

# ON FREE PRODUCTS OF CYCLIC ROTATION GROUPS

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We consider the group of rotations in three-dimensional Euclidean space, leaving the origin fixed. These rotations are represented by real orthogonal third-order matrices with positive determinant. It is known that this rotation group contains free non-abelian subgroups of continuous rank (see **1**).

In this paper we shall prove the following conjectures of J. de Groot (**1**, pp. 261–262):

**THEOREM 1.** *Two rotations with equal rotation angles  $\alpha$  and with arbitrary but different rotation axes are free generators of a free group, if  $\cos \alpha$  is transcendental.*

**THEOREM 2.** *A free product of at most continuously many cyclic groups can be isomorphically represented by a rotation group.*

More precisely: Theorem 2 is a special case of the following conjecture of J. de Groot (**1**, p. 262): A free product of at most continuously many rotation groups, each consisting of less than continuously many elements, can be isomorphically represented by a rotation group.

J. Mycielski at Wroclaw informed me that he, with S. Balcerzyk has proved a theorem, which includes our Theorem 2 as a special case; moreover, our Theorem 1 seems to intersect with a theorem proved by S. Balcerzyk.

*Preliminaries.* We define

$$A(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix}, \quad D(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let  $i_\sigma = \pm 1$ ,  $k_\sigma = 1, 2, 3, \dots$  ( $\sigma = 1, \dots, s$ ). Furthermore, we assume  $\cos \alpha$  is transcendental. Then we have

**LEMMA:** *No product  $P_s$  ( $s \geq 1$ ) of the form*

$$P_s = D(\theta_0)A^{i_1k_1}(\alpha)D(\theta_1)A^{i_2k_2}(\alpha) \dots A^{i_s k_s}(\alpha)D(\theta_s)$$

*is the identity, if one of the following conditions is satisfied for  $\sigma = 1, \dots, s - 1$ :*

(a)  $\theta_\sigma$  is not a multiple of  $\pi$ ;

(b)  $\theta_\sigma$  is not a multiple of  $2\pi$  and the exponents of  $A$  are of alternating sign:

$$i_{\sigma+1} = -i_\sigma.$$

*Proof:* We use the formulas ( $k > 0$ ):

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$$\begin{aligned}\cos k\alpha &= 2^{k-1}\cos^k\alpha + \dots, \\ \sin k\alpha &= \sin\alpha(2^{k-1}\cos^{k-1}\alpha + \dots), \\ \sin^2\alpha &= 1 - \cos^2\alpha,\end{aligned}$$

where  $\dots$  denote terms of lower degree in  $\cos\alpha$ , So we have

$$(1) \quad A^{i_\sigma k_\sigma}(\alpha)D(\theta_\sigma) = \begin{pmatrix} \cos\theta_\sigma, & -\sin\theta_\sigma, & 0 \\ q_\sigma \sin\theta_\sigma \cos\alpha + \dots, & q_\sigma \cos\theta_\sigma \cos\alpha + \dots, & -i_\sigma q_\sigma \sin\alpha + \dots \\ i_\sigma q_\sigma \sin\theta_\sigma \sin\alpha + \dots, & i_\sigma q_\sigma \cos\theta_\sigma \sin\alpha + \dots, & q_\sigma \cos\alpha + \dots \end{pmatrix},$$

where  $\dots$  denote terms of lower degree in  $\cos\alpha$  and  $\sin\alpha$  and

$$q_\sigma = (2\cos\alpha)^{k_\sigma-1}.$$

By induction with respect to  $\sigma$  we find that the elements of the matrices  $P_\sigma = (p_{ik}^\sigma)$  are polynomials in  $\cos\alpha$ , multiplied or not by a factor  $\sin\alpha$ . In particular the elements  $p_{32}^\sigma$  and  $p_{33}^\sigma$  obtain the form (we consider the leading terms only, denoting terms of lower degree by  $\dots$ ):

$$\begin{aligned}p_{32}^\sigma &= i_\sigma q_\sigma V_\sigma \cos\theta_\sigma \sin\alpha + \dots, \\ p_{33}^\sigma &= q_\sigma V_\sigma \cos\alpha + \dots.\end{aligned}$$

Indeed, for  $\sigma = 1$  we have  $V_1 = 1$  and multiplying  $P_\sigma$  with the matrix (1), where  $\sigma$  is replaced by  $\sigma + 1$ , we find

$$\begin{aligned}p_{32}^{\sigma+1} &= i_\sigma q_\sigma V_\sigma \cos\theta_\sigma \sin\alpha \cdot q_{\sigma+1} \cos\theta_{\sigma+1} \cos\alpha \\ &\quad + q_\sigma V_\sigma \cos\alpha \cdot i_{\sigma+1} q_{\sigma+1} \cos\theta_{\sigma+1} \sin\alpha + \dots \\ &= i_{\sigma+1} q_{\sigma+1} q_\sigma V_\sigma \cos\theta_{\sigma+1} \cos\alpha \sin\alpha (i_\sigma i_{\sigma+1} \cos\theta_\sigma + 1) + \dots \\ p_{33}^{\sigma+1} &= i_\sigma q_\sigma V_\sigma \cos\theta_\sigma \sin\alpha \cdot -i_{\sigma+1} q_{\sigma+1} \sin\alpha \\ &\quad + q_\sigma V_\sigma \cos\alpha \cdot q_{\sigma+1} \cos\alpha + \dots \\ &= q_\sigma q_{\sigma+1} V_\sigma \cos^2\alpha (i_\sigma i_{\sigma+1} \cos\theta_\sigma + 1) + \dots\end{aligned}$$

Hence,

$$V_{\sigma+1} = q_\sigma V_\sigma \cos\alpha (1 + i_\sigma i_{\sigma+1} \cos\theta_\sigma).$$

From this it follows that the coefficient of the leading term of  $p_{33}^{\sigma+1}$  does not vanish if

$$1 + i_\sigma i_{\sigma+1} \cos\theta_\sigma \neq 0 \quad (\sigma = 1, \dots, s-1),$$

that is, if (a) or (b) holds true.

Thus since  $p_{33}^{\sigma+1}$  is a polynomial in  $\cos\alpha$  and  $\cos\alpha$  is transcendental, the product  $P_s$  satisfying (a) or (b) obviously is unequal to the identity, by which the lemma is proved.

*Proof of Theorem 1.* Two rotations with rotation angles  $\alpha$ , the axes of which intersect under an angle  $\theta$ , may be represented by the matrices  $A = A(\alpha)$  and  $B = D(\theta)A(\alpha)D(-\theta)$ . Clearly the theorem is proved if we show that  $A$  and  $B$  generate a free non-abelian group when  $\cos\alpha$  is transcendental and  $\theta$  is not a multiple of  $\pi$ .

Since all non-trivial products of the elements  $A^{\pm 1}$  and  $B^{\pm 1}$  have the form  $P_s$  satisfying condition (a), they are not equal to the identity by virtue of the lemma, by which the theorem is proved.

*Proof of Theorem 2.* J. von Neumann **(2)** proved that the real numbers  $x_t$  defined by

$$x_t = \sum_{n=0}^{\infty} 2^{2^{[nt]} - 2^{n^2}} \quad (t > 0)$$

are algebraically independent over the field of rational numbers.

We define

$$(2) \quad \begin{cases} \phi_t = 2 \operatorname{arctg} x_t \\ \alpha = 2 \operatorname{arctg} x_1 \end{cases} \quad (0 < t < 1).$$

Then, according to a theorem of J. de Groot **(1)**, we have:

The continuously many rotations

$$B_t = D(\phi_t)A(\alpha)D(-\phi_t) \quad (0 < t < 1)$$

are free generators of a free rotation group.

Let  $(F)$  denote the group generated by the rotation  $F$ . We shall now prove:

*The group generated by the continuously many rotations*

$$F_t(\delta_t) = B_t D(\delta_t) B_t^{-1} \quad (0 < t < 1)$$

*is a free product of the cyclic groups  $(F_t(\delta_t))$ . This obviously implies Theorem 2.*

Consider any non-trivial product

$$\begin{aligned} & F_{t_1}^{m_1}(\delta_{t_1}) F_{t_2}^{m_2}(\delta_{t_2}) \dots F_{t_s}^{m_s}(\delta_{t_s}) \\ &= D(\phi_{t_1})A(\alpha)D(m_1\delta_{t_1})A^{-1}(\alpha)D(\phi_{t_2} - \phi_{t_1})A(\alpha)D(m_2\delta_{t_2}) \dots \\ & \quad A(\alpha)D(m_s\delta_{t_s})A^{-1}(\alpha)D(-\phi_{t_s}). \end{aligned}$$

We may assume that

$$m_k \delta_{t_k} \quad (k = 1, \dots, s)$$

is not a multiple of  $2\pi$ , for otherwise we have a trivial product. Furthermore, the numbers

$$\phi_{t_{k+1}} - \phi_{t_k} \quad (k = 1, \dots, s - 1)$$

are not multiples of  $2\pi$  by virtue of (2). Thus the product considered has the form  $P_{2,s}$  satisfying condition (b). According to the lemma this product is unequal to the identity, by which the theorem is proved.

REFERENCES

1. J. de Groot, *Orthogonal isomorphic representations of free groups*, Can. J. Math., 8 (1956), 256-262.
2. J. von Neumann, *Ein System algebraisch unabhängiger Zahlen*, Math. Ann., 99 (1928), 134-141.

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