* open and Φ is weak* upper semi-continuous at t there exists an open neighbourhood V of t where $V \subset U$ such that $\Phi(V) \cap \overline{B_{n_1}} = \emptyset$, and so $\Phi(V) \cap B_{n_1} = \emptyset$. Given $n \in \mathbf{N}$, the set

$$\{t \in A : \Phi(t) \cap B_n \neq \emptyset \text{ but for every open neighbourhood } U \text{ of } t \text{ there exists an open set } V \text{ of } U \text{ such that } \Phi(V) \cap B_n = \emptyset\}$$

is nowhere dense. We conclude that the subset of A where Φ is not norm lower semi-continuous is first category in A.

(ii) Consider $t_0 \in D$ and $x \in \Phi(t_0)$. Given an open ball W where $x \in W$, there exists an open ball V such that $x \in V \subset \overline{V} \subset W$. Since Φ is norm lower semi-continuous at t_0 there exists an open neighbourhood U of t_0 such that $\Phi(t) \cap V \neq \emptyset$ for each $t \in U$ and so $\Phi(t) \cap \overline{V} \neq \emptyset$ for each $t \in U$. Since $\overline{V}$ is weak* closed convex and Φ is minimal we have that $\Phi(U) \subset \overline{V}$ and so $\Phi(U) \subset W$. This implies that Φ is single-valued and norm upper semi-continuous at t_0. □

This proposition is a modification of [4, Proposition 2.3 and Theorem 2.6].

Given a continuous convex function ϕ on an open convex subset A of a Banach space X, the *subdifferential* of ϕ at $x \in A$ is the set

$$\partial\phi(x) = \{f \in X^* : f(y) \leqslant \lim_{\lambda \to 0+} \frac{\phi(x + \lambda y) - \phi(x)}{\lambda} \text{ for all } y \in X\},$$

and the *subdifferential mapping*, $x \mapsto \partial\phi(x)$, is a minimal weak* cusco from A into subsets of X^*. Now ϕ is *Fréchet differentiable* at $x \in A$ if there exists a continuous linear functional $\phi'(x)$ on X and given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left|\phi(x+y) - \phi(x) - \phi'(x)(y)\right| < \varepsilon \|y\| \quad \text{for all } \|y\| < \delta.$$

Further, ϕ is Fréchet differentiable at $x \in A$ if and only if the subdifferential mapping $x \mapsto \partial\phi(x)$ is single-valued and norm upper semi-continuous at x, [**9**, p.19].

So the proposition has implications for the differentiability of continuous convex functions.

COROLLARY 1.1. *A continuous convex function ϕ on an open convex subset A of a Banach space X where $\partial\phi(A)$ is separable is Fréchet differentiable on a dense G_δ subset of A.*

To utilise the full strength of our result we need the following proposition which is a generalisation of Gregory's method, [**2**, p.141], as developed by Veselý, [**10**, Theorem 2].

PROPOSITION 2. *A set-valued mapping Φ from an open subset A of a Banach space X into subsets of X^* is single-valued and norm upper semi-continuous at the points of a dense G_δ of A if for every closed separable subspace Y of X where $A \cap Y \neq \emptyset$, the restriction mapping $\Phi|_Y$ into subsets of Y^* is single-valued and norm upper semi-continuous at the points of a dense G_δ subset of $A \cap Y$.*

PROOF: Suppose that the set

$$H = \{x \in A : \Phi \text{ is not single-valued and norm upper semi-continuous at } x\}$$

is of the second Baire category. We construct a closed separable subspace Y of X where $A \cap Y \neq \emptyset$ and $\Phi|_Y$ is not single-valued and norm upper semi-continuous at each point of a non-empty open subset of $A \cap Y$. Now $H = \bigcup_{m=1}^{\infty} H_m$ where

$$H_m = \{x \in A : \text{ there exists an } f \in \Phi(x) \text{ and sequences } \{y_k\} \to x \text{ and } \{g_k\} \\ \text{where } g_k \in \Phi(y_k) \text{ such that } \|g_k - f\| > 1/m \text{ for all } k \in \mathbb{N}\}.$$

Since H is second category there exists an $m_0 \in \mathbb{N}$ and a non-empty open set G in A such that H_{m_0} is dense in G.

We construct inductively a sequence $\{Y_1, Y_2, \dots, Y_n, \dots\}$ of separable subspaces of X. Choose $x_0 \in H_{m_0} \cap G$. Now there exists an $f_0 \in \Phi(x_0)$ and sequences $\{y_k\} \to x_0$ and $\{g_k\}$ where $g_k \in \Phi(y_k)$ such that $\|g_k - f\| > 1/m_0$. Given $k \in \mathbb{N}$ there exists a $v_k \in X$, $\|v_k\| = 1$ such that $(g_k - f)(v_k) > 1/m_0$. Define

$$Y_0 = sp\{\{x_0\} \cup \{y_k\}_{k=1}^{\infty} \cup \{v_k\}_{k=1}^{\infty}\}.$$

Suppose we have defined $Y_0 \subset Y_1 \subset Y_2 \subset \ldots \subset Y_s$, separable subspaces of X. Now there exists a sequence $\{c_i^{(s)}\}_{i=1}^{\infty}$ dense in $Y_s \cap G$. Given $i \in \mathbb{N}$ there exists a sequence $\{x_{i,n}^{(s)}\}_{n=1}^{\infty} \subset H_{m_0} \cap G$ such that $\{x_{i,n}^{(s)}\} \to c_i^{(s)}$ as $n \to \infty$. By the definition of H_{m_0}, given $i, n \in \mathbb{N}$ there exists $f_{i,n}^{(s)} \in \Phi\left(x_{i,n}^{(s)}\right)$ and sequences $\{y_{i,n,k}^{(s)}\} \to x_{i,n}^{(s)}$ as $k \to \infty$ and $\{g_{i,n,k}^{(s)}\}$ where $\{g_{i,n,k}^{(s)}\} \in \Phi\left(y_{i,n,k}^{(s)}\right)$ such that $\left\|g_{i,n,k}^{(s)} - f_{i,n}^{(s)}\right\| > 1/m_0$. Given $k \in \mathbb{N}$ there exists $v_{i,n,k}^{(s)} \in X$, $\left\|v_{i,n,k}^{(s)}\right\| = 1$ such that $\left(g_{i,n,k}^{(s)} - f_{i,n}^{(s)}\right)\left(v_{i,n,k}^{(s)}\right) > 1/m_0$. Define

$$Y_{s+1} = sp\{Y_s \cup \{x_{i,n}^{(s)}\}_{i,n=1}^{\infty} \cup \{y_{i,n,k}^{(s)}\}_{i,n,k=1}^{\infty} \cup \{v_{i,n,k}^{(s)}\}_{i,n,k=1}^{\infty}\}.$$

Now $Y = \overline{\bigcup_{s=1}^{\infty} Y_s}$ is a closed separable subspace of X and $G \cap Y$ is a non-empty open subset of $A \cap Y$.

We show that $\Phi|_Y$ is nowhere single-valued and norm upper semi-continuous on $G \cap Y$. Consider $z \in G \cap Y$ and $\delta > 0$. There exists some $s \in \mathbb{N}$ and $u^{(s)} \in Y_s \cap G$ such that $\|z - u^{(s)}\| < \delta/4$ and some $i, n, k \in \mathbb{N}$ such that $\left\|u^{(s)} - c_i^{(s)}\right\| < \delta/4$, $\left\|c_i^{(s)} - x_{i,n}^{(s)}\right\| < \delta/4$ and $\left\|x_{i,n}^{(s)} - y_{i,n,k}^{(s)}\right\| < \delta/4$. Then $\left\|z - x_{i,n}^{(s)}\right\| < \delta$ and $\left\|z - y_{i,n,k}^{(s)}\right\| < \delta$. Given $Q : X^* \to Y^*$, the restriction mapping $f \mapsto f|_Y$, we have

$$Q\left(f_{i,n}^{(s)}\right) \in \Phi|_Y\left(x_{i,n}^{(s)}\right) \text{ and } Q\left(g_{i,n,k}^{(s)}\right) \in \Phi|_Y\left(y_{i,n,k}^{(s)}\right)$$

and

$$\begin{aligned}\left\|Q\left(f_{i,n}^{(s)}\right) - Q\left(g_{i,n,k}^{(s)}\right)\right\| &\geqslant \left(Q\left(f_{i,n}^{(s)}\right) - Q\left(g_{i,n,k}^{(s)}\right)\right)\left(v_{i,n,k}^{(s)}\right) \\ &= \left(f_{i,n}^{(s)} - g_{i,n,k}^{(s)}\right)\left(v_{i,n,k}^{(s)}\right) > 1/m_0.\end{aligned}$$

Therefore $\operatorname{diam} \Phi|_Y(B_Y(z;\delta)) \geqslant 1/m_0$ and we conclude that $\Phi|_Y$ is not single-valued and norm upper semi-continuous at z. □

This proposition says that for any set-valued mapping from an open subset of a Banach space into subsets of the dual, generic single-valuedness and norm upper semi-continuity is a separably determined property.

So we deduce Gregory's result for convex functions, [2, p141].

Corollary 2.1. *A continuous convex function ϕ on an open convex subset A of a Banach space X is Fréchet differentiable on a dense G_δ subset of A if for every closed separable subspace Y of X where $A \cap Y \neq \emptyset$, the continuous convex function $\phi|_Y$ is Fréchet differentiable on a dense G_δ subset of $A \cap Y$.*

Given a locally Lipschitz function ψ on an open subset A of a Banach space X, the *Clarke subdifferential* of ψ at $x \in A$ is the set

$$\partial\psi(x) = \{f \in X^* : f(y) \leqslant \limsup_{\substack{z \to x \\ \lambda \to 0+}} \frac{\psi(z+\lambda y) - \psi(z)}{\lambda} \text{ for all } y \in X\},$$

and the *Clarke subdifferential mapping* $x \mapsto \partial\psi(x)$ is a weak* cusco from A into subsets of X^*. The function ψ is said to be *uniformly strictly differentiable* at $x \in A$ if ψ is Fréchet differentiable at x and given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|\psi(z+y) - \psi(z) - \psi'(x)(y)| < \varepsilon \|y\| \quad \text{for all } \|z - x\| < \delta \text{ and all } \|y\| < \delta.$$

Now ψ is uniformly strictly differentiable at $x \in A$ if and only if the Clarke subdifferential mapping $x \mapsto \partial\psi(x)$ is single-valued and norm upper semi-continuous at x, [**3**, p.374].

A locally Lipschitz function ψ on an open subset A of a Banach space X is said to be *pseudo-regular* at $x \in A$ if

$$\limsup_{\lambda \to 0+} \frac{\psi(x+\lambda y) - \psi(x)}{\lambda} = \limsup_{\substack{z \to x \\ \lambda \to 0+}} \frac{\psi(z+\lambda y) - \psi(z)}{\lambda} \quad \text{for all } y \in X.$$

The Clarke subdifferential mapping of a pseudo-regular function is a minimal weak* cusco, [**8**, Theorem 2.5]. Clearly the restriction of a pseudo-regular function is also pseudo-regular.

So more generally, Proposition 2 has differentiability implications for pseudo-regular locally Lipschitz functions.

COROLLARY 2.2. *A pseudo-regular locally Lipschitz function ψ on an open subset A of a Banach space X is uniformly strictly differentiable on a dense G_δ subset of A if for every closed separable subspace Y of X where $A \cap Y \neq \emptyset$ the locally Lipschitz function $\psi|_Y$ is uniformly strictly differentiable on a dense G_δ subset of $A \cap Y$.*

A set-valued mapping Φ from an open subset A of a Banach space X into subsets of X^* is said to be *monotone* if

$$(f - g)(x - y) \geqslant 0 \quad \text{for all } x, y \in A,\ f \in \Phi(x),\ g \in \Phi(y).$$

The mapping is *maximal monotone* if its graph is not contained in the graph of any other monotone mapping with the same domain. A maximal monotone mapping is a minimal weak* cusco, [**9**, p105]. The restriction of a maximal monotone mapping is also maximal monotone.

We have our main result from Propositions 1 and 2.

Theorem 3. *Consider any Banach space X.*

(i) *Every continuous convex function ϕ on an open convex subset A of X is Fréchet differentiable on a dense G_δ subset of A if for every closed separable subspace Y of X where $A \cap Y \neq \emptyset$, $\partial\phi|_Y(A \cap Y)$ is separable in Y^*.*

(ii) *Every pseudo-regular locally Lipschitz function ψ on an open subset A of X is uniformly strictly differentiable on a dense G_δ subset of A if for every closed separable subspace Y of X where $A \cap Y \neq \emptyset$, $\partial\psi|_Y(A \cap Y)$ is separable in Y^*.*

(iii) *Every maximal monotone mapping Φ from an open subset A of X into subsets of X^* is single-valued and norm upper semi-continuous at the points of a dense G_δ subset of A if for every closed separable subspace Y of X where $A \cap Y \neq \emptyset$, $\Phi|_Y(A \cap Y)$ is separable in Y^*.*

We deduce the known results relating to the characterisation of Asplund spaces.

Corollary 3.1. *A Banach space X where every closed separable subspace has separable dual has the properties that*

(i) *every continuous convex function ϕ on an open convex subset A of X is Fréchet differentiable on a dense G_δ subset of A,*

(ii) *every pseudo-regular locally Lipschitz function ψ on an open subset A of X is uniformly strictly differentiable on a dense G_δ subset of A, and*

(iii) *every maximal monotone mapping Φ from an open subset A of X into subsets of X^* is single-valued and norm upper semi-continuous at the points of a dense G_δ subset of A,* [7].

We should note that the converse of Proposition 2 does not hold in general. It is well known that the Banach space ℓ_∞ contains a subspace isometrically isomorphic to ℓ_1, [**5**, p.225]. The norm on ℓ_∞ is Fréchet differentiable on a dense G_δ set, [**9**, p.94], however the norm on ℓ_1 is nowhere Fréchet differentiable, [**9**, p.8].

Finally, we should point out that Corollaries 2.1 and 2.2 are special cases of a very general result which holds for arbitrary mappings between Banach spaces, [**11**, p.481].

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