CORRIGENDUM: LIFTINGS OF JORDAN AND SUPER JORDAN PLANES

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(Received 7 March 2022; first published online 14 June 2022)


Keywords: Hopf algebras; Nichols algebras; Gelfand-Kirillov dimension

2020 Mathematics subject classification: Primary 16T05

1. Introduction

In the paper [1] we stated the classification of the pointed Hopf algebras with finite Gelfand-Kirillov dimension that are liftings of either the Jordan plane or the super Jordan plane over a nilpotent-by-finite group. But we overlooked one possibility, namely to deform degree one relations and therefore the classification in loc. cit. of liftings of Jordan planes is not complete. Here we fill the gap. It turns out that the missed example is essentially a Hopf algebra introduced by C. Ohn in 1992, see [3].

Throughout $k$ is an algebraically closed field of characteristic 0. Recall that $\mathcal{V}(1, 2)$ is the braided vector space with basis $x_1, x_2$ and braiding $c$ given by $c(x_i \otimes x_1) = x_1 \otimes x_i$, $c(x_i \otimes x_2) = (x_1 + x_2) \otimes x_i$, $i = 1, 2$. Here is the revised version of [1, Proposition 4.2].

**Proposition 1.** Let $G$ be a nilpotent-by-finite group and let $H$ be a pointed Hopf algebra with finite GKdim such that

\begin{itemize}
  \item $G(H) \simeq G$ and
  \item the infinitesimal braiding of $H$ is isomorphic to $\mathcal{V}(1, 2)$.
\end{itemize}
Then there exists a Jordanian YD-triple \( \mathcal{D} = (g, \chi, \eta) \) for \( \mathbb{k}G \) such that either

(I) \( H \cong \U(\mathcal{D}, 0) \) or \( H \cong \U(\mathcal{D}, 1) \), introduced in [1, §4.1]; or

(II) \( \chi = \varepsilon \) and there exists \( \xi \in \text{Der}_{{\varepsilon, \varepsilon}}(\mathbb{k}G, \mathbb{k}) \), \( \xi \neq 0 \), such that \( H \cong \U_{\xi}(\mathcal{D}, 0) \) or \( H \cong \U_{\xi}(\mathcal{D}, 1) \) see Definition 9; or

(III) \( \chi = \varepsilon \) and \( H \cong \U_{\text{Jordan}}(\mathcal{D}) \), see Definition 11.

Conversely, any of these Hopf algebras is pointed and has finite \( \text{GKdim} \), actually \( \text{GKdim} \mathbb{k}G + 2 \). See Lemmas 10, 12 and [1, Proposition 4.2]. Notice that if \( \chi = \varepsilon \) and \( \xi = 0 \), then \( \U_0(\mathcal{D}, \lambda) \cong \U(\mathcal{D}, \lambda) \), introduced in [1, §4.1].

The subspace of \( (g, 1) \) skew-primitive elements in a Hopf algebra in case (I) is decomposable as \( G \)-module, while in (II) is decomposable as \( \langle g \rangle \)-module but it is an indecomposable \( G \)-module, and in (III) it is an indecomposable \( \langle g \rangle \)-module. Thus Hopf algebras from different cases could not be isomorphic. Whether Hopf algebras in the same case are isomorphic is treated as in [1, §4.1].

This note is organized as follows. In §1.1 the minimal Hopf algebra missing in [1, Proposition 4.2] and its relation with [3] are described. In §1.2 we discuss the gap. Proposition 1 is proved in §1.3.

### Notation

We keep the notations from [1]. Let \( G \) be a group, let \( \mathbb{k}G \) be its group algebra and let \( \hat{G} \) be its group of characters. Given \( \chi \in \hat{G} \), recall that

\[
\text{Der}_{\chi, \chi}(\mathbb{k}G, \mathbb{k}) = \{ \eta \in (\mathbb{k}G)^* : \eta(ht) = \chi(h)\eta(t) + \chi(t)\eta(h) \quad \forall h, t \in G \}.
\]

A collection \( \mathcal{D} = (g, \chi, \eta) \in \mathbb{Z}(G) \times \hat{G} \times \text{Der}_{\chi, \chi}(\mathbb{k}G, \mathbb{k}) \) is a YD-triple for \( \mathbb{k}G \) if \( \eta(g) = 1 \). Then the vector space \( \mathcal{V}_g(\chi, \eta) \) with a basis \( (x_i)_{i \in I} \) belongs to \( \mathbb{k}G \mathcal{YD} \), with the coaction \( \delta(x_i) = g \otimes x_i, i \in I \), and the action given by

\[
h \cdot x_1 = \chi(h)x_1, \quad h \cdot x_2 = \chi(h)x_2 + \eta(h)x_1, \quad h \in \mathbb{k}G.
\]

When \( \chi(g) = 1 \) we say that \( \mathcal{D} = (g, \chi, \eta) \) is a Jordanian YD-triple.

Let \( L \) be a Hopf algebra. The \( \Delta, \varepsilon \) and \( S \) denote respectively the comultiplication, the counit and the antipode. The group of group-like elements is denoted by \( G(L) \). Also the space of \( (g, h) \)-primitive elements is \( \mathcal{P}_{g,h}(L) = \{ \ell \in L : \Delta(\ell) = \ell \otimes h + g \otimes \ell \} \), where \( g, h \in G(L) \), and \( \mathcal{P}(L) = \mathcal{P}_{1,1}(L) \) is the space of primitive elements. The adjoint action of \( G(L) \) on \( L \) is denoted by \( g \cdot \ell := g\ell g^{-1}, g \in G(L), \ell \in L \).

#### 1.1. The Jordanian enveloping algebra of \( s\ell(2) \)

Let \( \U_{\text{Jordan}} \) be the algebra generated by \( a_1, a_2, g, g^{-1} \) with defining relations

\[
g \pm 1 g \mp 1 = 1, \quad ga_1 = a_1 g + (g - g^2), \quad ga_2 = a_2 g + a_1 g.
\]
It is easy to see that \( \tilde{U}^{\text{jordan}} \) is a Hopf algebra by imposing \( g \in G(\tilde{U}^{\text{jordan}}) \) and \( a_1, a_2 \in \mathcal{P}_{g,1}(\tilde{U}^{\text{jordan}}) \). We introduce

\[
z = a_1a_2 - a_2a_1 - \frac{a_1^2}{2} + a_2 + \frac{1}{2}a_1 \in \tilde{U}^{\text{jordan}}
\]  

(1.2)

**Lemma 2.** The element \( z \) belongs to \( \mathcal{P}_{g^2,1}(\tilde{U}^{\text{jordan}}) \) and commutes with \( g \).

**Proof.** We compute

\[
\Delta(z) = a_1a_2 \otimes 1 + a_1 \ g \otimes a_2 + ga_2 \otimes a_1 + g^2 \otimes a_1a_2 \\
\quad - \ a_2a_1 \otimes 1 - a_2g \otimes a_1 - ga_1 \otimes a_2 - g^2 \otimes a_2a_1 \\
\quad - \ \frac{1}{2}a_1^2 \otimes 1 - \ \frac{1}{2}(a_1 \ g + ga_1) \otimes a_1 - \ \frac{1}{2}g^2 \otimes a_1^2 \\
\quad + \ a_2 \otimes 1 + g \otimes a_2 + \ \frac{1}{2}a_1 \otimes 1 + \ \frac{1}{2}g \otimes a_1 \\
\quad = \ z \otimes 1 + g^2 \otimes z + (a_1 \ g - ga_1 + g - g^2) \otimes a_2 \\
\quad + \ (ga_2 - a_2 \ g - \ \frac{1}{2}(a_1 \ g + ga_1) + \ \frac{1}{2}g - \ \frac{1}{2}g^2) \otimes a_1 \\
\quad = \ z \otimes 1 + g^2 \otimes z;
\]

here \( ga_2 - a_2 \ g - \ \frac{1}{2}(a_1 \ g + ga_1) + \ \frac{1}{2}g - \ \frac{1}{2}g^2 = \ \frac{1}{2}(a_1 \ g - ga_1 + (g - g^2)) = 0 \).

It remains to prove that \( \gamma(z) = 0 \), where \( \gamma \in \text{End}_k(\tilde{U}^{\text{jordan}}) \) is given by \( \gamma(x) = gxg^{-1} - x \), for all \( x \in \tilde{U}^{\text{jordan}} \). Note that

\[
\gamma(xy) = \gamma(x)(\gamma(y) + y) + x\gamma(y) \text{ for all } x, y \in \tilde{U}^{\text{jordan}}.
\]

From (1.1) we have that

\[
\gamma(a_1) = 1 - g, \quad \gamma(a_2) = a_1.
\]

Therefore,

\[
\gamma(z) = \gamma(a_1a_2 + \left( a_2 + \frac{1}{2}a_1 \right)(1 - a_1)) \\
\quad = \gamma(a_1)(\gamma(a_2) + a_2) + a_1\gamma(a_2) \\
\quad + \gamma \left( a_2 + \frac{1}{2}a_1 \right)(\gamma(1 - a_1) + 1 - a_1) + \left( a_2 + \frac{1}{2}a_1 \right) \gamma(1 - a_1).
\]

https://doi.org/10.1017/S0013091522000207 Published online by Cambridge University Press
By using (1.3) we obtain that
\[
\gamma(z) = (1 - g)(a_2 + a_1) + a_1^2
+ \left( a_1 + \frac{1}{2}(1 - g) \right) (g - a_1) + \left( a_2 + \frac{1}{2}a_1 \right) (g - 1)
= a_2 + a_1 - (a_2g + 2a_1g + g - g^2) + a_1^2 + \left( a_2 + \frac{1}{2}a_1 \right) (g - 1)
+ a_1g - a_1^2 + \frac{1}{2}(g - g^2) + \frac{1}{2}a_1(g - 1) + \frac{1}{2}(g - g^2).
\]
Now it follows easily that \( \gamma(z) = 0 \). \( \square \)

The Jordanian enveloping algebra of \( \mathfrak{sl}(2) \) is
\[
\mathfrak{U}^{\text{Jordan}} := \tilde{\mathfrak{U}}^{\text{Jordan}} / \langle z \rangle.
\]
By Lemma 2, \( \mathfrak{U}^{\text{Jordan}} \) is a Hopf algebra quotient of \( \tilde{\mathfrak{U}}^{\text{Jordan}} \). By abuse of notation the images of \( g, a_1, a_2 \) in \( \mathfrak{U}^{\text{Jordan}} \) are denoted by the same symbols.

**Remark 3.** For each \( \lambda \in \mathbb{k} \) let
\[
\mathfrak{U}_\lambda^{\text{Jordan}} := \tilde{\mathfrak{U}}^{\text{Jordan}} / \langle z - \lambda(1 - g^2) \rangle.
\]
Then \( \mathfrak{U}_\lambda^{\text{Jordan}} \) is a Hopf algebra, since \( z - \lambda(1 - g^2) \in \mathcal{P}_{g^2,1}(\tilde{\mathfrak{U}}^{\text{Jordan}}) \).

Let us now fix \( \lambda, \mu \in \mathbb{k} \). Let \( U \) be the algebra
\[
U = \mathbb{k}\langle g, g^{-1}, a_1, a_2 \rangle / \langle gg^{-1} - 1, g^{-1}g - 1 \rangle.
\]
Then \( U \) has a unique Hopf algebra structure such that \( g, g^{-1} \in G(U) \) and \( a_1, a_2 \in \mathcal{P}_{g,1}(U) \). Moreover, there exists a well-defined Hopf algebra map
\[
\varphi_{\lambda, \mu} : U \rightarrow \mathfrak{U}_\lambda^{\text{Jordan}}, \quad g \mapsto g, \ a_1 \mapsto a_1, \ a_2 \mapsto a_2 + \mu(1 - g).
\]
It is easily checked that
\[
ga_1 - a_1g - g + g^2, \quad ga_2 - (a_2 + a_1)g \in \ker \varphi_{\lambda, \mu}.
\]
Moreover, for \( z \in U \) defined as in (1.2) we obtain that
\[
\varphi_{\lambda, \mu}(z) - z = a_1\mu(1 - g) - \mu(1 - g)a_1 + \mu(1 - g^2).
\]
Since \( z = \lambda(1 - g^2) \in \mathfrak{U}_\lambda^{\text{Jordan}} \), we conclude that \( \varphi_{\lambda, \mu} \) induces a surjective Hopf algebra map
\[
\varphi_{\lambda, \mu} : \mathfrak{U}_{\lambda + \mu}^{\text{Jordan}} \rightarrow \mathfrak{U}_\lambda^{\text{Jordan}}.
\]
It follows that \( \varphi_{0, \lambda} : \mathfrak{U}_\lambda^{\text{Jordan}} \rightarrow \mathfrak{U}_\lambda^{\text{Jordan}} \) is a Hopf algebra isomorphism.
Remark 4. For any $\eta \in k$, the Hopf algebra $U_{\eta}$ was introduced by Christian Ohn in [3]; this is the algebra generated over $k$ by $K, Y, T^{\pm 1}$ with relations:

$$TT^{-1} = T^{-1} T = 1, \quad [K, T] = T^2 - 1, \quad [Y, T] = -\frac{\eta}{2}(KT + TK), \quad (1.6)$$

$$[K, Y] = -\frac{1}{2}(YT + TY + YT^{-1} + T^{-1} Y), \quad (1.7)$$

with the Hopf algebra structure of $U_{\eta}$ determined by $T \in G(U_{\eta})$ and $X, Y \in \mathcal{P}_{T^{-1}, T}(U_{\eta})$. It is easy to see that the the Hopf algebras $U_{\eta}$ with $\eta \neq 0$ are all isomorphic so we fix one of them. The apppellative Jordanian was introduced by Alev and Dumas to the best of our knowledge. We claim that $U^\text{Jordan}_\lambda$ is isomorphic to the Hopf subalgebra $\mathfrak{U}$ of $U_{\eta}$ generated by $x = KT^{-1}, y = YT^{-1}, g = T^{-2}$; (1.8)

we choose these variables to have $x, y \in \mathcal{P}_{g, 1}(\mathfrak{U})$. Now (1.6) implies

$$g \cdot x = x + 2(1 - g), \quad g \cdot y = y - 2\eta(x + (1 - g)). \quad (1.9)$$

We perform a new change of variables:

$$a_1 = \frac{1}{2} x, \quad a_2 = -\frac{1}{4} \eta y - \frac{1}{4} x;$$

these new variables satisfy (1.1). Now (1.7) translates succesively into

$$xy - yx = -2y - \eta x^2 + \frac{\eta}{4}(1 - g^2)$$

and then into

$$z = -\frac{1}{32}(1 - g^2).$$

That is, $\mathfrak{U} \simeq U^\text{Jordan}_{\frac{1}{12}}$.

Remark 5. The algebra $U_{\eta}$ can be described as an iterated Ore extension:

$$U_{\eta} = k[T^\pm][x; \delta][y; \sigma, D] \quad (1.10)$$

with $\delta$ a derivation of $k[T^\pm]$, $\sigma$ an automorphism of $k[T^\pm][x; \delta]$ and $D$ a $\sigma$-derivation of $k[T^\pm][x; \delta]$ defined by:

$$yT = y + (-\eta T x - \frac{\eta}{2}(T - T^{-1})) \quad = \sigma(T) \quad (1.12)$$

$$y x = (x + 2) y + \eta x^2 - \frac{\eta}{4}(1 - T^{-4}) \quad = D(x). \quad (1.13)$$
Proposition 6. There exist a derivation \( \delta_1 \) of \( R := \mathbb{k}[g, g^{-1}] \), a derivation \( \delta_2 \) of \( S := R[a_1; \text{id}, \delta_1] \) and an automorphism \( \sigma \) of \( S \) such that \( \mathcal{U}_{\text{jordan}} \) is isomorphic to the Ore extension \( S[a_2; \sigma, \delta_2] \).

Hence \( \mathcal{U}_{\text{jordan}} \) is a noetherian domain of Gelfand-Kirillov 3, and the monomials \( g^ia_1^ja_2^k \) form a PBW-basis of \( \mathcal{U}_{\text{jordan}} \).

Proof. We leave the verification of the first claim to the reader as a long but straightforward exercise: the derivations \( \delta_1 : R \rightarrow R, \delta_2 : S \rightarrow S \) satisfy

\[
\delta_1(g) = g^2 - g, \quad \delta_2(g) = -a_1g, \quad \delta_2(a_1) = \frac{1}{2}a_1(1 - a_1),
\]

and \( \sigma \) is given by \( \sigma(g) = g, \sigma(a_1) = a_1 + 1 \). The rest is standard. \( \square \)

Corollary 7. The Hopf algebra \( \mathcal{U}_{\text{jordan}} \) is pointed and \( \text{gr} \mathcal{U}_{\text{jordan}} \) is isomorphic to the bosonization of the Jordan plane by the group algebra of the infinite cyclic group. \( \square \)

1.2. The gap and how to fix it

We fix a group \( G \). Let \( H \) be a pointed Hopf algebra with coradical filtration \( (H_n)_{n \in \mathbb{N}_0} \) such that \( G(H) \simeq G \). Then \( H_1/H_0 \simeq V \# \mathbb{k}G \), where \( V \in \mathbb{k}G \text{YD} \) is the infinitesimal braiding of \( H \). For \( g \in G \), the space of \((g, 1)\) skew-primitives \( \mathcal{P}_{g,1}(H) \) satisfies

\[
\mathcal{P}_{g,1}(H) \cap H_0 = \mathbb{k}(1 - g) \quad \text{and} \quad \mathcal{P}_{g,1}(H)/\mathcal{P}_{g,1}(H) \cap H_0 \simeq V_g.
\]

Now assume that \( V \simeq V_g(\chi, \eta) \) for a YD-triple \( D = (g, \chi, \eta) \) over \( \mathbb{k}G \). Thus \( V = V_g \) and we have an exact sequence of \( G \)-modules

\[
0 \rightarrow \mathbb{k}(1 - g) \rightarrow \mathcal{P}_{g,1}(H) \xrightarrow{\cong} V_g(\chi, \eta) \rightarrow 0.
\]

Since \( g \in Z(G) \), one has \( \mathbb{k}(1 - g) \subset \mathcal{P}_{g,1}(H)^e \). Hence \( \chi \neq \varepsilon \) implies that

\[
\mathcal{P}_{g,1}(H) \simeq \mathbb{k}(1 - g) \oplus V_g(\chi, \eta)
\]

and we have a morphism of Hopf algebras \( \pi : T(V_g(\chi, \eta)) \rightarrow H \), where \( T(V_g(\chi, \eta)) = T(\mathcal{V}_g(\chi, \eta)) \# \mathbb{k}G \). In particular the proof of [1, Prop. 4.3] goes over without changes.

We assume for the rest of this Section that the infinitesimal braiding \( V \) of \( H \) is isomorphic to \( V_g(\varepsilon, \eta) \) for a YD-triple \( D = (g, \varepsilon, \eta) \) as Yetter-Drinfeld module over \( \mathbb{k}G \). Under this assumption, \( \mathcal{P}_{g,1}(H) \) might be indecomposable.

Example 8. The indecomposability of \( \mathcal{P}_{g,1}(H) \) could happen in other situations. Here is a simple example. Let \( A \) be the algebra generated by \( a, \gamma \pm 1 \), where \( \gamma^{-1} \) is the inverse of \( \gamma \) and the relation \( \gamma a \gamma^{-1} = a + (1 - \gamma) \) holds, so that \( A \) is not commutative. Then \( A \) is a pointed Hopf algebra by declaring that \( \gamma \) is a group-like and \( a \) a \((\gamma, 1)\) skew-primitive element. Observe that \( \mathcal{P}_{g,1}(A) \) is indecomposable. Let \( \Gamma \simeq \mathbb{Z} \). It can be shown that \( \text{gr} A \simeq T(V) \otimes \mathbb{k}\Gamma \), where \( V \) has dimension 1 and is the infinitesimal braiding of \( A \). But \( \mathcal{P}_{g,1}(A) \) is indecomposable and there is no surjective morphism of Hopf algebras \( T(V) \otimes \mathbb{k}\Gamma \rightarrow A \).
Back to our situation, let us pick \(a_1, a_2 \in \mathcal{P}_{g, 1}(H)\) such that \(\varpi(a_j) = x_j, j = 1, 2\) and set \(a_0 = 1 - g\). Then there are \(\zeta \in \text{Der}_{\varepsilon, \varepsilon}(kG, k)\) and a linear map \(\xi : kG \to k\) such that the action of \(h \in G\) on \(\mathcal{P}_{g, 1}(H)\) is given in the basis \((a_0, a_1, a_2)\) by

\[
\|h\| = \begin{pmatrix}
1 & \zeta(h) & \xi(h) \\
0 & 1 & \eta(h) \\
0 & 0 & 1
\end{pmatrix}.
\]  

(1.14)

Notice that \(\text{Der}_{\varepsilon, \varepsilon}(kG, k) = \text{Hom}_{\text{gpd}}(G, (k, +))\) and that \(\xi\) is a kind of differential operator of degree 2, meaning that

\[
\xi(hk) = \xi(h) + \zeta(h)\eta(k) + \xi(k) \quad \text{for all } h, k \in G.
\]  

(1.15)

Thus if \(\zeta \neq 0\), then the claim [1, Prop. 4.2, page p. 669, line 8] is not true. To correct this we consider the subalgebra \(A\) generated by \(g\) and \(\mathcal{P}_{g, 1}(H)\), a Hopf subalgebra of \(H\). The action of \(g\) on \(\mathcal{P}_{g, 1}(H) = \mathcal{P}_{g, 1}(A)\) in the basis \((a_0, a_1, a_2)\) is given by

\[
\|g\| = \begin{pmatrix}
1 & \zeta(g) & \xi(g) \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}.
\]  

(1.16)

As \(g \in Z(G)\), we have that \(\xi(gh) = \xi(hg)\) for all \(h \in G\), so (1.15) says that

\[
\zeta(h) = \eta(h)\zeta(g) \quad \text{for all } h \in G.
\]  

(1.17)

We consider two cases:

(A) \(\zeta(g) = 0\). Then \(\zeta = 0\) by (1.17) and \(\xi \in \text{Der}_{\varepsilon, \varepsilon}(kG, k)\) by (1.15).

(B) \(t := \zeta(g) \neq 0\), the Jordanian case. In the basis \((a_0, t^{-1}a_1, t^{-1}a_2 - t^{-2}\zeta(g)a_1)\), the action of \(g\) is given by \(\begin{pmatrix}1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1\end{pmatrix}\). We still denote the new basis by \((a_0, a_1, a_2)\);

that is, we may assume that \(\zeta(g) = 1, \xi(g) = 0\). By (1.17), \(\zeta = \eta\), and by (1.15),

\[
\xi(hk) = \xi(h) + \eta(h)\eta(k) + \xi(k) \quad \text{for all } h, k \in G.
\]  

We shall see that the following Hopf algebras exhaust the case (A).

**Definition 9.** Let \(\mathcal{D} = (g, \varepsilon, \eta)\) be a YD-triple, \(\xi \in \text{Der}_{\varepsilon, \varepsilon}(kG, k)\) and \(\lambda \in k\). We define \(\mathfrak{U}_\xi(\mathcal{D}, \lambda)\) as the algebra generated by \(h \in G, a_1, a_2\) with defining relations being those of \(G\) and

\[
ha_1 - a_1 h, \quad h \in G; \\
ha_2 - (a_2 + \eta(h)a_1 + \xi(h)(1 - g))h, \quad h \in G; \\
a_1a_2 - a_2a_1 - \frac{a_1^2}{2} - \lambda(1 - g^2).
\]

(1.18)

(1.19)

(1.20)

As we said already, \(\mathfrak{U}_0(\mathcal{D}, \lambda) \simeq \mathfrak{U}(\mathcal{D}, \lambda)\), introduced in [1, §4.1].
Lemma 10. \( \mathcal{U}_\xi(\mathcal{D}, \lambda) \) is a Hopf algebra with comultiplication determined by
\[
G(\mathcal{U}_\xi(\mathcal{D}, \lambda)) = G \quad \text{and} \quad a_1, a_2 \in \mathcal{P}_{g,1}(\mathcal{U}_\xi(\mathcal{D}, \lambda)).
\]
Thus \( \mathcal{U}_\xi(\mathcal{D}, \lambda) \) is pointed. The set \( \{a_1^m a_2^n h \mid m, n \in \mathbb{N}_0, \ h \in G \} \) is a basis of \( \mathcal{U}_\xi(\mathcal{D}, \lambda) \);
\[
\text{gr} \ \mathcal{U}_\xi(\mathcal{D}, \lambda) \simeq \mathcal{B}(\mathcal{V}(1, 2)) \# \mathbb{k}G \quad \text{and}
\]
\[
\text{GKdim} \ \mathcal{U}_\xi(\mathcal{D}, \lambda) = \text{GKdim} \mathbb{k}G + 2.
\]
In particular, if \( G \) is nilpotent-by-finite, then \( \text{GKdim} \mathcal{U}_\xi(\mathcal{D}, \lambda) < \infty \).

Proof. Left to the reader. \( \square \)

We shall see that the following Hopf algebras exhaust the case (B).

Definition 11. Let \( \mathcal{D} = (g, \varepsilon, \eta) \) be a YD-triple and define \( \xi \in (\mathbb{k}G)^* \) by \( \xi(h) = \frac{1}{2}(\eta(h)^2 - \eta(h)), \ h \in G \). We introduce \( \mathcal{U}^{\text{jordan}}(\mathcal{D}) \) as the algebra generated by \( h \in G, \ a_1, a_2 \) with defining relations those of \( G, (1.19) \) and
\[
ha_1 - (a_1 + \eta(h)(1 - g))h, \quad h \in G.
\] (1.21)
\[
a_1 a_2 - a_2 a_1 - \frac{a_1^2}{2} + a_2 + \frac{1}{2}a_1.
\] (1.22)

Observe that \( \xi \), needed in (1.19), satisfies (1.15) with \( \zeta = \eta \). The proof of the following Lemma is also standard.

Lemma 12. \( \mathcal{U}^{\text{jordan}}(\mathcal{D}) \) is a Hopf algebra with structure determined by
\[
G(\mathcal{U}^{\text{jordan}}(\mathcal{D})) = G \quad \text{and} \quad a_1, a_2 \in \mathcal{P}_{g,1}(\mathcal{U}^{\text{jordan}}(\mathcal{D})).
\]
Thus \( \mathcal{U}^{\text{jordan}}(\mathcal{D}) \) is pointed. The set \( \{a_1^m a_2^n h \mid m, n \in \mathbb{N}_0, \ h \in G \} \) is a basis of \( \mathcal{U}^{\text{jordan}}(\mathcal{D}) \);
\[
\text{gr} \ \mathcal{U}^{\text{jordan}}(\mathcal{D}) \simeq \mathcal{B}(\mathcal{V}(1, 2)) \# \mathbb{k}G \quad \text{and}
\]
\[
\text{GKdim} \ \mathcal{U}^{\text{jordan}}(\mathcal{D}) = \text{GKdim} \mathbb{k}G + 2.
\]
In particular, if \( G \) is nilpotent-by-finite, then \( \text{GKdim} \mathcal{U}^{\text{jordan}}(\mathcal{D}) < \infty \).

1.3. Proof of proposition 1

Let \( G \) be a nilpotent-by-finite group and let \( H \) be a pointed Hopf algebra with finite \( \text{GKdim} \) such that \( G(H) \simeq G \) and the infinitesimal braiding \( V \) of \( H \) is isomorphic to \( \mathcal{V}(1, 2) \). By [1, Lemma 2.3], there exists a unique YD-triple \( \mathcal{D} = (g, \chi, \eta) \) such that \( V \simeq \mathcal{V}_g(\chi, \eta) \) in \( \mathcal{YD} \). By [1, Lemma 3.7], \( \text{gr} H \simeq \mathcal{B}(\mathcal{V}(1, 2)) \# \mathbb{k}G \), hence \( H \) is generated by \( \mathcal{P}_{g,1}(H) \) and \( G \) as algebra.

If \( \chi \neq \varepsilon \), then the proof of [1, Prop. 4.1] implies that \( H \) is isomorphic either to \( \mathcal{U}(\mathcal{D}, 0) \) or \( \mathcal{U}(\mathcal{D}, 1) \), the Hopf algebras introduced in [1, §4.1].

Assume that \( \chi = \varepsilon \). Pick a basis \( (a_0 = 1 - g, a_1, a_2) \) such that any \( h \in G \) acts on \( \mathcal{P}_{g,1}(H) \) by (1.14) where \( \zeta \in \text{Der}_{\varepsilon, \varepsilon}(\mathbb{k}G, \mathbb{k}) \) and \( \xi \in (\mathbb{k}G)^* \) satisfies (1.15). Let \( A \) be the subalgebra generated by \( \mathcal{P}_{g,1}(H) \). As explained above we consider two cases.
Case (A): \( \zeta(g) = 0 \), thus \( \zeta = 0 \). Even if \([1, \text{Proposition 4.2}]\) does not apply in general since we may have \( \xi \neq 0 \), it does apply to \( A \) up to changing the base to \((a_0, \bar{a}_1, \bar{a}_2)\) where \( \bar{a}_2 := a_2 - \xi(g)a_1 \), see \((1.14)\). Call the new basis again \((a_0, a_1, a_2)\) by abuse of notation. Hence \( A \simeq \mathcal{U}(\mathcal{D}', \lambda) \) where \( \mathcal{D}' = \langle g, \chi(g), \eta(g) \rangle \) is a YD-triple over the subgroup \( \langle g \rangle \) of \( G \) and \( \lambda \in \{0, 1\} \). In particular the following equality holds in \( H \):

\[
a_2a_1 = a_1a_2 - \frac{1}{2}a_1^2 + \lambda(1-g^2).
\]

We first claim that \( A \) is stable under the action of \( G \). Indeed let \( G \) act on the free algebra generated by \( g^{\pm 1}, a_1, a_2 \), where \( G \) acts trivially on \( g \), and by \((1.14)\) on \( a_1, a_2 \). As \( g \) is central, the action of each \( h \in G \) preserves the defining ideal of \( A \), so \( G \) acts on \( A \).

We next claim that \( H \simeq A \times \mathbb{k}G/I \), where \( I \) is the ideal that identifies the two copies of \( g \) where \( \times \) stands for smash product. Indeed, the inclusions \( A \hookrightarrow H, \mathbb{k}G \hookrightarrow H \) induce a Hopf algebra map \( \psi : A \times \mathbb{k}G/I \rightarrow H \). As \( \text{gr } H \simeq \mathcal{B}(V) \# \mathbb{k}G \), \( H \) is generated by \( a_1, a_2 \) and \( G \), so \( \psi \) is surjective. On the other hand, \((A \times \mathbb{k}G/I)_1 \) is spanned by the set \( \{1 \otimes h, a_1 \otimes h, a_2 \otimes h | h \in G \} \). The image of this set under \( \psi \) is linearly independent, which implies that \( \psi_{|_{(A \times \mathbb{k}G/I)_1}} \) is injective. By \([2, 5.3.1]\), \( \psi \) is injective, and the claim follows. As a consequence, the set \( \{a_1^na_2^nh | n \in \mathbb{N}_0, h \in G \} \) is a basis of \( H \).

Finally, we see that there is a Hopf algebra map \( \mathcal{U}_\xi(D, \lambda) \rightarrow H \); since this map sends a basis to a basis, we conclude that \( H \simeq \mathcal{U}_\xi(D, \lambda) \).

Case (B). \( \zeta(g) \neq 0 \). As discussed above, we may assume that \( \zeta = \eta \). Recall that we are assuming that \( \text{GKdim } H < \infty \). We claim that

(i) There exists a Hopf algebra isomorphism \( A \simeq \mathcal{U}^{\text{Jordan}}, \text{cf. } (1.4) \).

(ii) \( \xi(h) = \frac{1}{2}(\eta(h)^2 - \eta(h)) \) for all \( h \in G \).

(iii) \( A \) is stable under the adjoint action of \( G \) and \( H \simeq A \times \mathbb{k}G/I \), where \( I \) is the ideal that identifies the two copies of \( g \).

(iv) The set \( \{a_1^na_2^nh | n \in \mathbb{N}_0, h \in G \} \) is a basis of \( H \) and \( H \simeq \mathcal{U}^{\text{Jordan}}(D) \).

(i): It is easy to see that there exists a Hopf algebra surjective map \( \tilde{\pi} : \mathcal{U}^{\text{Jordan}} \rightarrow A \), which applies \( g, a_1, a_2 \) to the corresponding elements of \( A \). Hence \( \pi(z) \in \mathcal{P}_{g_1,1}(A) \), by Lemma 2. Now, as \( g \neq g^2 \) and \( \text{gr } H \simeq \mathcal{B}(V) \# \mathbb{k}G \), we have that \( \mathcal{P}_{g_2,1}(H) = \mathcal{P}_{g^2,1}(H) \cap H_0 = \mathbb{k}(1-g^2) \); thus there exists \( \lambda \in \mathbb{k} \) such that \( \tilde{\pi}(z) = \lambda(1-g^2) \), which implies that \( \tilde{\pi} \) factors through a map \( \pi : \mathcal{U}^{\text{Jordan}} \rightarrow A \). The set \( \{g^k, a_1g^k, a_2g^k | k \in \mathbb{Z} \} \) is linearly independent in \( H \), so \( \pi_{|_{(\mathcal{U}^{\text{Jordan}})}} \) is injective. By \([2, 5.3.1]\), \( \pi \) is an isomorphism. Up to composing with \( \varphi_{0,\lambda} \), see Remark 3, we may assume that \( \lambda = 0 \).

(ii): Given \( h \in G \), let \( \gamma_h \in \text{End}_k H \) be given by

\[
\gamma_h(x) = hxh^{-1} - x \text{ for all } x \in H.
\]

Note that \( \gamma_h(xy) = \gamma_h(x)(\gamma_h(y) + y) + x\gamma_h(y) \) for all \( x, y \in H \). From \((1.14)\),

\[
\gamma_h(a_1) = \eta(h)(1-g), \quad \gamma_h(a_2) = \eta(h)a_1 + \xi(h)(1-g).
\]

\[(1.23)\]
Therefore,

\[
\gamma_h(z) = \eta(h)(1 - g)(\eta(h)a_1 + \xi(h)(1 - g) + a_2) + a_1(\eta(h)a_1 + \xi(h)(1 - g))
+ \left(\eta(h)a_1 + \xi(h)(1 - g) + \frac{1}{2}\eta(h)(1 - g)\right)(-\eta(h)(1 - g) + 1 - a_1)
\]

\[
- \left(a_2 + \frac{1}{2}a_1\right)\eta(h)(1 - g) = \left(\frac{1}{2}\eta(h) - \frac{1}{2}\eta(h)^2 + \xi(h)\right)(1 - g^2).
\]

By (i), \(z = 0\), so \(\gamma_h(z) = 0\). Thus, \(\xi(h) = \frac{1}{2}(\eta(h)^2 - \eta(h))\).

(iii): Let \(G\) act on the free algebra generated by \(g^\pm 1, a_1, a_2\), where \(G\) acts trivially on \(g\), and by (1.14) on \(a_1, a_2\). Each \(h \in G\) fixes the defining relations \(gg^{-1} - 1, g^{-1}g - 1, ga_1 - a_1 g - g + g^2, z\), and \(h \cdots (ga_2 - a_2 g - a_1 g) = ga_2 - a_2 g - a_1 g + \eta(h)(ga_1 - a_1 g - (1 - g)g)\), so the action descends to \(A\). The proof of (iv) is as in Case (A).

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\(\square\)
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**Acknowledgements.** N. A. thanks François Dumas for conversations on and help with the computations in this Note.

N. A. and I. A. were partially supported by CONICET, FONCyT-ANPCyT, Secyt (UNC). The work of N. A. was partially done during a visit to the University of Clermont. The three authors were partially supported by the Alexander von Humboldt Foundation through the Research Group Linkage Programme.

**Competing interests.** The authors declare none.

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