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RESEARCH ARTICLE

Modularity of arithmetic special divisors for unitary Shimura varieties (with an appendix by Yujie Xu)

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with an appendix by Yujie Xu [2]

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Abstract
We construct explicit generating series of arithmetic extensions of Kudla's special divisors on integral models of unitary Shimura varieties over CM fields with arbitrary split levels and prove that they are modular forms valued in the arithmetic Chow groups. This provides a partial solution to Kudla's modularity problem. The main ingredient in our construction is S. Zhang's theory of admissible arithmetic divisors. The main ingredient in the proof is an arithmetic mixed Siegel-Weil formula.

Contents

1. Introduction

Let E be a CM field, V a hermitian space over E of signature $(n,1),(n+1,0),\ldots,(n+1,0)$, and X a Shimura variety for $U(V)$. Let F be the maximal totally real subfield and $F_{>0}$ the set of totally positive elements of F. For $t \in F_{>0}$, we have a special divisor Z_t on X, following Kudla's work [Kud97a] for orthogonal Shimura varieties. Let $[Z_t]$ be the class of Z_t in the Chow group $\mathrm{Ch}^1(X)_{\mathbb{C}}$ of divisors on X with $\mathbb{C}$-coefficients. By Liu [Liu11a], the generating series

$$\text{constant term} + \sum_{t \in F_{>0}} [Z_t] q^t, \tag{1.1}$$

with a suitable constant term, is a $\mathrm{Ch}^1(X)_{\mathbb{C}}$-valued holomorphic modular form. Here, $q = \prod_{k=1}^{[F:\mathbb{Q}]} e^{2\pi i \tau_k}$ with $\tau = (\tau_k)_{k=1}^{[F:\mathbb{Q}]} \in \mathcal{H}^{[F:\mathbb{Q}]}$, where $\mathcal{H}$ is the usual upper half plane. This is an analog of the theorem of Borcherds [Bor99], Yuan, S. Zhang and W. Zhang [YZZ09] for orthogonal Shimura varieties, which was originally conjectured by Kudla [Kud97a]. In [Kud02, Kud03, Kud04], Kudla also raised the problem of finding (canonical) arithmetic extensions of special divisors on integral models of Shimura varieties to obtain a modular generating series, which is crucial for Kudla's program on arithmetic theta lifting.

The main result of this paper provides a solution to Kudla's modularity problem in the case that X is proper with arbitrary level structures at split places and certain lattice level structures at nonsplit places. The arithmetic extensions are defined using S. Zhang's theory of admissible arithmetic divisors. Slightly more explicitly, we construct a regular integral model $\mathcal{X}$ of X proper flat over $\mathcal{O}_E$. An admissible arithmetic divisor on $\mathcal{X}$ is an analog of an admissible Green function (i.e., one with harmonic curvature). Consider the normalized admissible extension $Z_t^{\overline{\mathcal{L}}}$ of Z_t, which is the Zariski closure at every finite place of E where the model is smooth. Let $[Z_t^{\overline{\mathcal{L}}}]$ be its class in the arithmetic Chow group. Then the generating series

$$\text{constant term} + \sum_{t \in F_{>0}} \left([Z_t^{\overline{\mathcal{L}}}] + \mathfrak{e}_t\right) q^t, \tag{1.2}$$

with a suitable constant term, is a holomorphic modular form. Here, $\mathfrak{e}_t$ is formed using coefficients of an explicit Eisenstein series and its derivative.

Previous to our work, solutions to Kudla's modularity problem were obtained using different methods by Kudla, Rapoport and Yang [Kud03] [KRY06] for quaternionic Shimura curves, by Bruinier, Burgos Gil, and Kühn [BBGK07] for Hilbert modular surfaces, over $\mathbb{Q}$ with minimal level structures, by Howard and Madapusi Pera [HMP20] for orthogonal Shimura varieties over $\mathbb{Q}$, and by Bruinier, Howard, Kudla, Rapoport and Yang [BHK+20a] for unitary Shimura varieties over imaginary quadratic fields, with self-dual lattice level structures. Compared to these results, we expect that the greater generality of the level structures in our result could be more useful for some purposes – for example, to approach modularity in higher codimensions following the inductive process in [YZZ09] for the generic fibers.

In the other direction, S. Zhang [Zha20] introduced the notion of L-liftings of divisor classes (on general polarized arithmetic varieties), and then deduced a solution to Kudla's modularity problem directly from the modularity results for the generic fibers in the first paragraph, regardless of level structures. The L-lifting of a divisor class is also admissible but 'normalized' in the level of arithmetic divisor classes using the Faltings heights. Our approach is an explicit alternative of S. Zhang's. In some applications, an explicit modular generating series as our (1.2) is necessary. For example, W. Zhang's proof of the arithmetic fundamental lemma [Zha21a] used the explicit result of [BHK+20a].

The main ingredient in the proof of our main result is an arithmetic mixed Siegel-Weil formula, which identifies the arithmetic intersection between the generating series (1.2) with a CM 1-cycle on $\mathcal{X}$ (associated to a 1-dimensional hermitian subspace of V) and an explicit modular form constructed from theta series and (derivatives of) Eisenstein series.

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Arithmetic mixed Siegel-Weil formulas appeared in the literature in different contexts. The first one appeared in the work of Gross and Zagier [GZ86, p 233, (9.3)] for generating series of Hecke operators on the square of a modular curve, and implies their celebrated formula relating heights of heegner points and derivatives of L-functions. This arithmetic mixed Siegel-Weil formula was partially generalized to quaternionic Shimura curves over totally real fields in the work of Yuan, S. Zhang and W. Zhang [YZZ13, 1.5.6] on the general Gross-Zagier formula. For certain orthogonal Shimura varieties over $\mathbb{Q}$, an arithmetic mixed Siegel-Weil formula was conjectured by Bruinier and Yang [BY09, Conjecture 1.3]. Its analog for unitary Shimura varieties over imaginary quadratic fields with certain self-dual lattice level structures was proved by Bruinier, Howard and Yang [BHY15, Theorem C].

In the rest of this introduction, we first state our main result in more detail. Then we discuss its proof. Finally, we mention two non-holomorphic modular variants of (1.2).

1.1. Main result

To state our main result, we need some preliminaries.

1.1.1. Admissible divisors

Let E be a number field, $\mathcal{X}$ a regular scheme (or more generally Deligne-Mumford stack) proper flat over $\operatorname{Spec}\mathcal{O}_E$ and $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ an ample hermitian line bundle on $\mathcal{X}$. At each infinite place v of E, equip the complex manifold $\mathcal{X}_{E_v}$ with the Kähler form that is the curvature form $\operatorname{curv}(\overline{\mathcal{L}}_{E_v})$. First, a Green function is admissible (introduced by Gillet and Soulé [GS90, 5.1] following Arakelov [Ara74]) if its curvature form α is harmonic; equivalently, on each connected component of $\mathcal{X}_{E_v}$, $\operatorname{curv}(\overline{\mathcal{L}}_{E_v})^{n-1}\wedge\alpha$ is proportional to $\operatorname{curv}(\overline{\mathcal{L}}_{E_v})^n$, where $n = \dim\mathcal{X}_{E_v}$. It is further normalized if on each connected component of $\mathcal{X}_{E_v}$, its pairing with (i.e., integration against) $\operatorname{curv}(\overline{\mathcal{L}}_{E_v})^n$ is 0. Second, at each finite place v, a divisor Y on $\mathcal{X}_{\mathcal{O}_{E_v}}$ is admissible if it has 'harmonic curvature' with respect to $\overline{\mathcal{L}}_{\mathcal{O}_{E_v}}$, in the sense that on each connected component of $\mathcal{X}_{\mathcal{O}_{E_v}}$, the linear form on the space of vertical divisors defined by intersecting with $Y\cdot c_1(\mathcal{L}_{\mathcal{O}_{E_v}})^{n-1}$ is proportional to the linear form defined by intersecting with $c_1(\mathcal{L}_{\mathcal{O}_{E_v}})^n$. We further call Y normalized if its vertical part has intersection pairing 0 with $c_1(\mathcal{L})^n$. Finally, an arithmetic divisor on $\mathcal{X}$ is (normalized) admissible if it is (normalized) admissible at every finite place and its Green function is (normalized) admissible. For a divisor Z on $\mathcal{X}_E$, we have the unique normalized admissible extension $Z^{\overline{\mathcal{L}}}$ on $\mathcal{X}$ (called the Arakelov lifting of Z in [Zha20]).

Let $\widehat{\operatorname{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X})$ be the space of admissible arithmetic divisors with $\mathbb{C}$-coefficients, modulo the $\mathbb{C}$-span of the principal ones. If $\mathcal{X}$ is connected, then the natural map

$$\widehat{\operatorname{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}) \to \operatorname{Ch}^1(\mathcal{X}_E)_{\mathbb{C}} \tag{1.3}$$

is surjective and has a 1-dimensional kernel. It is the pullback of $\widehat{\operatorname{Ch}}^1_{\mathbb{C}}(\operatorname{Spec}\mathcal{O}_E)\simeq\mathbb{C}$, where the isomorphism is by taking degrees. Then the $\mathfrak{e}_t\in\mathbb{C}\subset\widehat{\operatorname{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X})$ in (1.2) is understood in this way.

1.1.2. Shimura varieties and integral models

Let $V(\mathbb{A}_E^\infty)$ be the space of finite adelic points of V. For an open compact subgroup $K\subset U(V(\mathbb{A}_E^\infty))$, we have a $U(V)$-Shimura variety $\operatorname{Sh}(V)_K$ (which could be stacky) of level K defined over E. We assume that $\operatorname{Sh}(V)_K$ is proper; equivalently, $F\neq\mathbb{Q}$ or $F=\mathbb{Q}, n=1$ and V is nonsplit at some finite place.

Let $\Lambda\subset V(\mathbb{A}_E^\infty)$ be a hermitian lattice with stabilizer $K_\Lambda\subset U(V(\mathbb{A}_E^\infty))$. Let $K\subset K_\Lambda$ such that $K_v = K_{\Lambda,v}$ for v nonsplit in E. We construct a regular integral model $\mathcal{X}_E$ of $\operatorname{Sh}(V)_K$ proper flat over $\mathcal{O}_E$ under some conditions on E, F, Λ (Theorem 4.4.4). Our construction is largely suggested by Liu.

We have two constructions according to different conditions on E, F. First, assume that $E/\mathbb{Q}$ is tamely ramified. We have the normalization in $\operatorname{Sh}(V)_K$ of the flat model of $\operatorname{Sh}(V)_{K_\Lambda}$ of Kisin [Kis10], Kisin and Pappas [KP18] over $\mathcal{O}_{E,(v)}$, for every finite place v. We want to show their regularity and

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glue them to obtain a regular integral model $\mathcal{X}_K$ over $\mathcal{O}_E$. For this purpose, we use a certain regular PEL moduli space for a group closely related to $U(V)$ over the ring of integers of a reflex field E'/E, constructed by Rapoport, Smithling and W. Zhang [RSZ20]. Expectably, the moduli space and our integral models are closely related, as shown by Xu in Appendix B (the proof for the general level at split places was suggested by Liu). Second, replacing the tameness assumption by the assumption that $E/\mathbb{Q}$ is Galois or E is the composition of F with some imaginary quadratic field, which implies that $E' = E$, we can construct a regular integral model over $\mathcal{O}_{E,(v)}$ from the above moduli space, following [LTX+22]. Moreover, if both the tameness assumption and the replacement hold, the two constructions give the same model.

We remark that by our choice of Λ, $\mathcal{X}_{K_\Lambda}$ is smooth over $\mathcal{O}_E$ so that the finite part of the normalized admissible extension of a divisor on $\mathrm{Sh}(V)_{K_\Lambda}$ is the Zariski closure on $\mathcal{X}_{K_\Lambda}$.

1.1.3. Hodge bundles and CM cycles

Let $\mathcal{L}_{K_\Lambda}$ be an arbitrarily line bundle on $\mathcal{X}_{K_\Lambda}$ extending the Hodge (line) bundle on $\mathrm{Sh}(V)_{K_\Lambda}$. Let $\mathcal{L}_K$, denoted by $\mathcal{L}$ if K is clear from the context, be the pullback of $\mathcal{L}_{K_\Lambda}$ to $\mathcal{X}_K$. Let $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$, where $\|\cdot\|$ is the descent of the natural hermitian metric on the hermitian symmetric domain uniformizing $\mathrm{Sh}(V)_K$. It is compatible under pullbacks as K shrinks. Changing $\overline{\mathcal{L}}$, $c_1(\overline{\mathcal{L}}) \in \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{X})$ changes by an element in the pullback of $\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathrm{Spec}\,\mathcal{O}_E)$. (It is a special feature due to the smoothness of $\mathcal{X}_{K_\Lambda}$ over $\mathcal{O}_E$.) In particular, changing $\overline{\mathcal{L}}$ will not change the generating series (1.2). However, this fact does not play a role in our proof.

For a 1-dimensional hermitian subspace $W \subset V$, we have an associated 0-dimensional Shimura subvariety of $\mathrm{Sh}(V)$. On $\mathcal{X}_{K_\Lambda}$, let the 1-cycle $\mathcal{P}_{K_\Lambda}$ be its Zariski closure, divided by the degree of its genetic fiber (so that $\deg \mathcal{P}_{K_\Lambda,E} = 1$). Then $\mathcal{P}_{K_\Lambda}$ is independent of the choice of this subspace (Proposition 5.1.9). We do not need CM cycles at other levels.

1.1.4. Generating series

We start with the non-constant terms of the generating series (1.2).

For $x \in \mathbb{V}^\infty$ with norm in $F_{>0}$, the orthogonal complement of $\mathbb{A}_E^\infty x$ in $\mathbb{V}^\infty$ defines a (shifted) unitary Shimura subvariety $Z(x)_K$ of $\mathrm{Sh}(V)_K$ of codimension 1. For $t \in F_{>0}$ and a Schwartz function ϕ on $V(\mathbb{A}_E^\infty)$ invariant by K, the weighted special divisor is

$$Z_t = Z_t(\phi)_K = \sum_{x \in K\backslash \mathbb{V}^\infty,\ q(x)=t} \phi(x) Z(x)_K.$$

It is compatible under pullbacks as K shrinks.

Now we define $\mathfrak{e}_t$. Let $E(s,\tau)$ be the Siegel-Eisenstein series on $\mathcal{H}^{[F:\mathbb{Q}]}$ associated to ϕ. Its t-th Whittaker function $E_t(s,\tau)$ has a decomposition

$$E_t(s,\tau) = W_{\infty,t}(s,\tau) W_t^\infty(s)$$

into the infinite component and the finite component. Here, we choose the s-variable so that $E(0,\tau)$ is holomorphic of weight $n+1$, equivalently, the infinite component $W_{\infty,t}(0,\tau)$ is a multiple of q^t. (Note that $s=0$ is the critical point for the Siegel-Weil formula, but not the center for the functional equation.) The t-th Fourier coefficient of $E(0,\tau)$ is $\frac{E_t(0,\tau)}{q^t}$. Define

$$\mathfrak{e}_t = \frac{W_{\infty,t}(0,\tau)}{q^t} \frac{d}{ds}\Big|_{s=0} W_t^\infty(s) + \frac{E_t(0,\tau)}{q^t} \log \mathrm{Nm}_{F/\mathbb{Q}} t. \tag{1.4}$$

Then $\mathfrak{e}_t$ is independent of τ.

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We introduce a number that will appear in the constant term of the generating series. Let

$$\mathfrak{a} = c_1(\overline{\mathcal{L}}_{K_\Lambda}) \cdot \mathcal{P}_{K_\Lambda} + 2\frac{L'(0,\eta)}{L(0,\eta)} + \log|\mathrm{Disc}_E| - \mathfrak{b}[F:\mathbb{Q}] - \frac{[F:\mathbb{Q}]}{n},$$

where $\mathrm{Disc}_E \in \mathbb{Z}$ is the discriminant of $E/\mathbb{Q}$, and $\mathfrak{b} = -(1+\log 4)$ when $n=1$ and more complicated in general. See (3.30) and the remarking following it. We hope to compute the Faltings height $c_1(\overline{\mathcal{L}}_{K_\Lambda})\cdot\mathcal{P}_{K_\Lambda}$ based on [YZ18] in a future work. And we expect cancellation among the terms defining $\mathfrak{a}$ so that the definition of $\mathfrak{a}$ will be elementary and transparent.

Theorem 1.1.1 (Theorem 4.4.21, (4.28)). *If $\phi_v = 1_{\Lambda_v}$ at ramified places, the generating series (1.2) with the constant term being $\phi(0)\left(c_1(\overline{\mathcal{L}}^\vee) + \mathfrak{a}\right)$ is a holomorphic modular form on $\mathcal{H}^{[F:\mathbb{Q}]}$ of parallel weight $n+1$ valued in $\widehat{\mathrm{Ch}}^1_{\mathcal{L},\mathbb{C}}(\mathcal{X}_K)$. Here, we understand $\mathfrak{a}, \mathfrak{e}_t \in \mathbb{C} \subset \widehat{\mathrm{Ch}}^1_{\mathcal{L},\mathbb{C}}(\mathcal{X}_K)$ as discussed below (1.3).*

Since the formation of normalized admissible extension is compatible under flat pullbacks, the generating series (1.2) is compatible under pullbacks as K shrinks.

We note that the sum of the normalized admissible Green function for Z_t and $\mathfrak{e}_t$ recovers the Bruinier-Borcherds Green function used by Bruinier, Howard, Kudla, Rapoport and Yang [BHK+20a] for $F = \mathbb{Q}$, so that Theorem 1.1.1 is an analog of [BHK+20a, Theorem B]. Though the Bruinier-Borcherds construction can not be directly extended to a general F (as explained to us by Bruinier), our construction could be considered as an alternative generalization.

1.2. Sketch of the proof

Now we discuss the proof of Theorem 1.1.1. By the 1-dimensionality of the kernel of (1.3) and the modularity of the generic fiber of the generating series (1.2) (i.e., the generating series (1.1)), the modularity of (1.2) is equivalent to the modularity of the generating series of arithmetic intersection numbers between $[Z_t^{\overline{\mathcal{L}}}] + \mathfrak{e}_t$'s and a 1-cycle on $\mathcal{X}_K$ whose generic fiber has nonzero degree. (A similar strategy was used in [Kud03, KRY06].)

Assume that $K = K_\Lambda$ for simplicity and let us take the 1-cycle to be $\mathcal{P}_{K_\Lambda}$. Then this generating series of arithmetic intersection numbers $\left([Z_t^{\overline{\mathcal{L}}}] + \mathfrak{e}_t\right) \cdot \mathcal{P}_{K_\Lambda}$ is the arithmetic analog of the integration of a theta series of $U(1,1) \times U(V)$ along $U(W)\backslash U(W(\mathbb{A}_E))$, where $W \subset V$ is the 1-dimensional hermitian subspace defining $\mathcal{P}_{K_\Lambda}$. By the Siegel-Weil formula for $U(1,1) \times U(W)$, this integration is a theta-Eisenstein series (i.e., a linear combination of products of theta series and Eisenstein series).[1]

Let $\theta E(s,\tau)$ be the theta-Eisenstein series associated to ϕ and let $\theta E_t(s,\tau)$ be its t-th Whittaker coefficient. We study the holomorphic projections of $\theta E'(0,\tau)$ in order to match the above generating series of arithmetic intersection numbers, which is supposed to be holomorphic in view of our goal (Theorem 1.1.1). A priori, there are two holomorphic projections. One is the projection of $\theta E'(0,\tau)$ to the space of holomorphic cusp forms, which we call 'cuspidal holomorphic projection' and denote by $\theta E'_{\mathrm{chol}}(0,\tau)$. Let $\theta E'_{\mathrm{chol},t}(0,\tau)$ be its t-th Whittaker coefficient. The other is for $\theta E'_t(0,\tau)$ and purely at infinite places, which we call 'quasi-holomorphic projection' following [YZZ13] and denote by $\theta E'_{t,\mathrm{qhol}}(0,\tau)$. However, neither of them could be the desired match, since $\theta E'_{\mathrm{chol}}(0,\tau)$ has no constant term and $\theta E'_{t,\mathrm{qhol}}(0,\tau)$ is (in general) not the t-th Whittaker coefficient of a modular form. Thus, we compute their difference and find that $\theta E'_{t,\mathrm{qhol}}(0,\tau) - \theta E'_{\mathrm{chol},t}(0,\tau)$ is the sum of $2\mathfrak{e}_t q^t$ and the t-th Whittaker coefficient of a holomorphic Eisenstein series. See (3.31). In the case $n=1$, the computations of holomorphic projections have their roots in [GZ86]. The strategy we follow is outlined in [YZZ13, 6.4.3], and the explicit computation was done by Yuan [Yua22]. We largely follow [Yua22].

[1]The name actually comes from 'mixed Eisenstein-theta series' in [YZZ13]. We use a slightly different notation to cope with the later notation '$\theta E(s,\tau)$', which we will take derivative on the Eisenstein series and so that we make the 'E' closer to the s-variable.

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Let f be the sum of $-\frac{1}{2}\theta E'_{\mathrm{chol}}(0,\tau)$ and the negative of the holomorphic Eisenstein series in the last sentence. Then f is a holomorphic modular form on $\mathcal{H}^{[F:\mathbb{Q}]}$ of parallel weight $n+1$. The t-th Whittaker coefficient of f is $-\frac{1}{2}\frac{\theta E'_{t,\mathrm{qhol}}(0,\tau)}{q^t}+\mathfrak{e}_t$.

In 5.2.1, we define some explicit Schwartz functions ϕ'_v, for every ramified place v, which are 'error functions' due to ramification. Let g be sum of the theta-Eisenstein series associated to $\phi^v\phi'_v$'s. The following is our arithmetic mixed Siegel-Weil formula (Theorem 5.2.5) in the case $K=K_\Lambda$.

Theorem 1.2.1. *Assume that $\phi_v = 1_{\Lambda_v}$ for v nonsplit in E. The arithmetic intersection number $\left([Z_t^{\overline{\mathcal{L}}}]+\mathfrak{e}_t\right)\cdot\mathcal{P}_{K_\Lambda}$ is the t-th Fourier coefficient of $f-g-\frac{1}{n}E(0,\tau)$.*

In Theorem 5.2.5 for a general K, we use the pullback of $\mathcal{P}_{K_\Lambda}$ to $\mathcal{X}_K$, instead of natural CM cycles on $\mathcal{X}_K$, to simplify certain local computations. See Remark 5.2.6.

We remind the reader that in Theorem 5.2.5, we actually use the automorphic Green function for Z_t constructed by Bruinier [Bru02, Bru12] and Oda and Tsuzuki [OT03] (for $n=1$ and $F=\mathbb{Q}$, it was well known and used by Gross and Zagier [GZ86]). Its difference with the normalized admissible Green function is $\frac{1}{n}E_t(0,\tau)$ by Lemma 4.2.4 and the remark following it.

Let us remark on the innovation in proving the arithmetic mixed Siegel-Weil formula. We consider the difference of two CM cycles. The generic fiber of the difference has degree 0. Then the generating series of arithmetic intersection numbers is modular by the admissibility. (A similar observation was used in [MZ21] to generalize the arithmetic fundamental lemma. See also [Zha21b].) This modularity enables us to 'switch CM cycles' and thus avoid computing improper intersections directly. This idea is inspired by [YZZ13] and [Zha21a].

1.3. Non-holomorphic variants

We obtain a non-holomorphic modular variant of the generating series (1.2), where the sum of the normalized admissible Green function for Z_t and $\mathfrak{e}_t$ is replaced by Kudla's Green function [Kud97b]. See Theorem 4.4.24. This is an analog of [KRY06, Theorem A] [BHK+20a, Theorem 7.4.1]. Theorem 4.4.24 follows from Theorem 1.1.1, and the modularity of the differences between the generating series of two kinds of Green functions (Theorem 4.2.10). The latter (Theorem 4.2.10) is an analog of the main result of Ehlen and Sankaran [ES18] for $F=\mathbb{Q}$.

Note that Kudla's Green function is not admissible. We also obtain a non-holomorphic modular generating series with admissible Green functions (Theorem 4.4.21, (4.29)). This has not appeared in the literature yet, as far as we know.

2. Some notations and conventions

2.1.

For a number field F, let $\mathbb{A}_F$ be the ring of adeles of F and $\mathbb{A}_F^\infty$ the ring of finite adeles of F. For a finite place v of a number field F, let ϖ_{F_v} be a uniformizer of F_v. Let q_{F_v} be the cardinality of $\mathcal{O}_{F_v}/\varpi_{F_v}$. The discrete valuation is $v(\varpi_{F_v})=1$ and the absolute value $|\cdot|_{F_v}$ is $|\varpi_v|_{F_v}=q_{F_v}^{-1}$. For an infinite place, v is understood as a pair of complex embeddings. If v is real, the absolute value $|\cdot|_{F_v}$ is the usual one; if v is complex, the absolute value $|\cdot|_{F_v}$ is the square of the usual one. Their product is $|\cdot|_{\mathbb{A}_F}$. The symbol $|\cdot|$ without a subscript means the usual real or complex absolute value.

Below in this paper, E/F is always a CM extension. Let ∞ be the set of infinite places of F. Let $F_{>0}\subset F$ be the subset of totally positive elements. For a place v of F, E_v is understood as $E\otimes_F F_v$. The nontrivial Galois action will be denoted by $x\mapsto\bar{x}$, and the norm map $\mathrm{Nm}_{E/F}$ or its local version is abbreviated as Nm. Let η be the associated quadratic Hecke character of $F^\times\backslash\mathbb{A}_F^\times$ via the class field theory.

For a set of place S of F and a decomposable adelic object X over $\mathbb{A}_F$, we use X_S (resp. X^S) to denote the S-component (resp. component away from S) of X if the decomposition of X into the product of X_S and X^S is clear from the context. For example, $\mathbb{A}_F = \mathbb{A}_{F,S}\mathbb{A}_F^S$ and $\mathbb{A}_E = \mathbb{A}_{E,S}\mathbb{A}_E^S$ by regarding $\mathbb{A}_E$ as over $\mathbb{A}_F$. Here is another example which is ubiquitous in the paper: a function ϕ on the space of $\mathbb{A}_F$-points of an algebraic group over F that can be decomposed as $\phi = \phi_S \otimes \phi^S$, where ϕ_S (resp. ϕ^S) is a function on the set of $\mathbb{A}_{F,S}$-points (resp. $\mathbb{A}_F^S$-points) of the group. Note that such a decomposition of ϕ is not unique. By using these notations, we understand that we have fixed such a decomposition. See 2.3 for an example. If $S = \{v\}$, we write X_v (resp. X^v) for X_S (resp. X^S).

2.2.

All hermitian spaces are assumed to be nondegenerate. We always use $\langle \cdot, \cdot \rangle$ to denote a hermitian pairing and $q(x) = \langle x, x\rangle$ the hermitian norm if the underlying hermitian space (over E, E_v or $\mathbb{A}_E$) is indicated in the context. For a hermitian space V over E, we use V to denote $V(E)$ to lighten the notation if there is no confusion. For $t \in F$, let $V^t = \{v \in V : q(v) = t\}$. The same notation applies to a local or adelic hermitian space. We use $U(V)$ for both the algebraic group $U(V)$ and its group of F-points. Define

$$[U(V)] = U(V)\backslash U(V(\mathbb{A}_E)).$$

A hermitian space $\mathbb{V}/\mathbb{A}_E$ is called coherent (resp. incoherent) if its determinant belongs (resp. does not belong) to $F^\times \mathrm{Nm}(\mathbb{A}_E^\times)$; equivalently, $\mathbb{V} \simeq V(\mathbb{A}_E)$ for some (resp. no) hermitian space V/E. If $\mathbb{V}$ is incoherent of dimension 1, for a place v of F nonsplit in E, there is a unique hermitian space V/E such $V(\mathbb{A}_E^v) \simeq \mathbb{V}^v$. We call V the v-nearby hermitian space of $\mathbb{W}$.

2.3.

Let $\mathcal{S}(\mathbb{V})$ be the space of $\mathbb{C}$-valued Schwartz functions. For $v \in \infty$ such that $\mathbb{V}(E_v)$ is positive definite, the standard Gaussian function on $\mathbb{V}_\infty$ is $e^{-2\pi q(x)} \in \mathcal{S}(\mathbb{V}(E_v))$. If a hermitian space $\mathbb{V}/\mathbb{A}_E$ is totally positive-definite, let

$$\overline{\mathcal{S}}(\mathbb{V}) \subset \mathcal{S}(\mathbb{V})$$

be the subspace of functions of the form $\phi = \phi_\infty \otimes \phi^\infty$, where ϕ_∞ is the pure tensor of standard Gaussian functions over all infinite places and $\phi^\infty \in \mathcal{S}(\mathbb{V}^\infty)$ taking values in $\mathbb{C}$. For $\phi \in \mathcal{S}(\mathbb{V}^\infty)$, we always fix such a decomposition.

2.4.

Fix the additive character of $F\backslash\mathbb{A}_F$ to be $\psi := \psi_\mathbb{Q} \circ \mathrm{Tr}_{F/\mathbb{Q}}$, where $\psi_\mathbb{Q}$ is the unique additive character of $\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}$ such that $\psi_{\mathbb{Q},\infty}(x) = e^{2\pi i x}$. The additive character of $\mathbb{A}_E$ is $\psi_E := \psi \circ \mathrm{Tr}_{E/F}$. For $t \in \mathbb{A}_F$ (we in fact only use $t \in F$), let $\psi_t(b) = \psi(tb)$. For a place v of F and $t \in F_v$, Let $\psi_{v,t}(b) = \psi_v(tb)$. Then $\psi_{v,t} = \psi_{v,t_v}$ for $t \in \mathbb{A}_F$.

Fix the self-dual Haar measures for F_v and E_v. Then

$$d_{F_v^\times} x := \zeta_{F_v}(1)|x|_{F_v}^{-1} d_{F_v} x, \ d_{E_v^\times} x := \zeta_{E_v}(1)|x|_{E_v}^{-1} d_{E_v} x$$

are the induced Haar measures on $F_v^\times$ and $E_v^\times$. The subscripts will be omitted later in the paper. They induce the quotient measure on $E_v^\times/F_v^\times \simeq U(1)(F_v)$.

For $\phi \in \mathcal{S}(\mathbb{V}(E_v))$, the Fourier transform of ϕ (with respect to ψ and a Haar measure) is

$$\widehat{\phi}(x) = \int_{\mathbb{V}(E_v)} \phi(y)\psi_{E,v}(\langle x, y\rangle) dy.$$

We fix the self-dual Haar measure on $\mathbb{V}(E_v)$.

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2.5.

Let $G = U(1,1)$ be the unitary group over F of the standard skew-hermitian space over E of dimension 2; that is, the skew-hermitian form is given by the matrix

$$w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then $w \in G(F)$. We use w_v to denote the same matrix in $G(F_v)$

For $b \in \mathbb{G}_{a,F}$, let

$$n(b) = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}.$$

For $a \in \mathrm{Res}_{E/F}\mathbb{G}_{m,E}$, let

$$m(a) = \begin{bmatrix} a & 0 \\ 0 & \overline{a}^{-1} \end{bmatrix}.$$

Let $N = \{n(b) : b \in \mathbb{G}_{a,F}\} \subset G$, $M = \{m(a) : a \in \mathrm{Res}_{E/F}\mathbb{G}_{m,E}\} \subset G$, and $P = MN$ the subgroup of upper triangular matrices. Then G is generated P and w. The isomorphism $N \simeq \mathbb{G}_{a,F}$ induces an additive character and a Haar measure on $N(\mathbb{A}_F)$ which we fix in this paper.

Let $K_v^{\max}$ be the intersection of $G(F_v)$ with the standard maximal compact subgroup $\mathrm{GL}_2(E_v)$. Then $K_v^{\max}$ is a maximal compact subgroup of $G(F_v)$. For $v \in \infty$, $K_v^{\max}$ is the group of matrices

$$[k_1, k_2] := \frac{1}{2}\begin{bmatrix} k_1 + k_2 & -ik_1 + ik_2 \\ ik_1 - ik_2 & k_1 + k_2 \end{bmatrix},$$

where $k_1, k_2 \in E_v$ are of norm 1. We have the Iwasawa decomposition

$$G(F_v) = N(F_v)M(F_v)K_v^{\max}.$$

2.6.

For a place v of F, the local modulus character of $G(F_v)$ is given by

$$\delta_v(g) = |a|_{E_v}$$

if $g = n(b)m(a)k$ with $k \in K_v^{\max}$ under the Iwasawa decomposition. The global modulus character δ of $G(\mathbb{A}_F)$ is the product of the local ones. Since we will use results in [YZ18, YZZ13], where the subgroup $\mathrm{SL}_2 \subset G$ is used, we remind the reader that our modulus character, when restricted to $\mathrm{SL}_2(F_v) \subset G(F_v)$, is the square of the one in loc. cit..

2.7.

For $\mathfrak{w} = (\mathfrak{w}_v)_{v\in\infty}$, where $\mathfrak{w}_v$ is a pair of integers, let $\mathcal{A}(G, \mathfrak{w})$ be the space of smooth automorphic forms for G of weight $\mathfrak{w}$. Let $\mathcal{A}_{\mathrm{hol}}(G, \mathfrak{w})$ be the subspace of holomorphic automorphic forms. A characterization is as follows. For $v \in \infty$, $t \in F_{v,\geq 0}$ and a pair of integers (w_1, w_2), the standard holomorphic $\psi_{v,t}$-Whittaker function on $G(F_v)$ of weight (w_1, w_2) is

$$W_{v,t}^{(w_1,w_2)}(g) = e^{2\pi it(b+i|a|_{E_v})}|a|_{E_v}^{(w_1+w_2)/2}k_1^{w_1}k_2^{w_2},$$

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for $g = n(b)m(a)[k_1, k_2]$ under the Iwasawa decomposition. For $t \in \mathbb{A}_{F,\infty}$, let

$$W^{\mathfrak{w}}_{\infty,t} = \prod_{v\in\infty} W^{\mathfrak{w}_v}_{v,t}.$$

An automorphic form f on $G(\mathbb{A}_F)$ is holomorphic of weight $\mathfrak{w}$ if for $t \in F_{>0} \cup \{0\}$, its ψ_t-Whittaker function is a tensor of the finite and infinite component: $f_t = f_t^\infty \otimes W^{\mathfrak{w}}_{\infty,t}$, and for other $t \in F$, its ψ_t-Whittaker function is 0. In this case, we call the locally constant function f_t^∞ on $G(\mathbb{A}_F^\infty)$ the t-th Fourier coefficient of f. (Then $f_t^\infty(1)$ is the t-th Fourier coefficient in the sense of classical modular forms.)

For a subfield $C \subset \mathbb{C}$, let

$$\mathcal{A}_{\mathrm{hol}}(G, \mathfrak{w})_C \subset \mathcal{A}_{\mathrm{hol}}(G, \mathfrak{w})$$

be the C-subspace of automorphic forms f whose Fourier coefficients take values in C. (In the sense of classical modular forms, it means that the coefficients of the q-expansion of f along all cusps are in C.) Taking Fourier coefficients defines an embedding of C-vector spaces

$$\mathcal{A}_{\mathrm{hol}}(G, \mathfrak{w})_C \to \prod_{t\in F_{>0}\cup\{0\}} \mathrm{LC}\big(G(\mathbb{A}_F^\infty), C\big),$$

where $\mathrm{LC}\big(G(\mathbb{A}_F^\infty), C\big)$ means locally constant functions on $G(\mathbb{A}_F^\infty)$ valued in C. For a C-vector space X, we have the induced embedding

$$\mathcal{A}_{\mathrm{hol}}(G, \mathfrak{w})_C \otimes_C X \to \prod_{t\in F_{>0}\cup\{0\}} \mathrm{LC}\big(G(\mathbb{A}_F^\infty), X\big).$$

Define the t-th Fourier coefficient of an element in $\mathcal{A}_{\mathrm{hol}}(G, \mathfrak{w})_C \otimes_C X$ to be the t-th component of its image.

2.8.

Let $\mathbb{V}/\mathbb{A}_E$ be a hermitian space. For a character $\chi_{\mathbb{W}}$ of $E^\times\backslash\mathbb{A}_E^\times$ such that $\chi_{\mathbb{V},v}|_{F_v^\times} = \eta_v^{\dim \mathbb{V}}$ for every place v of F, the Weil representation $\omega = \omega_{\mathbb{V}}$ on $\mathcal{S}(\mathbb{V})$ is the restricted tensor product of local Weil representations of $G(F_v) \times U(\mathbb{V}(E_v))$ on $\mathcal{S}(\mathbb{V}(E_v))$. The local Weil representation (which we still denote by ω instead of ω_v if the meaning is clear from the context) of $G(F_v)$ is defined as follows: for $\phi \in \mathcal{S}(\mathbb{V}(E_v))$,

$$\begin{aligned}
\omega(m(a))\phi(x) &= \chi_{\mathbb{V},v}(a)|a|_{E_v}^{m/2}\phi(xa),\ a \in E_v^\times;\\
\omega(n(b))\phi(x) &= \psi_v(bq(x))\phi(x),\ b \in F_v;\\
\omega(w_v)\phi &= \gamma_{\mathbb{V}(E_v)}\widehat{\phi};\\
\omega(h)\phi(x) &= \phi(h^{-1}x),\ h \in U(\mathbb{V}(E_v)).
\end{aligned}$$

Here, $\gamma_{\mathbb{V}(E_v)}$ is the Weil index associated to ψ_v and $\mathbb{V}(E_v)$.

2.9.

For $v \in \infty$, define $\mathfrak{k}^{\chi_{\mathbb{V},v}}$ to be the unique integer such that

$$\chi_{\mathbb{V},v}(z) = z^{\mathfrak{k}^\chi}$$

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for $z \in E_v$ of norm 1. Define $\mathfrak{w}_{\chi_{\mathbb{V}}} = (\mathfrak{w}_{\chi_{\mathbb{V}},v})_{v\in\infty}$, where

$$\mathfrak{w}_{\chi_{\mathbb{V}},v} := \left(\frac{\dim \mathbb{V} + \mathfrak{k}_{\chi_{\mathbb{V}},v}}{2}, \frac{\dim \mathbb{V} - \mathfrak{k}_{\chi_{\mathbb{V}},v}}{2}\right).$$

3. Theta-Eisenstein series

First, we recall basic knowledge about Eisenstein series and theta series. Then we set up basic properties of theta-Eisenstein series (i.e., linear combinations of products of theta series and Eisenstein series). Finally, we study two kinds of holomorphic projections of theta-Eisenstein series. The origin of theta-Eisenstein series is in the work of Gross and Zagier[GZ86]. We largely follow the works of Yuan, S. Zhang and W. Zhang [Yua22, YZ18, YZZ13].

3.1. Eisenstein series and theta series

Let $\mathbb{W}$ be a hermitian space over $\mathbb{A}_E$ (with respect to the extension E/F). Let $\chi_{\mathbb{W}}$ be a character of $E^\times\backslash\mathbb{A}_E^\times$ such that $\chi_{\mathbb{V}}|_{\mathbb{A}_F^\times} = \eta$. We have the Weil representation $\omega_{\mathbb{W}}$, which we simply denote by ω.

3.1.1. Local Whittaker integrals

Let v be a place of F. For $t \in F_v$, $\phi \in \mathcal{S}(\mathbb{W}_v)$ and $g \in G(F_v)$, define the Whittaker integral

$$W_{v,t}(s, g, \phi) = \int_{N(F_v)} \delta_v(w_v n g)^s \omega(w_v n g)\phi(0)\psi_{-t}(n) dn. \tag{3.1}$$

We immediately have the following equations:

$$W_{v,t}(s, gk, \phi) = W_{v,t}(s, g, \omega(k)\phi),\ k \in K_v^{\max}, \tag{3.2}$$

$$W_{v,t}(s, n(b)g, \phi) = \psi_v(bt) W_{v,t}(s, g, \phi),\ b \in F_v. \tag{3.3}$$

Since $wn(b)m(a) = m(\overline{a}^{-1})wn(b\mathrm{Nm}(a)^{-1})$, $m(\overline{a}^{-1})n(b) = n(b')m(\overline{a}^{-1})$ for some b', and $\chi_{\mathbb{W},v}(\mathrm{Nm}(a)) = 1$, a direct computation gives

$$W_{v,t}(s, m(a)g, \phi) = |a|_{E_v}^{1-\dim \mathbb{W}/2-s} \chi_{\mathbb{W},v}(a) W_{v,\mathrm{Nm}(a)t}(s, g, \phi),\ a \in E_v^\times. \tag{3.4}$$

Lemma 3.1.1 [Ich04, Proposition 6.2][YZZ13, Proposition 2.7 (2)]. *Let $t \neq 0$.*

(1) *The set $\mathbb{W}_v^t$ is either empty or consists of one orbit of $U(\mathbb{W}_v)$.*

(2) *If $\mathbb{W}_v^t$ is empty, then $W_{v,t}(0, g, \phi) = 0$. Otherwise, for $x \in \mathbb{W}_v^t$, we have*

$$W_{v,t}(0, g, \phi) = \kappa \int_{U(\mathbb{W}_v)} \omega(g)\phi(h^{-1}x) dh,$$

for a nonzero constant κ.

(3) *If $\dim \mathbb{W} = 1$, with the measure fixed in 2.4, $\kappa = \frac{\gamma_{\mathbb{W}_v}}{L(1,\eta_v)}$.*

Assume $\dim \mathbb{W} = 1$.

Lemma 3.1.2. *Assume that v is a finite place and $\phi(0) = 0$. Then for t small enough, $W_{v,t}(s, g, \phi) = W_{v,0}(s, g, \phi)$ and is a holomorphic function.*

Proof. The proof is by the reasoning as the proof of [Qiu21, Lemma 4.2.4 (2)] □

We define the following normalization (following [YZZ13, 6.1.1]):

$$W_{v,t}^\circ = \gamma_{\mathbb{W}_v}^{-1} W_{v,t},\ t \neq 0. \tag{3.5}$$

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For $t = 0$, $W_{v,t}(s, g, \phi)$ has a possible pole at $s = 0$. And we take a different normalization (following [YZZ13, 6.1.1], and taking care of the difference between the modulus characters mentioned in 2.6),

$$W^{\circ}_{v,0}(s, g, \phi) := \gamma^{-1}_{\mathbb{W}_v} |\mathrm{Diff}_v \mathrm{Disc}_v|^{-1/2}_{F_v} \frac{L(2s+1, \eta_v)}{L(2s, \eta_v)} W_{v,0}(s, g, \phi). \tag{3.6}$$

Here, if v is a finite place, Diff_v is the different of $F_v/\mathbb{Q}_v$ and Disc_v is the discriminant of E_v/F_v, and if $v \in \infty$, $\mathrm{Diff}_v = \mathrm{Disc}_v = 1$.

Lemma 3.1.3. (1) *There is an analytic continuation of $W^{\circ}_0(s, g, \phi)$ to $\mathbb{C}$ such that $W^{\circ}_0(0, g, \phi) = \omega(g)\phi(0)$.*

(2) *If $E_v/\mathbb{Q}_v$ is unramified where $\mathbb{W}_v = E_v$ with $q = \mathrm{Nm}$, and $\phi = 1_{\mathcal{O}_{E_v}}$, then $W^{\circ}_0(s, g, \phi) = \delta_v(g)^{-s}\omega(g)\phi(0)$.*

(3) *If $v \in \infty$ and ϕ is the standard Gaussian function, then $W^{\circ}_0(s, g, \phi) = \delta_v(g)^{-s}\omega(g)\phi(0)$.*

Proof. (1) follows from [YZZ13, Proposition 6.1]. (2) follows from [Tan99, Proposition 2.1]. (3) follows from [YZ18, Lemma 7.6 (1)] (or its proof). □

3.1.2. Siegel-Eisenstein series

Now we come back to a general $\mathbb{W}$. For $\phi \in \mathcal{S}(\mathbb{W})$, we have a Siegel-Eisenstein series of G:

$$E(s, g, \phi) = \sum_{\gamma \in P(F)\backslash G(F)} \delta(\gamma g)^s \omega(\gamma g)\phi(0), \tag{3.7}$$

which is absolutely convergent if $\mathrm{Re} s > 1 - \dim \mathbb{W}/2$ and has a meromorphic continuation to the entire complex plane [Tan99]. Moreover, it is holomorphic at $s = 0$ [Tan99, Proposition 4.1].

Let $E_t(s, g, \phi)$ be the ψ_t-Whittaker function of $E(s, g, \phi)$. Let $W_t(s, g, \phi)$ be the global counterpart of the Whittaker integral (3.1) so that if ϕ is a pure tensor, then $W_t(s, g, \phi)$ is the product of $W_{v,t}(s, g_v, \phi_v)$ over all places of F. By the non-vanishing of $L(1, \eta)$ and Lemma 3.1.3, $W_0(s, g, \phi)$ has a meromorphic continuation to the entire complex plane, which is holomorphic at $s = 0$. Then we have

$$E_t(s, g, \phi) = W_t(s, g, \phi),\ t \neq 0; \tag{3.8}$$

$$E_0(s, g, \phi) = \delta(g)^s \omega(g)\phi(0) + W_0(s, g, \phi). \tag{3.9}$$

By Lemma 3.1.1 (2) and (3.8), for $h \in U(\mathbb{W})$, we have

$$E_t(0, g, \omega(h)\phi) = E_t(0, g, \phi). \tag{3.10}$$

For $t \neq 0$ and a pure tensor ϕ, define

$$E'_t(0, g, \phi)(v) = W'_{v,t}(0, g_v, \phi_v) \prod_{u \neq v} W_{u,t}(0, g_u, \phi_u). \tag{3.11}$$

Extend the definition to all Schwartz functions by linearity. Then

$$E'_t(0, g, \phi) = \sum_v E'_t(0, g, \phi)(v). \tag{3.12}$$

3.1.3. Coherent case: Siegel-Weil formula

Assume that $\mathbb{W} = W(\mathbb{A}_E)$ is coherent. For $\phi \in \mathcal{S}(W(\mathbb{A}_E))$ and $(g, h) \in G(\mathbb{A}_F) \times U(\mathbb{W})$, we define a theta series, which is absolutely convergent:

$$\theta(g, h, \phi) = \sum_{x \in W} \omega(g, h)\phi(x).$$

Then $\theta(g, h, \phi)$ is smooth, slowly increasing and $G(F) \times U(W)$-invariant.

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Assume that W is anisotropic. Then $E(s,g,\phi)$ is holomorphic at $s=0$, and the following equation is a special case of the regularized Siegel-Weil formula [Ich04, Theorem 4.2]:

$$E(0,g,\phi)=\frac{\kappa}{\mathrm{Vol}([U(W)])}\int_{[U(W)]}\theta(g,h,\phi)dh, \tag{3.13}$$

where $\kappa=2$ if $\dim W=1$ and $\kappa=1$ if $\dim W>1$.

3.1.4. Incoherent case: derivative

Assume that $\mathbb{W}$ is incoherent. By Lemma 3.1.1 (1), for $t\neq 0$, the summand in (3.12) corresponding to v is nonzero only if t is represented by $\mathbb{W}^v$. By the incoherence, $\mathbb{W}_v$ does not represent t. In particular, we have the following lemma.

Lemma 3.1.4. *For v split in E, the summand in (3.12) corresponding v is* 0*.*

Assume that $\dim\mathbb{W}=1$. Assume that ϕ is a pure tensor.

First, assume that $t\neq 0$. By the product formula for Hasse invariant and the Hasse principle, if t is represented by $\mathbb{W}^v$, then v is nonsplit in E and t is represented by the v-nearby hermitian space W of $\mathbb{W}$. See 2.2. By Lemma 3.1.1 (2) and Lemma 3.1.4, we have

$$E_t'(0,g,\omega(h)\phi)=E_t'(0,g,\phi) \tag{3.14}$$

for $h\in U(\mathbb{W}_v)$, where v is split in E.

Now we consider the constant term. Let

$$\mathfrak{c}=4\frac{L'(0,\eta)}{L(0,\eta)}+2\log|\mathrm{Disc}_E/\mathrm{Disc}_F|, \tag{3.15}$$

where $\mathrm{Disc}_E,\mathrm{Disc}_F\in\mathbb{Z}$ are the discriminants of E and F over $\mathbb{Q}$. By [YZ18, p 586] (note the difference by 2 between the s-variables in the L-factors in loc. cit. and (3.6)),

$$W_0'(s,g,\phi)=-\mathfrak{c}\omega(g)\phi(0)-\sum_v\omega(g^v)\phi^v(0)W_{v,0}^{\circ}{}'(0,g_v,\phi_v). \tag{3.16}$$

3.2. *Theta-Eisenstein series*

From now on, always assume $\dim\mathbb{W}=1$.

3.2.1. Definition

Let $V^\sharp/E$ be a hermitian space of dimension $n>0$. Let $\chi_{V^\sharp}$ be a character of $E^\times\backslash\mathbb{A}_E^\times$ such that $\chi_{V^\sharp}|_{\mathbb{A}_F^\times}=\eta^n$. Let $\mathbb{V}=\mathbb{W}\oplus V^\sharp(\mathbb{A}_E)$ be the orthogonal direct sum and $\chi_{\mathbb{V}}=\chi_{\mathbb{W}}\chi_{V^\sharp}$. We have the corresponding Weil representations. Below, we shall use ω to denote a Weil representation if the hermitian space is indicated in the context – for example, by the function that it acts on.

For $\phi\in\mathcal{S}(\mathbb{V})$, we define a theta-Eisenstein series $\theta E(s,g,\phi)$ on G associated the to the orthogonal decomposition $\mathbb{V}=\mathbb{W}\oplus V^\sharp(\mathbb{A}_E)$:

$$\theta E(s,g,\phi)=\sum_{\gamma\in P(F)\backslash G(F)}\delta(\gamma g)^s\sum_{x\in V^\sharp}\omega(\gamma g)\phi((0,x)). \tag{3.17}$$

If $\phi=\phi_1\otimes\phi_2$ with $\phi_1\in\mathcal{S}(\mathbb{W})$ and $\phi_2\in\mathcal{S}\Big(V^\sharp(\mathbb{A}_E)\Big)$, then

$$\theta E(s,g,\phi)=E(s,g,\phi_1)\theta(g,\phi_2). \tag{3.18}$$

For $t\in F$, let $\theta E_t(s,g,\phi)$ be the ψ_t-Whittaker function of $\theta E(s,g,\phi)$.

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3.2.2. Coherent case

Assume that $\mathbb{W} = W(\mathbb{A}_E)$ is coherent. The regularized Siegel-Weil formula (3.13) immediately implies the following 'mixed Siegel-Weil formula':

$$\theta E(0, g, \phi) = \frac{2}{\mathrm{Vol}([U(W)])} \int_{[U(W)]} \theta(g, h, \phi) dh. \tag{3.19}$$

(We will prove an arithmetic analog of (3.19) for $\mathbb{W}$ being incoherent in 5.2.) Then for $t \in F$,

$$\theta E_t(0, g, \phi) = \frac{2}{\mathrm{Vol}([U(W)])} \int_{[U(W)]} \sum_{x \in V^t} \omega(g)\phi(h^{-1}x) dh. \tag{3.20}$$

For ϕ invariant by $U(W(E_v))$, $v \in \infty$, the integration in (3.20) is a finite sum.

Lemma 3.2.1. *Let $t \in F_{>0}$ and ϕ a pure tensor. Let u be a finite place of F, $O \subset \mathbb{V}_u$ an open compact neighborhood of* 0 *and $\phi^O = \phi^u \otimes (\phi_u 1_{\mathbb{V}_u - O})$. Given $g \in G(\mathbb{A}_F^u)P(F_u)$, for O small enough, we have $\theta E_t(0, g, \phi) = \theta E_t(0, g, \phi^O)$.*

Proof. Write $g_u = m(a)n(b)$, where $a \in E_u^\times$. See 2.5. Then $\{ah_u^{-1}x : x \in V^t, h \in U(\mathbb{V})\} \subset \mathbb{V}_u^{a^2t}$. The latter is closed in $\mathbb{V}_u$ and does not contain 0. Thus, $O \cap \{ah_u^{-1}x : x \in V^t, h \in U(\mathbb{V})\} = \emptyset$ if O is small enough. Then the lemma follows from (3.20), the remark below it, and the definition of the Weil representation in 2.8. □

3.2.3. Incoherent case

Assume that $\mathbb{W}$ is incoherent.

For a place v of F nonsplit in E, let W be the v-nearby hermitian space of $\mathbb{W}$. For $\phi = \phi_1 \otimes \phi_2$ with $\phi_1 \in \mathcal{S}(\mathbb{W}_v)$, $\phi_2 \in \mathcal{S}\left(V^\sharp(E_v)\right)$ and $x = (x_1, x_2) \in V := W(E_v) \oplus V^\sharp(E_v)$ with $x_1 \neq 0$, let

$$W\theta_{v,x}(s, g, \phi) = \frac{L(1, \eta_v)}{\mathrm{Vol}(U(W(E_v))} W^\circ_{v,q(x_1)}(s, g, \phi_1)\omega(g)\phi_2(x_2). \tag{3.21}$$

This is a local analog of (3.18). Extend this definition to $\mathcal{S}(\mathbb{W}_v) \otimes \mathcal{S}\left(V^\sharp(E_v)\right) \subset \mathcal{S}(\mathbb{V}(E_v))$ by linearity. The inclusion is an equality unless $v \in \infty$. However, this subspace is enough for our purpose. (Besides, there is another definition of $W\theta_{v,x}$ for the whole $\mathcal{S}(\mathbb{V}(E_v))$. We will not need it.) For v nonsplit in E and $t \neq 0$, define

$$\theta E_t'(0, g, \phi)(v) = \frac{2}{\mathrm{Vol}([U(W)])} \int_{[U(W)]} \sum_{x \in V^t - V^\sharp} W\theta'_{v, h_v^{-1}x}(0, g_v, \phi_v)\omega(g^v)\phi^v(h^{v,-1}x) dh. \tag{3.22}$$

Note that the analog of (3.12) does not hold.

We study $W\theta'_{v,x}(0, g, \phi)$ following [YZZ13]. Indeed, the computation is only on the Eisenstein (i.e., Whittaker) part. We remind the reader of the difference between the modulus characters mentioned in 2.6. By (3.3), (3.4) and Lemma 3.1.1, we have the following lemma, which says that under the action of $P(F_v)$, $W\theta'_{v,x}(s_0, g, \phi)$ behaves in the same was as the Weil representation.

Lemma 3.2.2 [YZZ13, Lemma 6.6]. *The following relations hold:*

$$W\theta'_{v,x}(0, m(a)g, \phi) = \chi_{\mathbb{V},v}(a) |\det a|_{E_v}^{\dim \mathbb{V}/2} W\theta'_{v,ax}(0, g, \phi), \ a \in E_v^\times;$$

$$W\theta'_{v,x}(0, n(b)g, \phi) = \psi_v(bq(x)) W\theta'_{v,x}(0, g, \phi), \ b \in F_v.$$

Corollary 3.2.3. *For $a \in E^\times$, $\theta E_t'(0, m(a)g, \phi)(v) = \theta E'_{a^2t}(0, g, \phi)(v)$.*

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3.3. Holomorphic projections

We define quasi-holomorphic projection and cuspidal holomorphic projection, and compare them for theta-Eisenstein series (Lemma 3.3.3). After imposing Gaussian condition at infinite places in 3.3.4, we make the comparison more explicit in (3.31). Finally, after imposing the incoherence condition, we explicitly compute the quasi-holomorphic projection (Proposition 3.3.13).

3.3.1. Definitions

For $v \in \infty$, let $\mathfrak{w}_v$ be a pair of integers whose sum $|\mathfrak{w}_v|$ is ≥ 2. For $t \in F_{v,>0}$, let $W^{\mathfrak{w}_v}_{v,t}$ be the standard holomorphic Whittaker function of weight $\mathfrak{w}_v$ as in 2.7. Then

$$\int_{Z(F_v)N(F_v)\backslash G(F_v)} |W^{\mathfrak{w}_v}_{v,t}(h)|^2 dh = (4\pi)^{-|\mathfrak{w}_v|+1}\Gamma(|\mathfrak{w}_v|-1).$$

For $t \in F_{v,>0}$, a $\psi_{v,t}$-Whittaker function W on $G(F_v)$, and $g \in G(F_v)$, define

$$W_s(g) = \frac{(4\pi)^{|\mathfrak{w}_v|-1}}{\Gamma(|\mathfrak{w}_v|-1)} W^{\mathfrak{w}_v}_{v,t}(g) \int_{Z(F_v)N(F_v)\backslash G(F_v)} \delta(h)^s W(h)\overline{W^{\mathfrak{w}_v}_{v,t}(h)}dh.$$

If W_s has a meromorphic continuation to $s=0$, define the quasi-holomorphic projection

$$W_{\mathrm{qhol}} := \widetilde{\lim_{s\to 0}} W_s$$

of W of weight $\mathfrak{w}_v$. Here, $\widetilde{\lim_{s\to 0}}$ denotes the constant term at $s=0$.

Let $\mathfrak{w} = (\mathfrak{w}_v)_{v\in\infty}$, where $\mathfrak{w}_v$ is a pair of integers. For a continuous function $f : N(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}$ with ψ_t-Whittaker function f_t for $t \in F_{>0}$, let $f_{t,\mathrm{qhol}}$ be the quasi-holomorphic projection of f_t of weight $\mathfrak{w}$ at all infinite places (if it is well defined).

For an automorphic form f on $G(\mathbb{A}_F)$, the cuspidal holomorphic projection f_{chol} of weight $\mathfrak{w}$ of f is the L^2-orthogonal projection of f to the subspace $\mathcal{A}_{\mathrm{hol}}(G,\mathfrak{w})$ of cusp forms – that is, for every cusp form $\phi \in \mathcal{A}_{\mathrm{hol}}(G,\mathfrak{w})$, the Petersson inner product $\langle f,\phi\rangle$ equals $\langle f_{\mathrm{chol}},\phi\rangle$.

Lemma 3.3.1 [Liu11b, Proposition 6.2][YZZ13, Proposition 6.12]. *Assume that there exists $\epsilon > 0$ such that for $v \in \infty$ and $a \in E_v^\times$ with $|a|_{E_v} \to \infty$, we have*

$$f(m(a)g) = O_g\left(|a|_{E_v}^{|\mathfrak{w}_v|/2-\epsilon}\right),$$

where $m(a)$ is as in 2.5. Then for $t \in F_{>0}$, $f_{t,\mathrm{qhol}}$ is well defined and $f_{t,\mathrm{qhol}} = f_{\mathrm{chol},t}$.

3.3.2. Holomorphic projections of $\theta E'(0,g,\phi)$

Let $\mathfrak{w} = \mathfrak{w}_{\chi_{\mathbb{V}}}$ be as in 2.9. Then $|\mathfrak{w}_v| = n+1$. Holomorphic projections below are of weight $\mathfrak{w}$. Retrieve the notations in 3.2.1. Let $\theta E'_{\mathrm{chol}}(0,g,\phi)$ be the cuspidal holomorphic projection of the derivative $\theta E'(0,g,\phi)$. For $t \in F_{>0}$, let $\theta E'_{\mathrm{chol},t}(0,g,\phi)$ be its ψ_t-Whittaker function. Let $\theta E_t(s,g,\phi)$ be the ψ_t-Whittaker function of $\theta E(s,g,\phi)$. Let $\theta E'_{t,\mathrm{qhol}}(0,g,\phi)$ be the quasi-holomorphic projection of $\theta E'_t(0,g,\phi)$ if it is well defined. The difference between $\theta E'_{\mathrm{chol},t}(0,g,\phi)$ and $\theta E'_{t,\mathrm{qhol}}(0,g,\phi)$ is given as follows.

For $\phi = \phi_1 \otimes \phi_2 \in \overline{\mathcal{S}}(\mathbb{V})$ with $\phi_1 \in \mathcal{S}(\mathbb{W})$ and $\phi_2 \in \overline{\mathcal{S}}\left(V^\sharp(\mathbb{A}_E)\right)$, define

$$\theta E_{00}(s,g,\phi) = \delta(g)^s\omega(g)\phi(0) + W_0(s,g,\phi_1)\omega(g)\phi_2(0),$$

which is the product of $E_0(s,g,\phi_1)$ and the constant term of $\theta(g,\phi_2)$. See (3.9). The definition extends to general $\phi \in \mathcal{S}(\mathbb{V})$ by linearity. By (3.3) and (3.4), we can define an Eisenstein series

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$$J(s,g,\phi)=\sum_{\gamma\in P(F)\backslash G(F)}\theta E_{00}(s,\gamma g,\phi).$$

For $t\in F_{>0}$, let $J_t(s,g,\phi)$ be its ψ_t-Whittaker function. Let $J'_{t,\mathrm{qhol}}(0,g,\phi)$ be the quasi-holomorphic projection of the derivative $J'_t(0,g,\phi)$ if it is well defined.

Remark 3.3.2. In the notations of [Tan99], $\delta(g)^s\omega(g)\phi(0)$ is in the degenerate principal series $I(n/2+s,\chi_{\mathbb{V}})$, while $W_0(s,g,\phi_1)\omega(g)\phi_2(0)$ is in $I(n/2-s,\chi_{\mathbb{V}})$ by (3.3) and (3.4).

Lemma 3.3.3. *If one of $\theta E'_{t,\mathrm{qhol}}(0,g,\phi)$ and $J'_{t,\mathrm{qhol}}(0,g,\phi)$ is well defined, then so is the other one. In this case, $\theta E'_{\mathrm{chol},t}(0,g,\phi)=\theta E'_{t,\mathrm{qhol}}(0,g,\phi)-J'_{t,\mathrm{qhol}}(0,g,\phi)$.*

Proof. By the same proof of [YZZ13, Lemma 6.13], $\theta E'(0,m(a)g,\phi)-\theta E'_{00}(0,m(a)g,\phi)$ is exponentially decay, and $J'(0,m(a)g,\phi)-\theta E'_{00}(0,m(a)g,\phi)$ is exponentially decay up to the derivative at $s=0$ of the intertwining part of the constant term. The intertwining part of the constant term lies in $I(-n/2\pm s,\chi_{\mathbb{V}})$ so that its derivative at $s=0$ has growth rate $O_g\left(|a|_{E_v}^{-n/2+1/2+\epsilon}\right)$ (in the notations in Lemma 3.3.1). In particular, both differences satisfy the growth condition in Lemma 3.3.1. Thus, $\theta E'(0,g,\phi)-J'(0,g,\phi)$ satisfies the growth condition in Lemma 3.3.1. Since the cuspidal holomorphic projection of the Eisenstein series $J(s,g,\phi)$ is 0, the lemma follows. □

3.3.3. A new Eisenstein series

We introduce a new Eisenstein series in order to compute $J'(0,g,\phi)$. For $\phi_v=\phi_{v,1}\otimes\phi_{2,v}\in\mathcal{S}(\mathbb{V}(E_v))$ with $\phi_{v,1}\in\mathcal{S}(\mathbb{W}_v)$ and $\phi_{2,v}\in\mathcal{S}\left(V^\sharp(E_v)\right)$, define a function on $G(F_v)\times V^\sharp(E_v)$:

$$c(g,x,\phi_v)=W^{\circ}_{v,0}{}'(0,g,\phi_{1,v})\omega(g)\phi_{2,v}(x)+\log\delta_v(g)\omega(g)\phi_v(x). \tag{3.23}$$

For the moment, we only need

$$c(g,\phi_v):=c(g,0,\phi_v).$$

Extend this definition to the whole $\mathcal{S}(\mathbb{V}(E_v))$ linearly.

By (3.3), (3.4) and Lemma 3.1.3 (1), a direct computation shows the following lemma.

Lemma 3.3.4. *The function $c(g,\phi_v)$ on $G(F_v)$ is in the same principal series as $\omega(g)\phi_v(0)$; that is,*
(1) *$c(m(a)g,\phi_v)=\chi_{\mathbb{V},v}(a)|\det a|_{E_v}^{\dim\mathbb{V}/2}c(g,\phi_v)$ for $a\in E_v^\times$;*
(2) *$c(n(b)g,\phi_v)=c(g,\phi_v)$ for $b\in F_v$.*

Thus, we can define the following Eisenstein series in the case that ϕ is a pure tensor:

$$C(s,g,\phi)(v)=\sum_{\gamma\in P(F)\backslash G(F)}c(\gamma g_v,\phi_v)\omega(\gamma g^v)\phi^v(0).$$

Lemma 3.3.5. *For all but finitely many finite places, $c(g,\phi_v)=0$ for all g.*

Proof. If $v\in\infty$, by Lemma 3.1.3 (3) which says $W_0^{\circ}(s,g,\phi)=\delta_v(g)^{-s}\omega(g)\phi(0)$, we clearly have $c(g,\phi_v)=0$. The same is true if $E_v/\mathbb{Q}_v$ is unramified and $\phi=1_{\mathcal{O}_{E_v}}$ by Lemma 3.1.3 (2). These cases cover all but finitely many *finite* places. □

Let

$$C(s,g,\phi)=\sum_v C(s,g,\phi)(v),$$

where the sum is over these *finite* places of F. Let $C_t(s,g,\phi)$ be the ψ_t-Whittaker function of $C(0,g,\phi)$. The definitions can be obviously extended to the whole $\mathcal{S}(\mathbb{V})$ by linearity.

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By (3.16), a direct computation shows that

$$J'(0,g,\phi) = 2E'(0,g,\phi) - \mathfrak{c}E(0,g,\phi) - C(0,g,\phi). \tag{3.24}$$

3.3.4. Gaussian functions and holomorphy

Below in this section, assume that $\mathbb{V}$ is totally positive definite and $\phi = \phi_\infty \otimes \phi^\infty \in \overline{\mathcal{S}}(\mathbb{V})$. (So ϕ_∞ is Gaussian. See 2.3.) Let $v \in \infty$. Then

$$\omega([k_1,k_2])\phi_v = k_1^{w_1}k_2^{w_2}\phi_v \tag{3.25}$$

for $[k_1,k_2] \in K_v^{\max}$ as in 2.5 if $\mathfrak{w}_v = (w_1,w_2)$. (Indeed, first check (3.26) for $g = w_v$ and $g \in K_v^{\max}$ being diagonal, then for $g \in K_v^{\max}$ being anti-diagonal, and finally for general $g \in K_v^{\max}$.) Then by the Iwasawa decomposition, it is easy to check that for $g \in G(F_v), x \in \mathbb{V}(E_v)$,

$$\omega(g)\phi_v(x) = W_{v,q(x)}^{\mathfrak{w}_v}(g). \tag{3.26}$$

By (3.2) combined with (3.25), (3.3) and (3.4), $W_{v,t}(0,\cdot,\phi_v)$ is a multiple of $W_{v,t}^{\mathfrak{w}_v}$. Then by [YZZ13, Proposition 2.11 (2) (4)], we have

$$W_{v,t}(0,g,\phi_v) = \gamma_{\mathbb{V}(E_v)}\frac{(2\pi)^{n+1}}{\Gamma(n+1)}t^n W_{v,t}^{\mathfrak{w}_v}(g),\ t>0, \tag{3.27}$$

$$W_{v,t}(0,g,\phi_v) = 0,\ t \le 0. \tag{3.28}$$

Lemma 3.3.6. *Both $E(0,g,\phi)$ and $C(0,g,\phi)$ are holomorphic of weight $\mathfrak{w}$.*

Proof. For $E(0,g,\phi)$, use (3.8), (3.9), (3.26), (3.27) and (3.28). For $C(0,g,\phi)$, by Lemma 3.1.3 (3), $c(g,\phi_v) = 0$ for $v \in \infty$. Thus, $C(s,g,\phi)(v) = 0$ for $v \in \infty$. The rest of the proof is the same as the proof for $E(0,g,\phi)$. □

For $t \in F^\times$, let $E_t'(0,g,\phi)(v)$ be as in (3.11) so that we have the decomposition (3.12). Let

$$E_{t,\mathrm{f}}'(0,g,\phi) = \sum_{v\notin\infty} E_t'(0,g,\phi)(v). \tag{3.29}$$

By (3.27) and (3.28), if $t \in F_{>0}$, then $E_{t,\mathrm{f}}'(0,g,\phi)$ is a multiple of $W_{\infty,t}^{\mathfrak{w}}(g_\infty)$; otherwise, $E_{t,\mathrm{f}}'(0,g,\phi) = 0$. We call $E_{t,\mathrm{f}}'(0,g,\phi)$ the holomorphic part of the Whittaker function $E_t'(0,g,\phi)$.

3.3.5. Properties of Eisenstein series

We list some properties of the above Eisenstein series for later use. The reader may skip these properties for the moment.

By [YZ18, Lemma 7.6 (2)] (or its proof) and taking care of the difference between the modulus characters mentioned in 2.6, we have the following lemma.

Lemma 3.3.7. *Let E_v/F_v be split, $\mathbb{W}_v = E_v$ and $q = \mathrm{Nm}$. Assume $\phi_v = \phi_{v,1}\otimes\phi_{v,2}$, where $\phi_{v,1} = 1_{\mathcal{O}_{E_v}}$ and $\phi_{v,2} \in \mathcal{S}\left(V^\sharp(E_v)\right)$. Then $c(1,\phi_v) = 2\log|\mathrm{Diff}_v|_v\phi_v(0)$, where Diff_v is the different of $F_v/\mathbb{Q}_v$.*

We omit the routine proof of the following analog of (3.2).

Lemma 3.3.8. *For a place v of F and $k \in K_v^{\max}$, we have $E(s,g,\omega(k)\phi) = E(s,gk,\phi)$. The same relation holds for $C(s,g,\phi)$, $\theta E(s,g,\phi)$, $\theta E_{\mathrm{chol}}'(0,g,\phi)$, and their t-th Whittaker/Fourier coefficients, and $E_t'(0,g,\phi)(v)$ (thus, $E_{t,\mathrm{f}}'(0,g,\phi)$) for $t \in F^\times$.*

Lemma 3.3.9. (1) *We have* $E_0(0,g,\phi) = \omega(g)\phi(0)$.

(2) *If, moreover,* $\mathbb{V}$ *is incoherent, then for a finite place* v, $C_0(0,g,\phi)(v) = c(g_v,\phi_v)\omega(g^v)\phi^v(0)$.

(3) *In (2), assume that* ϕ_v *is supported outside* $V^\sharp(E_v)$ *for* v *in a set* S *of two places of* F *and* $g \in P(\mathbb{A}_{F,S})G(\mathbb{A}_F^S)$. *Then* $E_0(0,g,\phi) = 0$ *and* $C_0(0,g,\phi) = 0$.

Proof. (1) If $F = \mathbb{Q}$, $n = 1$ and $\mathbb{V}^\infty$ is not split at some finite place, it is proved in [YZZ13, Proposition 2.9 (3)]. If $F \neq \mathbb{Q}$ so that we have at least 2 infinite places, its proof is similar to the one in [YZZ13, Proposition 2.9 (3)] by using (3.9) and (3.28) (for $t = 0$).

(2) The proof is similar, with the fact that $\mathbb{V}$ is not split at (at least) 2 places outside v by the incoherence. See also [Yua22, p 65-66].

(3) follows from (1)(2) directly. □

3.3.6. Compute $J'_{t,\mathrm{qhol}}(0,g,\phi)$

Let $t \in F_{>0}$. For $v \in \infty$, let $E'_{t,\mathrm{qhol}}(0,g,\phi)(v)$ be the quasi-holomorphic projection of $E'_t(0,g,\phi)(v)$. By (3.24), Lemma 3.3.6 and the discussion below it, to compute $J'_{t,\mathrm{qhol}}(0,g,\phi)$, we only need to compute $E'_{t,\mathrm{qhol}}(0,g,\phi)(v)$.

Consider the quasi-holomorphic projection $W'_{v,t,\mathrm{qhol}}(0,g,\phi_v)$ of $W'_{v,t}(0,g,\phi_v)$. By definition, it is a multiple of $W^{\mathfrak{w}_v}_{v,t}(g)$. Then by (3.27),

$$b_{v,t} := \frac{W'_{v,t,\mathrm{qhol}}(0,g,\phi_v)}{W_{v,t}(0,g,\phi_v)}$$

is a well defined constant $b_{v,t}$. We define

$$\mathfrak{b} = b_{v,1}. \tag{3.30}$$

Remark 3.3.10. The constant $\mathfrak{b}$ can be explicitly computed using [YZZ13, Proposition 2.11] and [Yua22, Lemma 3.3] in principle. For example, if $n = 1$, then $\mathfrak{b} = -(1 + \log 4)$. (This is twice of the corresponding number in [Yua22, Lemma 3.3 (2)] due to the difference between the modulus characters mentioned in 2.6.) It is more complicated in general. The full computation could be tedious, and the result in a previous version of our paper actually contains a mistake. (Fortunately, we will not need the explict number of $\mathfrak{b}$.) Ziqi Guo (student of Yuan, author of [Yua22]) pointed this out to us and informed us that he will give full details on this in his upcoming work.

Lemma 3.3.11. *We have* $b_{v,t} = b_{v,1} + \log|t|_v$ *and* $b_{v,1}$ *is independent of* v.

Proof. The lemma follows direct computations with the following ingredients. For the equation, use (3.4) and (3.27). Note that the dependence on v is on the weight $\mathfrak{w}_v$. We use (3.2) and (3.26) for $g \in K_v^{\max}$. □

Then $E'_{t,\mathrm{qhol}}(0,g,\phi)(v) = (\mathfrak{b} + \log|t|_v)E_t(0,g,\phi)$. Since both $E(0,g,\phi)$ and $C(0,g,\phi)$ in (3.24) are holomorphic of weight $\mathfrak{w}$, we have

$$J'_{t,\mathrm{qhol}}(0,g,\phi) = (2\mathfrak{b}[F:\mathbb{Q}] - \mathfrak{c})E_t(0,g,\phi) - C_t(0,g,\phi) + 2\Big(E'_{t,\mathrm{f}}(0,g,\phi) + E_t(0,g,\phi)\log \mathrm{Nm}_{F/\mathbb{Q}} t\Big).$$

Combined with Lemma 3.3.3, we have

$$\begin{aligned}&\theta E'_{t,\mathrm{qhol}}(0,g,\phi) - 2\Big(E'_{t,\mathrm{f}}(0,g,\phi) + E_t(0,g,\phi)\log \mathrm{Nm}_{F/\mathbb{Q}} t\Big)\\ &\quad = \theta E'_{\mathrm{chol},t}(0,g,\phi) + (2\mathfrak{b}[F:\mathbb{Q}] - \mathfrak{c})E_t(0,g,\phi) - C_t(0,g,\phi).\end{aligned} \tag{3.31}$$

3.3.7. Quasi-holomorphic projection of $\theta E_t'(0,g,\phi)$

Assume that $\mathbb{W}/\mathbb{E}$ is incoherent. For $v\in\infty$ and W the v-nearby hermitian space of $\mathbb{W}$, let $V=W\oplus V^\sharp$. For $x\in V(E_v)-V^\sharp(E_v)$, define

$$\widetilde{W\theta}_s(x)=\frac{\Gamma(s+n)}{\Gamma(n)(4\pi)^s}P_s(-q(x_1)),$$

where x_1 is the projection of x to $W(E_v)$ (so that $x_1\neq 0$), and

$$P_s(t):=\int_{u=1}^{\infty}\frac{1}{u(1+tu)^{s+n}}du,\ t>0. \tag{3.32}$$

(For $s\in\mathbb{C}$ with $\mathrm{Re}s>-n$, $P_s(t)$ converges absolutely.) Define

$$\theta E_{t,s}'(0,g,\phi)(v)=\frac{2W_{v,t}^{\mathfrak{w}}(g_v)}{\mathrm{Vol}([U(W)])}\int_{[U(W)]}\sum_{x\in V^t-V^\sharp}\widetilde{W\theta}_s(h_v^{-1}x)\omega^v(g^v)\phi^v\left(h^{v,-1}x\right)dh. \tag{3.33}$$

Lemma 3.3.12. *For $s\in\mathbb{C}$ with $\mathrm{Re}s>0$, (3.33) converges absolutely and $\theta E_{t,s}'(0,g,\phi)(v)$ is holomorphic on s. Moreover, $\theta E_{t,s}'(0,g,\phi)(v)$ admits a meromorphic continuation to $\{s\in\mathbb{C},\mathrm{Re}s>-1\}$ with at most a simple pole at $s=0$.*

Then the constant term $\widetilde{\lim_{s\to0}}\theta E_{t,s}'(0,g,\phi)(v)$ of $\theta E_{t,s}'(0,g,\phi)(v)$ at $s=0$ is well defined (and used in the following proposition). We will prove Lemma 3.3.12 after Lemma 6.1.10, by comparing $\theta E_{t,s}'(0,g,\phi)(v)$ to a Green function with s-variable, which has a meromorphic continuation.

Proposition 3.3.13. *Let $t\in F_{>0}$ and let $\phi\in\overline{\mathcal{S}}(\mathbb{V})$ be a pure tensor.*

(1) *We have*

$$\begin{aligned}\theta E_{t,\mathrm{qhol}}'(0,g,\phi)=&-\sum_{v\notin\infty,\ \text{nonsplit}}\theta E_t'(0,g,\phi)(v)-\sum_{v\in\infty}\widetilde{\lim_{s\to0}}\theta E_{t,s}'(0,g,\phi)(v)\\&-\left(4\frac{L_{\mathrm{f}}'(0,\eta)}{L_{\mathrm{f}}(0,\eta)}+2\log|\mathrm{Disc}_E/\mathrm{Disc}_F|\right)\sum_{x\in(V^\sharp)^t}\omega(g)\phi(x)\\&-\sum_{v\notin\infty}\sum_{x\in(V^\sharp)^t}c(g,x,\phi_v)\omega(g^v)\phi^v(x)\\&+\sum_{x\in(V^\sharp)^t}(2\log\delta(g^\infty)+\log|t^\infty|)\omega(g)\phi(x)\end{aligned}. \tag{3.34}$$

Here, $L_{\mathrm{f}}(s,\eta)$ is the finite part of $L(s,\eta)$.

(2) *Assume that ϕ_v is supported outside $V^\sharp(E_v)$ for v in a set S of two places of F and $g\in P(\mathbb{A}_{F,S})G(\mathbb{A}_F^S)$. Then we have*

$$\theta E_{t,\mathrm{qhol}}'(0,g,\phi)=-\sum_{v\notin\infty,\ \text{nonsplit in }E}\theta E_t'(0,g,\phi)(v)-\sum_{v\in\infty}\widetilde{\lim_{s\to0}}\theta E_{t,s}'(0,g,\phi)(v). \tag{3.35}$$

Proof. The proof of (1) is almost identical with [YZ18, Theorem 7.2] and is omitted. (Note that [YZ18, Assumption 7.1] in [YZ18, Theorem 7.2] is only used to identify the quasi-holomorphic projection with the cuspidal holomorphic projection and does not play a role in computing the quasi-holomorphic projection). (2) follows from (1) immediately. □

Lemma 3.3.14. *Let $t\in F_{>0}$ and let ϕ be a pure tensor such that ϕ^∞ is $\overline{\mathbb{Q}}$-valued. Let u be a finite place of F of residue characteristic p. Let $O\subset\mathbb{V}_u$ be an open compact neighborhood of 0 and*

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$\phi^O = \phi^u \otimes (\phi_u 1_{\mathbb{V}_u - O})$. *Given* $g \in G(\mathbb{A}_F^u)P(F_u)$, *for* O *small enough, we have*

$$\frac{\theta E'_{t,\mathrm{qhol}}(0,g,\phi)}{W^{\mathfrak{w}}_{\infty,t}(g_\infty)} = \frac{\theta E'_{t,\mathrm{qhol}}(0,g,\phi^O)}{W^{\mathfrak{w}}_{\infty,t}(g_\infty)} (\mathrm{mod}\, \overline{\mathbb{Q}} \log p).$$

Proof. The proof is similar to the one of Lemma 3.2.1, except that we further need (3.22), (3.26), (3.33) and (3.34). □

By (3.34), using (3.22) and (3.33), for $h \in U(\mathbb{W}_v)$ where v is split in E,

$$\theta E'_{t,\mathrm{qhol}}(0,g,\omega(h)\phi) = \theta E'_{t,\mathrm{qhol}}(0,g,\phi). \tag{3.36}$$

4. Special divisors

This section is about special divisors on unitary Shimura varieties. It consists of 4.1-4.4. First, we define their generating series which are modular. Second, we introduce their Green functions and show the modularity of the differences between the generating series of different kinds of Green functions. Third, we raise two modularity problems for their admissible extensions on integral models. Finally, we propose a precise conjecture and state our modularity theorems.

4.1. Generating series

Let $\mathbb{V}$ be a totally positive-definite incoherent hermitian space over $\mathbb{A}_E$ (with respect to the extension E/F) of dimension $n+1$ where $n > 0$. Fix an infinite place $v_0 \in \infty$ of F. Let V_0 be the unique hermitian space over E such that $V_0(\mathbb{A}_E^v) \simeq \mathbb{V}^v$ and $V_0(E_v)$ is of signature $(n,1)$. For an open compact subgroup K of $U(\mathbb{V}^\infty)$, let $\mathrm{Sh}(\mathbb{V})_K$ be the n-dimensional smooth unitary Shimura variety associated to $U(V_0)$ of level K over E (see [LTX+22, 3.2]), which we allow to be a Deligne-Mumford stack. (It is expected that $\mathrm{Sh}(\mathbb{V})_K$ does not depend on the choice of v_0. See [LL21, Remark 1.2].) See (4.4) for its usual complex uniformization.

∘ From now on, we always assume that $\mathrm{Sh}(\mathbb{V})_K$ is proper.

Equivalently, $F \neq \mathbb{Q}$, or $F = \mathbb{Q}, n = 1$ and $\mathbb{V}^\infty$ is not split at some finite place.

4.1.1. Simple special divisors

Let $\mathbb{V}^\infty_{>0} \subset \mathbb{V}^\infty$ be the subset of x's such that $q(x) \in F_{>0}$. For $x \in \mathbb{V}^\infty_{>0}$, let $x^\perp$ be the orthogonal complement of $\mathbb{A}_E^\infty x$ in $\mathbb{V}^\infty$. Regard $U(x^\perp)$ as a subgroup of $U(\mathbb{V}^\infty)$. Then we have a finite morphism

$$\mathrm{Sh}(x^\perp)_{U(x^\perp)\bigcap K} \to \mathrm{Sh}(\mathbb{V})_K, \tag{4.1}$$

explicated in [Kud97a, (2.4)] and [LL21, Definition 4.1]. The proper pushforward defines a divisor $Z(x)_K$ on $\mathrm{Sh}(\mathbb{V})_K$ that is called a simple special divisor. The following observation is trivial.

Lemma 4.1.1. *We have* $Z(x)_K = Z(kxa)_K$ *for every* $a \in E^\times, k \in K$.

Let L_K be the Hodge line bundle on $\mathrm{Sh}(\mathbb{V})_K$. See 4.2.1 for the description in terms of the complex uniformization of $\mathrm{Sh}(\mathbb{V})_K$. Let $c_1(L_K^\vee)$ be the first Chern class of the dual of L_K, and $[Z(x)_K] \in \mathrm{Ch}^1(\mathrm{Sh}(\mathbb{V})_K)$ the class of $Z(x)_K$. For $\phi \in \overline{\mathcal{S}}(\mathbb{V})^K$ the subspace of K-invariant functions, define a formal generating series of divisor classes:

$$z(\phi)_K = \phi(0)c_1(L_K^\vee) + \sum_{x \in K\backslash \mathbb{V}^\infty_{>0}} \phi(x_\infty x)[Z(x)_K],$$

where $x_\infty \in \mathbb{V}_\infty$ such that $q(x_\infty) = q(x) \in F_{>0}$.

Let $\mathfrak{w} = \mathfrak{w}_{\chi_{\mathbb{V}}}$ which is defined in 2.9.

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Theorem 4.1.2 [Liu11a, Theorem 3.5]. *For every $\phi \in \overline{\mathcal{S}}(\mathbb{V})^K$, we have*

$$z(\omega(\cdot)\phi)_K \in \mathcal{A}_{\mathrm{hol}}(G, \mathfrak{w}) \otimes \mathrm{Ch}^1(\mathrm{Sh}(\mathbb{V})_K)_{\mathbb{C}}.$$

Note that $\dim \mathrm{Ch}^1(\mathrm{Sh}(\mathbb{V})_K)_{\mathbb{C}} < \infty$.

4.1.2. Weighted special divisors

For $\phi \in \overline{\mathcal{S}}(\mathbb{V})^K$ and $t \in F_{>0}$, define the weighted special divisor

$$Z_t(\phi)_K = \sum_{x \in K\backslash\mathbb{V}^\infty,\ q(x)=t} \phi(x_\infty x) Z(x)_K,$$

which is a finite sum. Lemma 4.1.1 implies the following lemma.

Lemma 4.1.3. *For every $a \in E^\times$, $Z_t(\phi)_K = Z_{a^2 t}(\omega(a)\phi)_K$.*

Let $\phi = \phi_\infty \otimes \phi^\infty$ be as in 2.3. We define another weighted special divisor

$$Z_t(\phi^\infty)_K = \sum_{x \in K\backslash\mathbb{V}^\infty,\ q(x)=t} \phi^\infty(x) Z(x)_K.$$

By (3.26), for $g \in G(\mathbb{A}_F)$, we have

$$Z_t(\omega(g)\phi)_K = Z_t(\omega(g^\infty)\phi^\infty)_K W^{\mathfrak{w}}_{\infty,t}(g_\infty). \tag{4.2}$$

Then $[Z_t(\omega(\cdot)\phi^\infty)_K]$ is the t-th Fourier coefficient of $z(\omega(\cdot)\phi)_K$. See 2.7.

Lemma 4.1.4. *Let $\phi \in \overline{\mathcal{S}}(\mathbb{V})$ be a pure tensor. Let u be a finite place of F, $O \subset \mathbb{V}_u$ an open compact neighborhood of* 0 *and $\phi^O = \phi^u \otimes (\phi_u 1_{\mathbb{V}_u - O})$. Given $g \in G(\mathbb{A}_F^u)P(F_u)$, for O small enough, if K fixes ϕ^O, then $Z_t(\omega(g)\phi)_K = Z_t(\omega(g)\phi^O)_K$.*

Proof. The lemma is an analog to Lemma 3.2.1, and the proof is also similar. We record the proof for the reader's convenience. Write $g_u = m(a)n(b)$, where $a \in E_u^\times$. See 2.5. Then $\{ax : x \in \mathbb{V}_u^t\} \subset \mathbb{V}_u^{a^2 t}$. The latter is closed in $\mathbb{V}_u$ and does not contain 0. Thus, $O \cap \{ax : x \in \mathbb{V}_u^t\} = \emptyset$ if O is small enough. Then the lemma follows from the definition of $Z_t(\omega(g)\phi)_K$ and the Weil representation formula in 2.8. □

4.1.3. Change level

For $K \subset K'$, let $\pi_{K,K'} : \mathrm{Sh}(\mathbb{V})_K \to \mathrm{Sh}(\mathbb{V})_{K'}$ be the natural projection.

Lemma 4.1.5 [Kud97a, PROPOSITION 5.10][Liu11a, Corollary 3.4]. *We have $\pi^*_{K,K'} Z_t(\phi)_{K'} = Z_t(\phi)_K$ and $\pi^*_{K,K'} L_{K'} = L_K$. In particular, $\pi^*_{K,K'} z(\phi)_{K'} = z(\phi)_K$.*

Remark 4.1.6. We have $\pi_{K,K',*} Z(x)_K = d(x) Z(x)_{K'}$ where $d(x)$ is the degree of $Z(x)_K$ over $Z(x)_{K'}$. It is easy to check that $d(x)$ is not constant near 0. In particular, it does not extend to a smooth function on $\mathbb{V}^\infty$. Thus, for a general ϕ, there seems no $\phi' \in \overline{\mathcal{S}}(\mathbb{V})^{K'}$ such that $\pi_{K,K',*} Z_t(\phi)_K$ is of the form $Z_t(\phi')_{K'}$ for every t.

Below, let $L = L_K$, $Z(x) = Z(x)_K$, $Z_t(\phi) = Z_t(\phi)_K$ and $z(\phi) = z(\phi)_K$ for simplicity if K is clear from the context.

4.2. *Green functions*

We use complex uniformization to defined automorphic Green functions for special divisors. They are admissible. We compare them with the normalized admissible Green functions and show the modularity of the differences between their generating series. Then we recall Kudla's Green functions and prove the modularity of the differences between the generating series of normalized admissible Green functions and Kudla's Green functions.

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4.2.1. Complex uniformization

For nonnegative integers p, q, let $\mathbb{C}^{p,q}$ be the $p+q$ dimensional hermitian space associated to the hermitian matrix $\mathrm{diag}(-1_p, 1_q)$. Let $U(\mathbb{C}^{1,n})$ be the unitary group of $\mathbb{C}^{1,n}$ (so of signature $(n,1)$ in the usual convention). Let $\mathbb{B}_n$ be the complex open unit ball of dimension n. Embed $\mathbb{B}_n$ in $\mathbb{P}(\mathbb{C}^{1,n})$ as the set of negative lines as follows: $[z_1, \ldots, z_n] \in \mathbb{B}_n$, where $\sum_{i=1}^n |z_i|^2 < 1$, is the line containing the vector $(1, z_1 \ldots, z_n)$. Then $U(\mathbb{C}^{1,n})$ acts on $\mathbb{B}_n$ naturally and transitively. Let Ω be the tautological bundle of negative lines on $\mathbb{B}_n$, and $\overline{\Omega}$ the hermitian line bundle with the metric induced from the negative of the hermitian form. The Chern form of $\overline{\Omega}$ is a $U(\mathbb{C}^{1,n})$-invariant Kähler form

$$c_1(\overline{\Omega}) = \frac{1}{2\pi i}\partial\bar{\partial}\log(1 - \sum_{i=1}^n |z_i|^2). \tag{4.3}$$

For an arithmetic subgroup $\Gamma \subset U(\mathbb{C}^{1,n})$, on (the orbifold) $\Gamma\backslash\mathbb{B}_n$, we have the descent $\overline{\Omega}_\Gamma$ of $\overline{\Omega}$ whose (orbifold) Chern curvature form is the descent of $c_1(\overline{\Omega})$. Define the degree

$$\deg(\overline{\Omega}_\Gamma) = \int_{\Gamma\backslash\mathbb{B}_n} c_1(\overline{\Omega}_\Gamma)^n.$$

If $\Gamma\backslash\mathbb{B}_n$ is compact, this degree is the usual degree via intersection theory.

For $x \in \mathbb{C}^{1,n}$, let $\mathbb{B}_x \subset \mathbb{B}_n$ be the subset of negative lines perpendicular to x. Later, we will use the following function on $\mathbb{B}_n$ measuring 'the distance to $\mathbb{B}_x$':

$$R_x(z) = -\frac{|\langle x, \widetilde{z}\rangle|^2}{\langle \widetilde{z}, \widetilde{z}\rangle},$$

where $\widetilde{z}$ is a nonzero vector contained in the line z. In particular, $R_x > 0$ outside $\mathbb{B}_x$. If $q(x) < 0$, then $x^\perp$ is of signature $(n,0)$ so that $\mathbb{B}_x = \emptyset$. Below, until 4.2.3, assume $q(x) > 0$. Then $x^\perp$ is of signature $(n-1,1)$. So $\mathbb{B}_x$ is a complex unit ball of dimension $n-1$. Assume that $\Gamma \cap U(x^\perp)$ is an arithmetic subgroup of $U(x^\perp)$. Let $C(x,\Gamma)$ be the pushforward of the fundamental cycle by

$$\big(\Gamma \cap U(x^\perp)\big)\backslash\mathbb{B}_x \to \Gamma\backslash\mathbb{B}_n.$$

Define

$$\deg_{\overline{\Omega}_\Gamma}(C(x,\Gamma)) = \int_{C(x,\Gamma)} c_1(\overline{\Omega}_\Gamma)^{n-1}.$$

Now we can uniformize Shimura varieties and special divisors. Let v be an infinite place of F. Let V be the unique hermitian space over E such that $V(\mathbb{A}_E^v) \simeq \mathbb{V}^v$ and $V(E_v) \simeq \mathbb{C}^{1,n}$. By [LL21, Lemma 5.5], we have the complex uniformization:

$$\mathrm{Sh}(\mathbb{V})_{K,E_v} \simeq U(V)\backslash(\mathbb{B}_n \times U(\mathbb{V}^\infty)/K). \tag{4.4}$$

Then the Hodge bundle L_{E_v} is the descent of $\Omega \times 1_{U(\mathbb{V}^\infty)/K}$. Let $\overline{L}_{E_v}$ be the descent of $\overline{\Omega} \times 1_{U(\mathbb{V}^\infty)/K}$. Let $h_1, \ldots, h_m$ be a set of representatives of $U(V)\backslash U(\mathbb{V}^\infty)/K$. Let $\Gamma_{h_j} = U(V) \cap h_j K h_j^{-1}$. Then (4.4) decomposes as the disjoint union of $\Gamma_{h_j}\backslash\mathbb{B}_n$'s. Then for $t \in F_{>0}$, we have (see [Kud97a, PROPOSITION 5.4])

$$Z_t(\phi^\infty)_{E_v} = \sum_{j=1}^m \sum_{x \in \Gamma_{h_j}\backslash V^t} \phi^\infty(h_j^{-1}x) C(x, \Gamma_{h_j}). \tag{4.5}$$

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4.2.2. Admissible Green functions

Admissible Green functions are Green functions with harmonic curvatures. See Appendix A.3. Admissible Green functions for special divisors are constructed by Bruinier [Bru02, Bru12], Oda and Tsuzuki [OT03]. For $F = \mathbb{Q}$ and $n = 1$, it appeared in the work of Gross and Zagier [GZ86] for $n = 1$. We learned the following explicit computation from S. Zhang.

First, we start by working on $\mathbb{B}_n$. Let $x_0 = (0, \dots, 0, 1) \in \mathbb{C}^{1,n}$. Then $\mathbb{B}_{x_0} \subset \mathbb{B}_n$ consists of points $(z_1, \dots, z_{n-1}, 0)$'s. We want a $U(x_0^\perp)$-invariant smooth function G on $\mathbb{B}_n - \mathbb{B}_{x_0}$ such that $G(z_1, \dots, z_n) + \log|z_n|^2$ extends smoothly to $\mathbb{B}_n$, $\lim_{|z|\to 1} G(z) = 0$, and G is a solution of the following Laplacian equation:

$$\left(\frac{i}{2\pi}\partial\bar{\partial}G\right)c_1(\Omega)^{n-1} = \frac{s(s+n)}{2}G \cdot c_1(\Omega)^n. \tag{4.6}$$

The quotient of $\mathbb{B}_n - \mathbb{B}_{x_0}$ by $U(x_0^\perp)$ is isomorphic to $(1, \infty)$ via

$$z = [z_1, \dots, z_n] \mapsto t(z) := 1 + R_{x_0}(z) = \frac{1 - \sum_{i=1}^{n-1}|z_i|^2}{1 - \sum_{i=1}^{n}|z_i|^2}.$$

Thus, we look for $G = Q(t(z))$, where Q is a smooth function on $(1, \infty)$ such that $Q(t) + \log(t-1)$ extends to a smooth function on $\mathbb{R}$. By (4.3) and the $U(x_0^\perp)$-invariance, (4.6) is reduced to the following hypergeometric differential equation

$$\left(t - t^2\right)\frac{d^2Q}{dt^2} + (n - (n+1)t)\frac{dQ}{dt} + s(s+n)Q = 0,\ t \in (1, \infty).$$

For $\mathrm{Re}s > -1$, there is a unique solution Q_s such that $Q_s(t) + \log(t-1)$ extends to a smooth function on $\mathbb{R}$ and $\lim_{t\to\infty} Q_s(t) = 0$:

$$Q_s(t) = \frac{\Gamma(s+n)\Gamma(s+1)}{\Gamma(2s+n+1)t^{s+n}}F\left(s+n, s+1, 2s+n+1; \frac{1}{t},\right),\ t > 1 \tag{4.7}$$

where F is the hypergeometric function. (See also [OT03, 2.5.3]. When $n = 1$, our Q_s is the Legendre function of the second kind in [GZ86, 238] up to shifting s by 1).

For a general x with $q(x) > 0$, we have the following Green function for $\mathbb{B}_x$:

$$G_{x,s}(z) = Q_s(1 + R_x(z)),\ z \in \mathbb{B}_n - \mathbb{B}_x.$$

We will need the following explicit formula later: if $x = (x_1, x_2) \in \mathbb{C}^{1,n} = \mathbb{C}^{1,0} \oplus \mathbb{C}^{0,n}$, then

$$G_{x,s}([0, \dots, 0]) = Q_s(1 - q(x_1)). \tag{4.8}$$

Second, we define Green functions for $C(x, \Gamma)$ on arithmetic quotient of $\mathbb{B}_n$. Let Γ be an arithmetic subgroup of $U(\mathbb{C}^{1,n})$. Let

$$g_s = \sum_{\gamma\in\Gamma}\gamma^* G_{x,s}.$$

Lemma 4.2.1 [OT03, Proposition 3.1.1, Remark 3.1.1, Remark 3.2.1]. *For $s \in \mathbb{C}$ with* $\mathrm{Re}s > 0$*, the sum g_s converges absolutely and defines a smooth function on $\Gamma\backslash\mathbb{B}_n - C(x, \Gamma)$. Moreover, g_s is holomorphic on s.*

It is easy to see that g_s with $\mathrm{Re}s > 0$ is a Green function for $C(x, \Gamma)$.

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Theorem 4.2.2 [OT03, Theorem 7.8.1]. (1) *There is a meromorphic continuation of g_s to $s \in \mathbb{C}$ with a simple pole at $s = 0$ and residue $-\frac{\deg_{\overline{\Omega}_\Gamma}(C(x,\Gamma))}{\deg(\overline{\Omega}_\Gamma)}$.*

(2) *The function $\widetilde{\lim_{s\to 0}} g_s$ is an admissible Green function for $C(x,\Gamma)$.*

Recall that $\widetilde{\lim_{s\to 0}}$ denotes taking the constant term at $s = 0$.

Remark 4.2.3. (1) We read the residue κ in [OT03, Theorem 7.8.1 (3)] as follows. Beside the obvious differences between the choices of Green function here and in [OT03] (more precisely, s-variables and signs), the Kähler form and 'volume form' here and in [OT03] are different. First, the Kähler form on the bottom of [OT03, 514] is $\pi c_1(\overline{\Omega})$ in our notation. Second, [OT03, Theorem 7.8.1 (3)] uses volumes to express the residue, while we use degrees of line bundles to express the residue. The volume form for $\mathbb{B}_n$ on the top of [OT03, 515] is $\frac{\pi^n}{n!}c_1(\overline{\Omega})^n$ in our notation. Third, there is a π missing in (the numerator of) the residue κ in [OT03, Theorem 7.8.1 (3)]. It should be easy to spot from [OT03, Proposition 3.1.2, Lemma 7.2.2].

(2) When $n = 1$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ via $\mathrm{SU}(\mathbb{C}^{1,1}) \simeq \mathrm{SL}_2(\mathbb{R})$, we have $\deg(\overline{\Omega}_\Gamma) = \frac{1}{12}$. See [K01, 4.10]. Thus, for $\Gamma_0(N)$ a standard congruence subgroup of Γ, Theorem 4.2.2 (1) coincides with the residue $\frac{-12}{[\Gamma:\Gamma_0(N)]}$ in [GZ86, p 239, (2.13)]. Theorem 4.2.2 (1) is not used in any other place of the paper.

By [OT03, Proposition 3.1.2] and taking care of the differences in the remark, we have

$$\int_{\Gamma\backslash\mathbb{B}_n} g_s c_1(\overline{\Omega})^n = \frac{n\deg_{\overline{\Omega}_\Gamma}(C(x,\Gamma))}{s(s+n)}.$$

Thus,

$$\int_{\Gamma\backslash\mathbb{B}_n} (\widetilde{\lim_{s\to 0}} g_s) c_1(\overline{\Omega})^n = -\frac{\deg_{\overline{\Omega}_\Gamma}(C(x,\Gamma))}{n}. \tag{4.9}$$

Finally, we define Green functions for $Z_t(\phi)$, $t \in F_{>0}$. Let $v \in \infty$ and V as in 4.2.1. For $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 0$, consider the following formal sum for $(z,h) \in \mathbb{B}_n \times U(\mathbb{V}^\infty)$:

$$\mathcal{G}_{Z_t(\phi^\infty)_{E_v},s}(z,h) := \sum_{x\in V^t} \phi^\infty\left(h^{-1}x\right) G_{x,s}(z).$$

By (4.5) and Lemma 4.2.1, if (z,h) is not over $Z_t(\phi^\infty)_{E_v}$ via (4.4), the formal sum is defined and absolutely convergent. Thus, $\mathcal{G}_{Z_t(\phi^\infty)_{E_v},s}$ descends to $\mathrm{Sh}(\mathbb{V})_{K,E_v} - Z_t(\phi^\infty)_{E_v}$ via (4.4), which we still denote by $\mathcal{G}_{Z_t(\phi^\infty)_{E_v},s}$. By (4.5) and Theorem 4.2.2, $\mathcal{G}_{Z_t(\phi^\infty)_{E_v},s}$ is a Green function for $Z_t(\phi^\infty)_{E_v}$. For $g \in G(\mathbb{A}_F)$, let

$$\mathcal{G}_{Z_t(\omega(g)\phi)_{E_v},s} = \mathcal{G}_{Z_t(\omega(g^\infty)\phi^\infty)_{E_v},s} W^{\mathfrak{w}}_{\infty,t}(g_\infty),$$

which is a Green function for $Z_t(\omega(g)\phi)_{E_v}$ by (4.2). Define the automorphic Green functions for $Z_t(\phi^\infty)_{E_v}$ and $Z_t(\omega(g)\phi)_{E_v}$ to be

$$\mathcal{G}^{\mathrm{aut}}_{Z_t(\phi^\infty)_{E_v}} = \widetilde{\lim_{s\to 0}} \mathcal{G}_{Z_t(\phi^\infty)_{E_v},s}, \quad \mathcal{G}^{\mathrm{aut}}_{Z_t(\omega(g)\phi)_{E_v}} = \widetilde{\lim_{s\to 0}} \mathcal{G}_{Z_t(\omega(g)\phi)_{E_v},s}, \tag{4.10}$$

respectively, which are admissible by Theorem 4.2.2.

Let $\mathcal{G}^{\overline{L}_{E_v}}_{Z_t(\phi)_{E_v}}$ be the normalized admissible Green function for $Z_t(\phi)_{E_v}$ with respect to $\overline{L}_{E_v}$ as in Definition A.3.3. In particular, its integration against $c_1(\overline{L}_{E_v})^n$ is 0.

Lemma 4.2.4. (1) *We have*

$$\mathcal{G}^{\mathrm{aut}}_{Z_t(\omega(g)\phi)_{E_v}} - \mathcal{G}^{\overline{L}_{E_v}}_{Z_t(\omega(g)\phi)_{E_v}} = -\frac{1}{n}\frac{c_1(L)^{n-1}\cdot Z_t(\omega(g)\phi)}{c_1(L)^n}.$$

Both sides are independent of K.

(2) *We have*

$$\frac{1}{n}\omega(g)\phi(0) + \sum_{t\in F_{>0}}\left(\mathcal{G}^{\mathrm{aut}}_{Z_t(\omega(g)\phi)_{E_v}} - \mathcal{G}^{\overline{L}_{E_v}}_{Z_t(\omega(g)\phi)_{E_v}}\right) \in \mathcal{A}_{\mathrm{hol}}(G,\mathfrak{w}).$$

Proof. The equation in (1) follows from (4.9) and the independence of the right-hand side follows from the projection formula. (2) follows from Theorem 4.1.2. □

Remark 4.2.5. The automorphic form in Lemma 4.2.4 (2) can be made explicit:

$$c_1(L)^{n-1}\cdot z(\omega(g)\phi) = -c_1(L)^n E(0,g,\phi). \tag{4.11}$$

This is a geometric version of the Siegel-Weil formula (3.13). It is stated in [Kud97a, COROLLARY 10.5] for the orthogonal case; the proof carries over to the unitary case.

4.2.3. Kudla's Green function

We recall Kudla's Green functions for special divisors [Kud97b], following [Liu11a, 4C] in the unitary case. We consider simple special divisors $Z(x)$'s, instead of weight special divisors $Z_t(\phi)$'s. We extend the definition of special divisors as follows. For $x\in\mathbb{V}^\infty$ such that $q(x)\in F^\times - F_{>0}$, let $Z(x)=0$.

First, we work on $\mathbb{B}_n$. For $v\in\infty$, V as in the end of 4.2.1 with respect to v, $g\in G(F_v)$ and $x\in V$, define

$$G^{\mathrm{Kud}}(x,g)(z) = -\mathrm{Ei}(-2\pi\delta_v(g)R_x(z)),\ z\in\mathbb{B}_n\backslash\mathbb{B}_x,$$

where the exponential integral $\mathrm{Ei}(t)=\int_{-\infty}^t \frac{e^s}{s}ds$ on $t\in(-\infty,0)$ has a log-singularity at 0. If $q(x)\neq 0$ so that $\mathbb{B}_x$ is either empty or a complex unit ball of dimension $n-1$, $G^{\mathrm{Kud}}(x,g)$ is a Green function for $\mathbb{B}_x$. If $q(x)<0$, equivalently $\mathbb{B}_x$ is empty, then G^{Kud} is smooth.

Now we work on $\mathrm{Sh}(\mathbb{V})_K$. Let $x\in\mathbb{V}^\infty$ with $q(x)\in F^\times$. For $v\in\infty$, if $u(q(x))>0$ for every $u\in\infty-\{v\}$, by the Hasse-Minkowski theorem and Witt's theorem, there exists $h\in U(\mathbb{V}^\infty)$ and $x^{(v)}\in V-\{0\}$, where V is as in the last paragraph, such that $x=h^{-1}x^{(v)}$. Define

$$\mathcal{G}^{\mathrm{Kud}}_{Z(x)_{E_v}}(g) = \sum_{j=1}^m \sum_{y\in U(V)x^{(v)}\cap h_jKh^{-1}x^{(v)}} G^{\mathrm{Kud}}(y,g), \tag{4.12}$$

where $h_1,...,h_m$ is a set of representatives of $U(V)\backslash U(\mathbb{V}^\infty)/K$. By the decomposition analogous to (4.5) (see the second equation on [Kud97a, p 56]) and [Liu11a, Proposition 4.9]), $\mathcal{G}^{\mathrm{Kud}}_{Z(x)_{E_v}}(g)$ is absolutely convergent and descends to $\mathrm{Sh}(\mathbb{V})_{K,E_v}$ via (4.4). And it is a Green function for $Z(x)_{E_v}$.

Besides Kudla's Green functions, we will need their projections to the constant function 1 to modify the normalized Green function. See 4.4.3.

Definition 4.2.6. For $x\in\mathbb{V}^\infty$ with $q(x)\in F^\times$, $g\in G(\mathbb{A}_{F,\infty})$ and $v\in\infty$, let

$$\mathfrak{k}(x,g_v) = \frac{1}{\deg(\overline{L}_{E_v})}\int_{\mathrm{Sh}(\mathbb{V})_{K,E_v}} \mathcal{G}^{\mathrm{Kud}}_{Z(x)_{E_v}}(g_v)c_1(\overline{L}_{E_v})^n.$$

Note that if $u(q(x))<0$ for some $u\in\infty-\{v\}$, then $\mathcal{G}^{\mathrm{Kud}}_{Z(x)_{E_v}}(g)=0$. Thus, if $q(x)$ is negative at more than one infinite places, then $\mathcal{G}^{\mathrm{Kud}}_{Z(x)_{E_v}}(g)=0$ and $\mathfrak{k}(x,g_v)=0$ for every $v\in\infty$.

The following can be read from [GS19, (1.12), Theorem 1.2, (1.18), (1.19), Proposition 5.9].

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Theorem 4.2.7. *Let $E_t'(0,g,\phi)(v)$ be as in (3.11). For $t \in F_{>0}$ and $v \in \infty$, we have*

$$
\begin{aligned}
&- W^{\mathfrak{w}}_{\infty,t}(g_\infty) \sum_{x \in K\backslash \mathbb{V}^\infty,\ q(x)=t} \omega(g^\infty)\phi^\infty(x)\mathfrak{k}(x, g_v) \\
&\quad = E_t'(0,g,\phi)(v) - E_t(0,g,\phi)(\log \pi - (\log \Gamma)'(n+1) + \log v(t)).
\end{aligned}
$$

For $t \in F^\times$ with $v(t) < 0$ for exactly one infinite place v, we have

$$
-W^{\mathfrak{w}}_{\infty,t}(g_\infty) \sum_{x \in K\backslash \mathbb{V}^\infty,\ q(x)=t} \omega(g^\infty)\phi^\infty(x)\mathfrak{k}(x, g_v) = E_t'(0,g,\phi)(v).
$$

And for $t = 0$, we have

$$
\omega(g)\phi(0)\log \delta_\infty(g_\infty) = E_0'(0,g,\phi).
$$

We remind the reader that ∞ is the set of infinite places of F. And our formulas differ from [GS19] by a factor $1/2$ since in loc. cit., the authors use the set of infinite places of E.

4.2.4. Modularity of difference of Green functions

We need a more general notion of modular forms.

Definition 4.2.8. Let V be a topological $\mathbb{C}$-vector space and V^* the continuous dual. Let $\mathcal{A}(G, \mathfrak{w}, V)$ be the space of smooth V-valued function f on $G(\mathbb{A})$ such that for every $l \in V^*$, we have $l \circ f \in \mathcal{A}(G, \mathfrak{w})$.

Remark 4.2.9. Note that $l \circ f$ is automatically smooth.

Clearly, if V is the topological direct sum of V_1, V_2 and $f_i \in \mathcal{A}(G, \mathfrak{w}, V_i)$, then $f_1 + f_2 \in \mathcal{A}(G, \mathfrak{w}, V)$.

Now we define the formal generating series of Green functions. Recall that ∞ is the set of infinite places of F, of cardinality $[F : \mathbb{Q}]$. Let

$$
E_\infty := E \otimes_{\mathbb{Q}} \mathbb{R} \simeq \prod_{v \in \infty} E_v,
$$

which is the product of $[F : \mathbb{Q}]$ many copies of $\mathbb{C}$. Then

$$
\mathrm{Sh}(\mathbb{V})_{K,E_\infty} := \mathrm{Sh}(\mathbb{V})_K \otimes_E E_\infty = \mathrm{Sh}(\mathbb{V})_K \otimes_{\mathbb{Q}} \mathbb{R}
$$

is the (disconnected) complex manifold that is the disjoint union of all base changes of $\mathrm{Sh}(\mathbb{V})_K$ to E_v's (each base change itself may not be connected either!).

Let

$$
\mathcal{G}^{\overline{\mathcal{L}}}(g,\phi) = \sum_{t \in F_{>0}} \sum_{v \in \infty} \mathcal{G}^{\overline{L}_{E_v}}_{Z_t(\omega(g)\phi)_{E_v}},
$$

$$
\mathcal{G}^{\mathrm{Kud}}(g,\phi) = \sum_{x \in \mathbb{V}^\infty,\ q(x) \in F^\times} \omega(g^\infty)\phi^\infty(x) W^{\mathfrak{w}}_{\infty,q(x)}(g_\infty) \sum_{v \in \infty} \mathcal{G}^{\mathrm{Kud}}_{Z(x)_{E_v}}(g),
$$

which are formal generating series of smooth functions on $\mathrm{Sh}(\mathbb{V})_{K,E_\infty}$ with logarithmic singularities along the same formal generating series of special divisors. Then $\mathcal{G}^{\overline{\mathcal{L}}}(g,\phi) - \mathcal{G}^{\mathrm{Kud}}(g,\phi)$ is a formal generating series valued in $C^\infty(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})$, the space of smooth $\mathbb{C}$-valued functions on $\mathrm{Sh}(\mathbb{V})_{K,E_\infty}$.

Let $E_1 := \mathcal{O}_{\mathrm{Sh}(\mathbb{V})_K}(\mathrm{Sh}(\mathbb{V})_K)$, which is a finite field extension of E (since $\mathrm{Sh}(\mathbb{V})_K$ is connected). Then we have a morphism $\mathrm{Sh}(\mathbb{V})_K \to \operatorname{Spec} E_1$. By Stein factorization, $\mathrm{Sh}(\mathbb{V})_K$, as a variety over E_1, is geometrically connected. So the connected components of $\mathrm{Sh}(\mathbb{V})_{K,E_\infty} = \mathrm{Sh}(\mathbb{V})_K \otimes_{\mathbb{Q}} \mathbb{R}$ are exactly indexed by the underlying set of $\operatorname{Spec} E_1 \otimes_{\mathbb{Q}} \mathbb{R}$, equivalently, the set of conjugate pairs of infinite places of E_1 (which, as a finite field extension of the CM field E, has only complex embeddings but no real embeddings). Let $\mathrm{LC}(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})$ be the space of locally constant functions on $\mathrm{Sh}(\mathbb{V})_{K,E_\infty}$. Then we have the canonical isomorphism from $\mathrm{LC}(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})$ to the product copies of $\mathbb{C}$ indexed by the set of conjugate pairs of infinite places of E_1. Now we can embed $\mathcal{O}_{E_1}^\times$ in $\mathrm{LC}(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})$ via this isomorphism and the Dirichlet regulator map, so that the $\mathbb{C}$-span of the image of $\mathcal{O}_{E_1}^\times$, denoted by $\mathbb{C}\mathcal{O}_{E_1}^\times$ is of codimension 1 in $\mathrm{LC}(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})$ by Dirichlet's unit theorem. Let $\mathbb{C}\mathcal{O}_{E_1}^\times$ be this span. Let

$$\overline{C^\infty}(\mathrm{Sh}(\mathbb{V})_{K,E_\infty}) = C^\infty(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})/\mathbb{C}\mathcal{O}_{E_1}^\times.$$

Equip $\overline{C^\infty}(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})$ with the quotient of the L^∞-topology. Define an embedding

$$\mathbb{C} \simeq \mathrm{LC}(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})/\mathbb{C}\mathcal{O}_{E_1}^\times \subset \overline{C^\infty}(\mathrm{Sh}(\mathbb{V})_{K,E_\infty}) \tag{4.13}$$

by mapping $a \in \mathbb{C}$ to the constant function a on $\mathrm{Sh}(\mathbb{V})_{K,E_v}$ for some $v \in \infty$ (rather than $\mathrm{Sh}(\mathbb{V})_{K,E_{1,w}}$ for some infinite place w of E_1). Below, by a complex number in $\overline{C^\infty}(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})$, we understand it as the image by (4.13).

Theorem 4.2.10. *Let $E'_{t,\mathrm{f}}(0,g,\phi)$ be as in (3.29). For $g \in G(\mathbb{A})$, the generating series of $\overline{C^\infty}(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})$-valued functions on $G(\mathbb{A})$*

$$\begin{aligned}\mathcal{D}(g) := \sum_{t\in F_{>0}} &\left(E'_{t,\mathrm{f}}(0,g,\phi) + E_t(0,g,\phi)\log \mathrm{Nm}_{F/\mathbb{Q}}t\right) + \left(\mathcal{G}^{\mathcal{L}}(g,\phi) - \mathcal{G}^{\mathrm{Kud}}(g,\phi)\right)\\ &- \omega(g)\phi(0)(-\log\delta_\infty(g_\infty) + [F:\mathbb{Q}](\log\pi - (\log\Gamma)'(n+1)))\end{aligned}$$

pointwise converges to an element in $\mathcal{A}\left(G,\mathfrak{w},\overline{C^\infty}(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})\right)$.

Remark 4.2.11. (1) It might be interesting to study whether Theorem 4.2.10 still holds if we put Fréchet topology on $\overline{C^\infty}(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})$, and if we require stronger convergence.

(2) Theorem 4.2.10 is a strengthened analog of the main result of Ehlen and Sankaran [ES18], which is for $F = \mathbb{Q}$.

Proof. We follow [KRY06] and [MZ21]. Let $C^\infty(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})^\circ$ be the L^2-orthogonal complement of $\mathrm{LC}(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})$ in $C^\infty(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})$, endowed with L^∞-topology. Then $\overline{C^\infty}(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})$ is the topological direct sum of $C^\infty(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})^\circ$ and $\mathrm{LC}(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})/\mathbb{C}\mathcal{O}_{E_1}^\times$.

Let the generating series $\mathcal{D}^\circ$ be the projection of $\mathcal{D}$ to $C^\infty(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})^\circ$. By [MZ21, Lemma 3.8][2] and the same argument in the proof [KRY06, Theorem 4.4.4],[3] for every $g \in G(\mathbb{A})$, $\mathcal{D}^\circ(g)$ converges in $L^\infty\left(\mathrm{Sh}(\mathbb{V})_{K,E_\infty}\right)$. Particularly, $\mathcal{D}^\circ(g)$ converges in $L^2\left(\mathrm{Sh}(\mathbb{V})_{K,E_\infty}\right)$ since $\mathrm{Sh}(\mathbb{V})_{K,E_\infty}$ is compact. So we can consider $\mathcal{D}^\circ$ (identified as its limit) as a function on $G(\mathbb{A}) \times \mathrm{Sh}(\mathbb{V})_{K,E_\infty}$. Then by the argument in the proof of [MZ21, Theorem 3.9], $\mathcal{D}^\circ \in C^\infty\left(G(\mathbb{A}) \times \mathrm{Sh}(\mathbb{V})_{K,E_\infty}\right)$. Also note that for every $g \in G(\mathbb{A})$, $\mathcal{D}^\circ(g) \in C^\infty\left(\mathrm{Sh}(\mathbb{V})_{K,E_\infty}\right)^\circ$. Thus, $\mathcal{D}^\circ$ is a smooth $C^\infty\left(\mathrm{Sh}(\mathbb{V})_{K,E_\infty}\right)^\circ$-valued function on $G(\mathbb{A})$. See [Tré67, Theorem 40.1 and Corollary]. Then the argument in the proof of either [KRY06, Theorem 4.4.4] or [MZ21, Theorem 3.9] shows that $\mathcal{D}^\circ \in \mathcal{A}\left(G,\mathfrak{w},\overline{C^\infty}(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})^\circ\right)$.

Consider the projection from $\overline{C^\infty}(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})$ to $\mathrm{LC}(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})/\mathbb{C}\mathcal{O}_{E_1}^\times \simeq \mathbb{C}$ – that is,

$$f \mapsto \sum_{v\in\infty} \frac{1}{\deg(\overline{L}_{E_v})}\int_{\mathrm{Sh}(\mathbb{V})_{K,E_v}} f\cdot c_1(\overline{L}_{E_v})^n.$$

[2] It is a priori only at the level of generating series of functions, but will be at the level of true L^2-functions after this paragraph.

[3] The convergence in loc. cit. was not stated explicitly. It could come from the general property of Laplacian spectral decomposition of a smooth function on a compact manifold, applied to the product of S^1 and a compact Shimura curve in loc. cit.

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By Theorem 4.2.7 and that $E_0(0, g, \phi) = \omega(g)\phi(0)$ (see Lemma 3.3.9 (1)), the projection of $\mathcal{D}$ (defined on each of its terms) pointwise converges to an element in $\mathcal{A}(G, \mathfrak{w})$. Since $\mathcal{D}$ is the sum of this projection and $\mathcal{D}^\circ$, the theorem follows. □

4.3. Modularity problems

We will raise two modularity problems for admissible extensions of special divisors. Before that, we recall some notions and Kudla's modularity problem.

4.3.1. Preliminaries

A (regular) integral model of $\mathrm{Sh}(\mathbb{V})_K$ over an integral domain R with fraction field E is a (regular) Deligne-Mumford stack proper flat over $\mathrm{Spec}\, R$ with a fixed isomorphism of its generic fiber to $\mathrm{Sh}(\mathbb{V})_K$. An isomorphism between integral models is an isomorphism over $\mathrm{Spec}\, R$ that respects the fixed isomorphisms to $\mathrm{Sh}(\mathbb{V})_K$.

Let $\mathcal{X}_K$ be a regular integral model of $\mathrm{Sh}(\mathbb{V})_K$ over $\mathrm{Spec}\, \mathcal{O}_E$. Let $\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{X}_K)$ be the Chow group of arithmetic divisors with $\mathbb{C}$-coefficients. See Definition A.3.4. In particular, we have an isomorphism

$$\deg : \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathrm{Spec}\, \mathcal{O}_E) \simeq \mathbb{C}$$

by taking degrees (see Remark A.3.5), and an arithmetic intersection pairing

$$\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{X}_K) \times Z_1(\mathcal{X}_K)_{\mathbb{C}} \to \mathbb{C}, \ (\widehat{z}, Y) \mapsto \widehat{z} \cdot Y.$$

See Appendix A.4. Here, we recall that $Z_1(\mathcal{X}_K)$ is the group of 1-cycles on $\mathcal{X}_K$.

Let $\mathcal{L} = \mathcal{L}_K$ be an extension of $L = L_K$ to $\mathcal{X}_K$, which we allow to be a line bundle. Let $\overline{\mathcal{L}}$ be $\mathcal{L}$ equipped with a hermitian metric. Let $c_1(\overline{\mathcal{L}}^\vee) \in \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{X}_K)$ be the first arithmetic Chern class of the dual of $\overline{\mathcal{L}}$. See Example A.3.7.

4.3.2. Kudla's problem

We consider the following modularity problem of Kudla [Kud02, Kud03, Kud04]: find an arithmetic divisor $\widehat{\mathcal{Z}}(x)$ on $\mathcal{X}$ extending $Z(x)$, explicitly and canonically, such that

$$\omega(g)\phi(0)c_1(\overline{\mathcal{L}}^\vee) + \sum_{x \in K\backslash \mathbb{V}^\infty_{>0}} \omega(g)\phi(x_\infty x)\widehat{\mathcal{Z}}(x),$$

where $x_\infty \in \mathbb{V}_\infty$ such that $q(x_\infty) = q(x) \in F_{>0}$ lies in $\mathcal{A}_{\mathrm{hol}}(G, \mathfrak{w}) \otimes \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{X}_K)$. The existence of such $\widehat{\mathcal{Z}}(x)$ is obvious, by choosing a section of the natural surjection $\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{X}_K) \to \mathrm{Ch}^1(\mathrm{Sh}(\mathbb{V})_K)_{\mathbb{C}}$. However, it is only defined at the level of divisor classes, and not explicit.

4.3.3. Admissible extensions

We consider the above modularity problem for admissible extensions. In particular, we assume Assumption A.1.1 for $\mathcal{X}_K$. (Also recall that $\mathcal{X}_K$ is connected, as $\mathrm{Sh}(\mathbb{V})_K$ is.) Assume that $\mathcal{L}$ is ample. Let $\widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K) \subset \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{X}_K)$ be the subgroup of arithmetic divisors that are admissible with respect to $\overline{\mathcal{L}}$. See Definition A.3.4. By Lemma A.3.6, the natural map

$$\widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K) \to \mathrm{Ch}^1(\mathcal{X}_{K,E})_{\mathbb{C}} \tag{4.14}$$

is surjective, and the kernel is the image of the pullback

$$\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathrm{Spec}\, \mathcal{O}_E) \simeq \widehat{\mathrm{Ch}}^1_{\mathbb{C}}\big(\mathrm{Spec}\, \mathcal{O}_{\mathcal{X}_K}(\mathcal{X}_K)\big)_{\mathbb{C}} \hookrightarrow \widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K). \tag{4.15}$$

In particular, $\widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K)$ is finite dimensional.

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Definition 4.3.1. Define an embedding $\mathbb{C} \hookrightarrow \widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K)$ as the composition of the inverse $\deg^{-1}$: $\mathbb{C} \simeq \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathrm{Spec}\,\mathcal{O}_E)$ of taking degree and (4.15). Below, by a complex number in $\widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K)$, we understand it as the image by this embedding.

Remark 4.3.2. The intersection $\widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K) \cap \overline{C^\infty}(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})$ in $\widehat{\mathrm{Ch}}_{\mathbb{C}}(\mathcal{X}_K)$ is $\mathbb{C}$, where $\mathbb{C}$ is in $\overline{C^\infty}(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})$ via (4.13).

Let $\widehat{Z}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K)$ the group of admissible arithmetic divisors. See Definition A.3.1. For a divisor Z on $\mathrm{Sh}(\mathbb{V})_K \simeq \mathcal{X}_{K,E}$, let

$$Z^{\overline{\mathcal{L}}} \in \widehat{Z}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K) \tag{4.16}$$

be the normalized admissible extension of Z with respect to $\overline{\mathcal{L}}$. See Definition A.3.3. Let $[Z(x)^{\overline{\mathcal{L}}}]$ be its class in $\widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K)$. Then a preiamge of $[Z(x)]$ via (4.14) is of the form $[Z(x)^{\overline{\mathcal{L}}}] + e(x)$ for some $e(x) \in \mathbb{C}$. Note that $c_1(\overline{\mathcal{L}}^\vee) \in \widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K)$. See Example A.3.7.

Problem 4.3.3. Find $a \in \mathbb{C}$ and $e = \{e(x) \in \mathbb{C}\}_{x \in K\backslash \mathbb{V}^\infty_{>0}}$ explicitly such that for every $\phi \in \overline{\mathcal{S}}(\mathbb{V})^K$, the generating series

$$z(\omega(g)\phi)^{\overline{\mathcal{L}}}_{e,a} := \omega(g)\phi(0)\Big(c_1(\overline{\mathcal{L}}^\vee) + a\Big) + \sum_{x \in K\backslash \mathbb{V}^\infty_{>0}} \omega(g)\phi(x_\infty x)\Big([Z(x)^{\overline{\mathcal{L}}}] + e(x)\Big), \tag{4.17}$$

where $x_\infty \in \mathbb{V}_\infty$ such that $q(x_\infty) = q(x) \in F_{>0}$, lies in $\mathcal{A}_{\mathrm{hol}}(G,\mathfrak{w}) \otimes \widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K)$. Moreover, $e(x)$ should to be naturally decomposed into a sum of 'local components' such that the v-component should be 0 at all but finitely many places and ∞-component should be independent of the choice of the regular integral model.

In other words, we want a modular generating series by modifying each $[Z(x)^{\overline{\mathcal{L}}}]$ by an explicit constant once for all ϕ. A weaker statement is to allow the modification to depend on g, ϕ.

Problem 4.3.4. Find $a \in \mathbb{C}$ and a smooth function $e_t(g,\phi)$ on $G(\mathbb{A}_F)$ for $\phi \in \overline{\mathcal{S}}(\mathbb{V})^K$ explicitly such that the generating series

$$\omega(g)\phi(0)\Big(c_1(\overline{\mathcal{L}}^\vee) + a\Big) + \sum_{t \in F_{>0}} \Big([Z_t(\omega(g)\phi)^{\overline{\mathcal{L}}}] + e_t(g,\phi)\Big) \tag{4.18}$$

lies in $\mathcal{A}_{\mathrm{hol}}(G,\mathfrak{w}) \otimes \widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K)$. Moreover, $e_t(g,\phi)$ should to be naturally decomposed into a sum of 'local components' such that the v-component should be 0 at all but finitely many places and ∞-component should be independent of the choice of the regular integral model.

Remark 4.3.5. In (4.17) and (4.18), one may replace $\omega(g)\phi(0)\Big(c_1(\overline{\mathcal{L}}^\vee) + a\Big)$ by $\omega(g)\phi(0)c_1(\overline{\mathcal{L}}^\vee)$ by adding a suitable multiple of the degree of $z(\omega(g)\phi)$, which is modular by Theorem 4.1.2 and can be made explicit by (4.11). However, we keep the freedom to have 'a' to get the decomposition of $e_t(g,\phi)$ as we will see in Section 4.4.

By Theorem 4.1.2 and Lemma A.3.6 (1), we immediately have the following lemma.

Lemma 4.3.6. *For $\mathcal{P} \in Z_1(\mathcal{X}_K)_{\mathbb{C}}$ such that* $\deg \mathcal{P}_E = 0$,

$$z(\omega(g)\phi)^{\overline{\mathcal{L}}}_{e,a} \cdot \mathcal{P} \in \mathcal{A}_{\mathrm{hol}}(G,\mathfrak{w}) \tag{4.19}$$

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Remark 4.3.7. (1) This lemma is called almost modularity in [MZ21, Theorem 4.3].

(2) In the case that $K = K_\Lambda$ with a different Λ, for Kudla-Rapoport arithmetic divisors and $\mathcal{P} \in Z_1(\mathcal{X}_K)_{\mathbb{C}}$ with $\mathcal{P}_E = 0$, the analogous statement is proved in [Zha21b, Theorem 14.6].

When $\deg \mathcal{P}_E \neq 0$, the truth of (4.19) is in fact equivalent to the modularity of $z(\omega(g)\phi)^{\overline{\mathcal{L}}}_{e,a}$ by Lemma 4.3.8 below. We will not use exactly this 'numerical criterion', but use Lemma 4.3.8 in a more sophisticated way to prove our modularity results in 5.2.4.

Lemma 4.3.8. *Let X be a $\mathbb{C}$-vector space, $x \in X$ nonzero, and l a linear form on X such that $l(x) = 1$. Let f be a formal generating series of functions on $G(\mathbb{A})$ valued in X and $\overline{f}$ the corresponding formal generating series of functions on $G(\mathbb{A})$ valued in $X/\mathbb{C}x$. Assume that $\overline{f} \in \mathcal{A}(G, \mathfrak{w}) \otimes X/\mathbb{C}x$. Then $f \in \mathcal{A}(G, \mathfrak{w}) \otimes X$ if and only if $l \circ f \in \mathcal{A}(G, \mathfrak{w})$.*

Proof. Define a section $\mathfrak{s}$ of the projection $X \to X/\mathbb{C}x$ by $\mathfrak{s} : z \mapsto s(z) - l(s(z)) \cdot x$, where s is any section (and $\mathfrak{s}$ is independent of the choice of s). Then $f = \mathfrak{s}(\overline{f}) + (l \circ f) \cdot x$, and the lemma follows. □

4.4. *Conjecture and theorems*

First, we define specific integral models. Then we define explicit admissible extensions. Then we will propose a precise conjecture. Finally, we state our modularity theorems.

4.4.1. Integral models

Let us at first set up some notations and assumptions that are needed to construct our integral models. For a finite place v of F and an $\mathcal{O}_{E_v}$-lattice Λ_v of $\mathbb{V}(E_v)$, the dual lattice is defined as $\Lambda_v^\vee = \{x \in \mathbb{V}(E_v) : \langle x, \Lambda_v\rangle \subset \mathcal{O}_{E_v}\}$. Then Λ_v is called

- self-dual if $\Lambda_v = \Lambda_v^\vee$;
- ϖ_{E_v}-modular if $\Lambda_v^\vee = \varpi_{E_v}^{-1}\Lambda_v$;
- almost ϖ_{E_v}-modular if $\Lambda_v^\vee \subset \varpi_E^{-1}\Lambda_v$ and the inclusion is of colength 1.

Assume the following assumption in the rest of the paper.

Assumption 4.4.1. (1) At least one of the following three conditions hold:

(1.a) every finite place of E is at most tamely ramified over $\mathbb{Q}$;
(1.b) $E/\mathbb{Q}$ is Galois;
(1.c) E is the composition of F with some imaginary quadratic field.

(2) Every finite place v of F ramified over $\mathbb{Q}$ or of residue characteristic 2 is unramified in E.
(3) At every finite place v of F inert in E, there is a self-dual lattice Λ_v in $\mathbb{V}(E_v)$.
(4) At every finite place v of F ramified in E, there is a π_v-modular (resp. almost π_v-modular) lattice Λ_v in $\mathbb{V}(E_v)$ if n is odd (resp. n is even).

We will classify $\mathbb{V}$ containing such Λ in Remark 5.1.2 below.

At every place v split in E, let Λ_v be a self-dual lattice in $\mathbb{V}(E_v)$. Let

$$\Lambda = \prod_v \Lambda_v \subset \mathbb{V}^\infty.$$

Let $K_\Lambda \subset U(\mathbb{V}^\infty)$ be the stabilizer of Λ.

Definition 4.4.2. Let $\widetilde{K_\Lambda}$ be the directed poset of compact open subgroups $K \subset K_\Lambda$, under the inclusion relation, such that for a finite place v of F,

(1) if v is non-split in E, then $K_v = K_{\Lambda,v}$;
(2) if v is split in E, then K_v is a principal congruence subgroup of $K_{\Lambda,v}$.

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Under Assumption 4.4.1 (1.a), for $K \in \widetilde{K}_\Lambda$ and a finite place v of E, we have an integral model $\mathcal{S}_v$ of $\mathrm{Sh}(\mathbb{V})_K$ over $\operatorname{Spec} \mathcal{O}_{E,(v)}$, as constructed in (B.2.11), B.2.25 and B.2.28. (Note that we can always choose a CM type satisfying the matching condition (B.2.20).) We remind the reader of the differences between the notations on the fields here and in Appendix B.

Lemma 4.4.3. (1) *The integral model $\mathcal{S}_v$ is regular.*

(2) *If $K = K_\Lambda$, there is an ample $\mathbb{Q}$-line bundle $\mathcal{P}_v$ on $\mathcal{S}_v$ extending L_{K_Λ}.*

Proof. (1) We apply B.2.26. We use M to denote the reflex field in B.2.26, which is defined in (B.1.15) (and denoted by E there), to avoid confusion. We use N to denote the field extension of the reflex field in B.2.26 (denoted by L there). By the finite étaleness of the moduli space of relative dimenison 0 as in B.1.23, we may choose N/M to be unramified at v [Sta18, Tag 04GL]. Then by [RSZ20] and B.2.26, $\mathcal{S}_{v,\mathcal{O}_{N,(v)}}$ is regular for a place v of N over v. By the descent of regularity under faithfully flat morphism [Sta18, Tag 033D], the lemma follows.

(2) Let N/E be a finite Galois extension such that v is unramified in N and every connected component of $\mathcal{S}_{v,\mathcal{O}_{N,(v)}}$ is geometrically connected. By its construction (B.2.11), every connected component is a quotient of a connected component of an integral model of Hodge type over $\mathcal{O}_{N,(v)}$ by a finite group action. The integral model of Hodge type is a closed subscheme of the integral Siegel moduli space [Xu21, Xu25], on which we have a well-known ample Hodge line bundle. The restriction is an ample line bundle on each geometrically connected component of the integral model of Hodge type. Taking norm along the quotient map by the finite group action, we get an an ample line bundle on every component of $\mathcal{S}_{v,\mathcal{O}_{N,(v)}}$. See [BLR90, Section 6, Theorem 7 and Example B] and [Vis04] for the stack case. Then taking norm map along the quotient map by the $\mathrm{Gal}(N/E)$, we have an ample line bundle $\mathcal{P}'$ on $\mathcal{S}_v$. Dividing $\mathcal{P}'$ by the product of the order of the finite group and $[N : E]$, we get the desired ample line bundle on $\mathcal{S}_v$. □

Under Assumption 4.4.1 (1.b) or (1.c), the reflex field (B.1.15), in our notations, is E. For $K \in \widetilde{K}_\Lambda$ and a finite place v of E, both sides of the morphism (B.1.30) (and the generalizations as in B.2.17 or B.2.28) are equipped with a natural action of the finite group $Z^{\mathbb{Q}}\left(\mathbb{A}_{\mathbb{Q}}^{\{p\}\cup\infty}\right)/Z^{\mathbb{Q}}(\mathbb{Z}_{(p)})K^p_{Z^{\mathbb{Q}}}$ compatible with the morphism. See [LTX+22, Definition 4.2.2]. Moreover, taking quotients by this finite group, we get a new morphism whose target is isomorphic to $\operatorname{Spec} \mathcal{O}_{E,(v)}$. Then the source of the new morphism is an integral model of $\mathrm{Sh}(\mathbb{V})_K$ over $\operatorname{Spec} \mathcal{O}_{E,(v)}$, whose regularity is assured by [RSZ20] and faithfully flat descent of regularity. We also denote this integral model by $\mathcal{S}_v$. Moreover, if Assumption 4.4.1 (1.a) also holds, then the construction here and in Lemma 4.4.3 coincides, by B.2.26.

We want to glue these models to obtain a regular integral model of $\mathrm{Sh}(\mathbb{V})_K$ over $\operatorname{Spec} \mathcal{O}_E$.

Theorem 4.4.4. *Assume Assumption 4.4.1.*

(1) *For $K \in \widetilde{K}_\Lambda$, there is a regular integral model $\mathcal{X}_K$ of $\mathrm{Sh}(\mathbb{V})_K$ over $\operatorname{Spec} \mathcal{O}_E$ such that $\mathcal{X}_{K,\mathcal{O}_{E,(v)}} \simeq \mathcal{S}_v$ as integral models. See 4.3.2. Moreover, Assumption A.1.1 holds for $\mathcal{X}_K$.*

(2) *For $K \subset K'$ in $\widetilde{K}_\Lambda$, there is a unique finite flat morphism $\pi_{K,K'} : \mathcal{X}_K \to \mathcal{X}_{K'}$ extending the natural morphism $\mathrm{Sh}(\mathbb{V})_K \to \mathrm{Sh}(\mathbb{V})_{K'}$.*

(3) *Regard $\mathcal{X}_K$ as an $\mathcal{X}_{K_\Lambda}$-scheme and $\mathrm{Sh}(\mathbb{V})_K$ as an $\mathrm{Sh}(\mathbb{V})_{K_\Lambda}$-scheme via $\pi_{K,K'}$. There is a unique action of K_Λ/K (note that K is normal in K_Λ) on the $\mathcal{X}_{K_\Lambda}$-scheme $\mathcal{X}_K$ extending the standard action of K_Λ/K on the $\mathrm{Sh}(\mathbb{V})_{K_\Lambda}$-scheme $\mathrm{Sh}(\mathbb{V})_K$ by 'right translation'.*

Proof. The construction of $\mathcal{X}_K$ is as follows. Continue to use the notation M in the proof of Lemma 4.4.3. First, consider the analog of the morphism (B.1.30) over $\mathcal{O}_M[R^{-1}]$, where R is a finite set of finite places of $\mathbb{Q}$ such that $K_v = K_{\Lambda,v}$ for v not over R (in particular, we do not need the generalizations as in B.2.17 or B.2.28). See, for example, [RSZ20, 5.1] with extra level structures over R (similar to (B.1.29)). Denote this morphism by $\mathcal{M} \to \mathcal{M}_0$. We use Galois descent to construct a model of $\mathrm{Sh}(\mathbb{V})_K$ outside finitely many finite places. Let N/M be a finite extension, Galois over E, such that the base change of $\mathcal{M}_0$ to $\operatorname{Spec} \mathcal{O}_N[R^{-1}]$ is a finite disjoint union of $\operatorname{Spec} \mathcal{O}_N[R^{-1}]$. Let $S \supset R$ be a finite set of finite places of $\mathbb{Q}$ such that $\operatorname{Spec} \mathcal{O}_N[S^{-1}] \to \operatorname{Spec} \mathcal{O}_E[S^{-1}]$ is unramified. Then the fiber

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of $\mathcal{M}_{\mathcal{O}_N[S^{-1}]} \to \mathcal{M}_{0,\mathcal{O}_N[S^{-1}]}$ over a chosen $\operatorname{Spec}\mathcal{O}_N[S^{-1}]$ is a regular Deligne-Mumford stack $\mathcal{M}^S$ proper over $\operatorname{Spec}\mathcal{O}_N[S^{-1}]$ with generic fiber $\operatorname{Sh}(\mathbb{V})_{K,N}$. By Zariski's main theorem (for stacks which easily follows from the scheme version), after possibly enlarging S, we may assume that the action of the finite group $\operatorname{Gal}(N/E)$ on $\operatorname{Sh}(\mathbb{V})_{K,N}$ extends to an action on $\mathcal{M}^S$. By [BLR90, Section 6, Example] (and [Vis04] for the stack case), after possibly enlarging S, $\mathcal{M}^S$ descends to $\operatorname{Spec}\mathcal{O}_E[S^{-1}]$. Let $\mathcal{Y}^S$ be the resulted Deligne-Mumford stack. By construction and B.2.26, we have $\mathcal{Y}^S_{\mathcal{O}_{E,(v)}} \simeq \mathcal{S}_v$ for $v \notin S$. Now, let $\mathcal{X}_K$ be the glueing of $\mathcal{Y}^S$ and $\mathcal{S}_v$ with $v \in S$. Then $\mathcal{X}_{K,\mathcal{O}_{E,(v)}} \simeq \mathcal{S}_v$ for every finite place v.

We check Assumption A.1.1. For $K' \subset K$ small enough with $K'^S = K^S$, the resulted Deligne-Mumford stack $\mathcal{Y}'^S$ is representable by [RSZ20, 5.2]. Note that $\operatorname{Spec}\mathcal{O}_N[S^{-1}] \to \operatorname{Spec}\mathcal{O}_E[S^{-1}]$ is unramified (so finite étale). Then Assumption A.1.1 (1) holds (with the current S) by construction. Similarly, Assumption A.1.1 (2) holds.

The uniqueness in (2) and (3) follows from the separatedness of the models. The existence follows from the construction and corresponding properties of the PEL type integral models in [RSZ20, Theorem 5.4]. We omit the details. □

Remark 4.4.5. If we drop condition (2) in Definition 4.4.2 on K, Theorem 4.4.4 (1)(2) still holds. Indeed, to construct $\mathcal{X}_K$, choose $K_1 \in \widetilde{K}_\Lambda$ such that $K_1 \subset K$. Let $\mathcal{X}_K$ be the quotient of $\mathcal{X}_{K_1}$ by K/K_1 where is the action is Theorem 4.4.4 (3).

By [RSZ20, Theorem 5.2] and our construction, we deduce the following lemma.

Lemma 4.4.6. *If $K_v = K_{\Lambda,v}$, then $\mathcal{X}_{K,\mathcal{O}_{E,(v)}}$ is smooth over* $\operatorname{Spec}\mathcal{O}_{E,(v)}$.

Remark 4.4.7. We may relax Assumption 4.4.1 (3) by allowing Λ_v to be almost self-dual. See B.2.28. Then Lemma 4.4.3 still holds. However, Lemma 4.4.6 does not hold any more.

Corollary 4.4.8. *A $\mathbb{Q}$-line bundle (in particular a line bundle) $\mathcal{L}_{K_\Lambda}$ on $\mathcal{X}_{K_\Lambda}$ extending L_{K_Λ} is ample. Moreover, it is unique as a $\mathbb{Q}$-line bundle.*

Proof. By Lemma 4.4.6, the corresponding divisors of two different extensions differ by a $\mathbb{Q}$-linear combination of special fibers of $\mathcal{X}_{K_\Lambda}$, which is 0 in $\operatorname{Ch}^1(\mathcal{X}_{K_\Lambda})_{\mathbb{Q}}$. The uniqueness follows. Now the ampleness follows from Lemma 4.4.3 (2). □

Below, by a line bundle, we mean a $\mathbb{Q}$-line bundle.

4.4.2. Admissible extensions

We want to define admissible extensions that are compatible as the level changes. So we consider a system of integral models.

Definition 4.4.9. (1) Let $\widetilde{\mathcal{X}}$ be the system $\{\mathcal{X}_K\}_{K\in\widetilde{K}_\Lambda}$ of integral models with transition morphisms $\pi_{K,K'}$ as in Theorem 4.4.4 (1)(2).

(2) For $h \in K_\Lambda$, define the 'right translation by h' automorphism on $\mathcal{X}_K$ to be the action of h as in Theorem 4.4.4 (3).

(3) Fix an arbitrarily ($\mathbb{Q}$-)line bundle $\mathcal{L}_{K_\Lambda}$ on $\mathcal{X}_{K_\Lambda}$ extending L_{K_Λ}. Let $\mathcal{L}_K = \pi^*_{K,K_\Lambda}\mathcal{L}_{K_\Lambda}$.

We use $\mathcal{L}$ to denote the compatible-under-pullback system $\{\mathcal{L}_K\}_{K\in\widetilde{K}_\Lambda}$ of ample line bundles on $\widetilde{\mathcal{X}} = \{\mathcal{X}_K\}_{K\in\widetilde{K}_\Lambda}$. The ampleness follows from the ampleness of $\mathcal{L}_{K_\Lambda}$ and finiteness of π_{K,K_Λ}. While using $\mathcal{L}$ for $\mathcal{X}_K$, we mean $\mathcal{L}_K$ so that it has the same meaning as before (we previously used $\mathcal{L}$ as the abbreviation of $\mathcal{L}_K$). Let $\overline{\mathcal{L}}_K$ be the corresponding hermitian line bundle with the hermitian metric as in 4.2.1, and define $\overline{\mathcal{L}}$ accordingly.

By Proposition A.3.8 – that is, the compatibility of the formation of admissible (Chow) cycles under flat pullback – we can define the direct limit group of admissible arithmetic divisors along the directed poset $\widetilde{K}_\Lambda$

$$\widehat{Z}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}}) = \varinjlim_{K\in\widetilde{K}_\Lambda} \widehat{Z}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K), \tag{4.20}$$

$$\widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}}) = \varinjlim_{K\in\widetilde{K}_\Lambda} \widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K), \tag{4.21}$$

where the transition maps are $\pi^*_{K,K'}$'s. By the projection formula (which gives that the composition of pullback by pushforward is the multiplication by the degree of the finite flat morphism), the natural maps

$$\widehat{Z}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K) \to \widehat{Z}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}}),\ \mathrm{Ch}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K) \to \mathrm{Ch}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}})$$

are injective and we understand the formers as subspaces of the latters, respectively. The embeddings $\mathbb{C} \hookrightarrow \widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K)$ (Definition 4.3.1) are then the same embedding

$$\mathbb{C} \hookrightarrow \widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}}). \tag{4.22}$$

Remark 4.4.10. If we drop condition (2) in Definition 4.4.2 as in Remark 4.4.10, (4.20)–(4.22) do not change.

Definition 4.4.11. Let $\overline{\mathcal{S}}(\mathbb{V})^{\widetilde{K}_\Lambda} \subset \overline{\mathcal{S}}(\mathbb{V})$ consist of functions invariant by some $K \in \widetilde{K}_\Lambda$.

For $t \in F_{>0}$, $\phi \in \overline{\mathcal{S}}(\mathbb{V})^{\widetilde{K}_\Lambda}$ and $K \in \widetilde{K}_\Lambda$ stabilizing ϕ, consider the normalized admissible extension $Z_t(\omega(g)\phi)^{\overline{\mathcal{L}}}_K$ of $Z_t(\omega(g)\phi)_K$ (as in (4.16)). By Lemma 4.1.5 and Proposition A.3.8,

$$Z_t(\omega(g)\phi)^{\overline{\mathcal{L}}} := Z_t(\omega(g)\phi)^{\overline{\mathcal{L}}}_K \in \widehat{Z}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K) \subset \widehat{Z}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}}) \tag{4.23}$$

is independent of K. And by definition,

$$c_1(\overline{\mathcal{L}}^\vee) := c_1(\overline{\mathcal{L}}^\vee_K) \in \mathrm{Ch}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K) \subset \mathrm{Ch}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}})$$

is independent of K.

For $h \in K_\Lambda$, the 'right translation by h' in $\mathrm{Aut}(\mathcal{X}_K/\mathcal{X}_{K_\Lambda})$ fixes $\mathcal{L}_K$ by definition. Thus, it induces an automorphism on $\widehat{Z}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}})$ and $\widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}})$. Since it sends $Z_t(g,\phi)$ to $Z_t(g,\omega(h)\phi)$, by Proposition A.3.8, we have the following lemma.

Lemma 4.4.12. *The 'right translation by h' automorphism is the identity map on the image of $\mathbb{C} \hookrightarrow \widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}})$, fixes $c_1(\overline{\mathcal{L}}^\vee)$, and sends $Z_t(g,\phi)^{\overline{\mathcal{L}}}$ to $Z_t(g,\omega(h)\phi)^{\overline{\mathcal{L}}}$.*

4.4.3. Conjecture

We define generating series of admissible arithmetic divisors in two ways, essentially by choosing different Green functions. (In fact, there will be a third one in 5.2.2.) The first (resp. second) definition is made toward Problem 4.3.4 (resp. Problem 4.3.3).

Recall the holomorphic part $E'_{t,\mathrm{f}}(0,g,\phi)$ of the derivative of $E_t(s,g,\phi)$ at $s=0$. See (3.29).

Definition 4.4.13. For $\phi \in \overline{\mathcal{S}}(\mathbb{V})^{\widetilde{K}_\Lambda}$, $t \in F_{>0}$ and $g \in G(\mathbb{A}_F)$, let

$$z_t(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{e}} = [Z_t(\omega(g)\phi)^{\overline{\mathcal{L}}}] + \left(E'_{t,\mathrm{f}}(0,g,\phi) + E_t(0,g,\phi)\log \mathrm{Nm}_{F/\mathbb{Q}} t\right) \in \widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}}).$$

For $a \in \mathbb{C}$ and $g \in G(\mathbb{A}_F)$, let

$$z(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{e},a} = \omega(g)\phi(0)\left(c_1(\overline{\mathcal{L}}^\vee) + a\right) + \sum_{t\in F_{>0}} z_t(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{e}}.$$

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Remark 4.4.14. (1) We rewrite the definition of $z_t(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{e}}$ as follows, which is closer to the notation '$\left([Z_t^{\overline{\mathcal{L}}}]+\mathfrak{e}_t\right)q^t$' in (1.2). For $t\in F_{>0}$, we have the t-th Fourier coefficient $\frac{E_t(0,g,\phi)}{W^{\mathfrak{w}}_{\infty,t}(1)}$, $g\in G(\mathbb{A}_F^\infty)$, of $E(0,g,\phi)$. See 2.7. We similarly and formally consider $\frac{E'_{t,\mathrm{f}}(0,g,\phi)}{W^{\mathfrak{w}}_{\infty,t}(1)}$, $g\in G(\mathbb{A}_F^\infty)$ as the 't-th Fourier coefficient' of $E'_{t,\mathrm{f}}(0,g,\phi)$. Let

$$\mathfrak{e}_t(g,\phi^\infty)=\frac{E'_t(0,g,\phi)}{W^{\mathfrak{w}}_{\infty,t}(1)}+\frac{E_{t,\mathrm{f}}(0,g,\phi)}{W^{\mathfrak{w}}_{\infty,t}(1)}\log \mathrm{Nm}_{F/\mathbb{Q}}t,\ g\in G(\mathbb{A}_F^\infty). \tag{4.24}$$

Then $\mathfrak{e}_t(1,\phi^\infty)$ is the constant $\mathfrak{e}_t$ in (1.4). And we can rewrite

$$z_t(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{e}}=\left([Z_t(\omega(g^\infty)\phi^\infty)^{\overline{\mathcal{L}}}]+\mathfrak{e}_t(g^\infty,\phi^\infty)\right)W^{\mathfrak{w}}_{\infty,t}(g_\infty).$$

(2) We also expect that the (proposed) infinite component of $e_t(g,\phi)$ in Problem 4.3.4 is $E'_{t,\mathrm{f}}(0,g,\phi)+E_t(0,g,\phi)\log\mathrm{Nm}_{F/\mathbb{Q}}t$ for any unitary Shimura variety. In particular, in the situation of this section, the v-component is 0 for every finite place v. It is related to Lemma 4.4.6. (The smoothness is 'exotic' at places where E/F is ramified.)

By (3.10), (3.14) and Lemma 4.4.12, we have the following lemma.

Lemma 4.4.15. *For $h\in K_\Lambda$, the ''right translation by h' automorphism on $\mathcal{X}_K$ sends $z(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{e},a}$ to $z(g,\omega(h)\phi)^{\overline{\mathcal{L}}}_{\mathfrak{e},a}$.*

Now we specify the correct constant a to be used in $z(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{e},a}$. Let

$$\begin{aligned}\mathfrak{a}&=\log|\mathrm{Disc}_F|-\frac{[F:\mathbb{Q}]}{n}-\left(\mathfrak{b}[F:\mathbb{Q}]-\frac{1}{2}\mathfrak{c}\right)-c_1(\overline{\mathcal{L}}^\vee_{K_\Lambda})\cdot\mathcal{P}_{K_\Lambda}\\&=c_1(\overline{\mathcal{L}}_{K_\Lambda})\cdot\mathcal{P}_{K_\Lambda}+2\frac{L'(0,\eta)}{L(0,\eta)}+\log|\mathrm{Disc}_E|-\mathfrak{b}[F:\mathbb{Q}]-\frac{[F:\mathbb{Q}]}{n},\end{aligned} \tag{4.25}$$

where $\mathfrak{b}$ is as in (3.30), $\mathfrak{c}$ is as in (3.15), Disc_F is the discriminant of $F/\mathbb{Q}$, and $\mathcal{P}_{K_\Lambda}\in Z^1(\mathcal{X}_{K_\Lambda})$ is a CM cycle to be precisely defined in Definition 5.1.10.

Remark 4.4.16. (1) The definition of $\mathfrak{a}$ is rather complicated. The reader may skip it for the moment.

(2) Looking at [YZ18, Theorem 1.7] and [YZ18, p 590], one can deduce cancellation between the terms in the definition of $\mathfrak{a}$ if one can, expectably, relate the Shimura varieties here and in loc. cit..

Conjecture 4.4.17. *Assume Assumption 4.4.1. Let $\phi\in\overline{\mathcal{S}}(\mathbb{V})^{\widetilde{K}_\Lambda}$. For $g\in G(\mathbb{A})$, we have*

$$z(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{e},\mathfrak{a}}\in\mathcal{A}_{\mathrm{hol}}(G,\mathfrak{w})\otimes\widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}}). \tag{4.26}$$

For the second definition, we modify the normalized admissible extensions of special divisors $Z(x)$'s instead of weighted special divisors $Z_t(\omega(g)\phi)^{\overline{\mathcal{L}}}$. Unfortunately, the modification does not only depend on x, and the result generating series is not holomorphic.

Definition 4.4.18. For $x\in\mathbb{V}^\infty$ with $q(x)\in F^\times$, $g\in G(\mathbb{A}_{F,\infty})$ and $K\in\widetilde{K}_\Lambda$, let $Z(x)^{\mathcal{L}}$ be the divisor on $\mathcal{X}_K$ that is the normalized admissible extension of $Z(x)$ with respect to $\mathcal{L}$, and $\mathcal{G}^{\overline{L}_{E_v}}_{Z(x)_{E_v}}$ the normalized admissible Green function for $Z(x)_{E_v}$ with respect to $\overline{L}_{E_v}$. Let

$$Z(x,g)^{\overline{\mathcal{L}}}_{K,\mathfrak{k}}=\left(Z(x)^{\mathcal{L}},\left(\mathcal{G}^{\overline{L}_{E_v}}_{Z(x)_{E_v}}+\mathfrak{k}(x,g_v)\right)_{v\in\infty}\right)\in\widehat{Z}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K)\subset\widehat{Z}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}}).$$

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Here, $\mathfrak{k}(x, g_v)$ is defined in Definition 4.2.6. For $\phi \in \overline{\mathcal{S}}(\mathbb{V})^K$, $a \in \mathbb{C}$ and $g \in G(\mathbb{A}_F)$, let

$$z(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{k},a} = \omega(g)\phi(0)\Big(c_1(\overline{\mathcal{L}}^\vee) - \log\delta_\infty(g_\infty) + [F:\mathbb{Q}](\log\pi - (\log\Gamma)'(n+1)) + a\Big) + \sum_{x\in\mathbb{V}^\infty,\ q(x)\in F^\times} \omega(g^\infty)\phi^\infty(x)[Z(x,g_\infty)^{\overline{\mathcal{L}}}_{K,\mathfrak{k}}]W^{\mathfrak{w}}_{\infty,q(x)}(g_\infty).$$

Though $Z(x,g)^{\overline{\mathcal{L}}}_{K,\mathfrak{k}}$, as an element in $\widehat{Z}^1_{\mathcal{L},\mathbb{C}}(\widetilde{\mathcal{X}})$, depends on K, $z(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{k},a}$ does not. Indeed, by Lemma 3.3.9 (1), $E_0(0,g,\phi) = \omega(g)\phi(0)$. Then by Theorem 4.2.7, we have

$$z(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{e},a} - z(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{k},a} = E'(0,g,\phi) - E(0,g,\phi)[F:\mathbb{Q}](\log\pi - (\log\Gamma)'(n+1)). \tag{4.27}$$

Then since $z(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{e},a}$ does not depend on K, neither does $z(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{k},a}$.

The above reasoning also shows that (4.26) is equivalent to

$$z(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{k},\mathfrak{a}} \in \mathcal{A}(G,\mathfrak{w}) \otimes \widehat{\mathrm{Ch}}^1_{\mathcal{L},\mathbb{C}}(\widetilde{\mathcal{X}}).$$

4.4.4. Modularity theorems

We need some notations to state our theorems. Let $\mathfrak{Ram}$ be the set of finite places of F nonsplit in E, that are ramified in E or over $\mathbb{Q}$. Let

$$\mathbb{G} = P(\mathbb{A}_{F,\mathfrak{Ram}})G(\mathbb{A}_F^{\mathfrak{Ram}}).$$

Let

$$\overline{\mathcal{S}}(\mathbb{V})^{\widetilde{K}_\Lambda}_{\mathfrak{Ram}} \subset \overline{\mathcal{S}}(\mathbb{V})^{\widetilde{K}_\Lambda}$$

be the span of pure tensors ϕ such that for every finite place v of F nonsplit in E, $\phi_v = \omega(g)1_{\Lambda_v}$ for some $g \in \mathbb{G}_v$. In particular, we have no condition for $\phi \in \overline{\mathcal{S}}(\mathbb{V})^{\widetilde{K}_\Lambda}_{\mathfrak{Ram}}$ over split places. The following proposition and remark further show that we have no condition for ϕ outside $\mathfrak{Ram}$.

Proposition 4.4.19. *For a finite place v of F inert in E (so that Λ_v is self-dual by Assumption 4.4.1) such that $\chi_{\mathbb{V},v}$ is unramified, the span of $\{\omega(g)1_{\Lambda_v}, g \in G(F_v)\}$ is $\mathcal{S}(\mathbb{V}(E_v))^{K_{\Lambda,v}}$.*

Proof. If v has residue characteristic $\neq 2$, this is a special case of [How79, Theorem 10.2]. In general, let $K_v^{\max} \subset G(F_v)$ be as in (2.5). Embedding $\mathcal{S}(\mathbb{V}(E_v))^{K_{\Lambda,v}\times K_v^{\max}}$ in an induced representation as [Ral82, (3.1)]. It is routine to show that $\mathcal{S}(\mathbb{V}(E_v))^{K_{\Lambda,v}\times K_v^{\max}}$ is generated by 1_{Λ_v} as a module over the Hecke algebra of bi-$K_v^{\max}$-invariant Schwartz functions on $G(F_v)$. Then the proposition follows from Kudla's supercuspidal support theorem for big theta lift. See [GI14, Proposition 5.2] □

Remark 4.4.20. We may choose $\chi_{\mathbb{V}}$ such $\chi_{\mathbb{V},v}$ is unramified if v is inert in E.

Let $\mathcal{A}_{\mathrm{hol}}(\mathbb{G},\mathfrak{w})$ and $\mathcal{A}(\mathbb{G},\mathfrak{w})$ be the restrictions of $\mathcal{A}_{\mathrm{hol}}(G,\mathfrak{w})$ and $\mathcal{A}(G,\mathfrak{w})$ to $\mathbb{G}$, respectively.

Theorem 4.4.21. *Assume Assumption 4.4.1. Let $\phi \in \overline{\mathcal{S}}(\mathbb{V})^{\widetilde{K}_\Lambda}_{\mathfrak{Ram}}$. For $g \in \mathbb{G}$, we have*

$$z(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{e},\mathfrak{a}} \in \mathcal{A}_{\mathrm{hol}}(\mathbb{G},\mathfrak{w}) \otimes \widehat{\mathrm{Ch}}^1_{\mathcal{L},\mathbb{C}}(\widetilde{\mathcal{X}}), \tag{4.28}$$

and

$$z(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{k},\mathfrak{a}} \in \mathcal{A}(\mathbb{G},\mathfrak{w}) \otimes \widehat{\mathrm{Ch}}^1_{\mathcal{L},\mathbb{C}}(\widetilde{\mathcal{X}}). \tag{4.29}$$

In fact, by (4.27), (4.28) and (4.29) are equivalent. Theorem 4.4.21 will be proved in 5.2.4.

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Remark 4.4.22. (1) The restriction of (4.28) to $g \in G(\mathbb{A}_\infty) \subset \mathbb{G}$ gives Theorem 1.1.1, with a less strict condition on ϕ_v, $v \in \mathfrak{Ram}$. Remark 4.4.10 allows K_v to be arbitrary for v split in E.

(2) By the density of $\mathbb{G} \subset G(F)\backslash G(\mathbb{A}_F)$, the restriction gives

$$\mathcal{A}_{\mathrm{hol}}(G, \mathfrak{w}) \simeq \mathcal{A}_{\mathrm{hol}}(\mathbb{G}, \mathfrak{w}), \ \mathcal{A}(G, \mathfrak{w}) \simeq \mathcal{A}(\mathbb{G}, \mathfrak{w}).$$

In particular, $z(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{e},\mathfrak{a}}$ with $g \in \mathbb{G}$ extends uniquely to an element in $\mathcal{A}_{\mathrm{hol}}(G, \mathfrak{w}) \otimes \widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}})$. Conjecture 4.4.17 predicts that this extension is $z(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{e},\mathfrak{a}}$ with $g \in G(\mathbb{A})$. However, it seems not trivial to check this prediction. The same discussion applies to $z(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{f},\mathfrak{a}}$.

Definition 4.4.23. For $x \in \mathbb{V}^\infty$ with $q(x) \in F^\times$ and $g \in G(\mathbb{A}_{F,\infty})$, let

$$Z(x,g)^{\mathcal{L},\mathrm{Kud}} = \left(Z(x)^{\mathcal{L}}_K, \left(\mathcal{G}^{\mathrm{Kud}}_{Z(x)_{E_v}}(g)\right)_{v\in\infty}\right) \in \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{X}_K).$$

Here, Kudla's Green function $\mathcal{G}^{\mathrm{Kud}}_{Z(x)_{E_v}}$ is defined in (4.12).

For $\phi \in \overline{\mathcal{S}}(\mathbb{V})^K$, $a \in \mathbb{C}$ and $g \in G(\mathbb{A}_F)$, let

$$\begin{aligned} z(g,\phi)^{\mathcal{L},\mathrm{Kud}}_a &= \omega(g)\phi(0)\Big(c_1(\overline{\mathcal{L}}^\vee) - \log\delta_\infty(g_\infty) + [F:\mathbb{Q}](\log\pi - (\log\Gamma)'(n+1)) + a\Big) \\ &+ \sum_{x\in\mathbb{V}^\infty,\ q(x)\in F^\times} \omega(g^\infty)\phi^\infty(x)[Z(x,g_\infty)^{\mathcal{L},\mathrm{Kud}}]W^{\mathfrak{w}}_{\infty,q(x)}(g_\infty). \end{aligned}$$

It is directly to check that $z(g,\phi)^{\mathcal{L},\mathrm{Kud}}_a$ is compatible under pullbacks by $\pi_{K,K'}$'s.

Define the topology on $\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{X}_K)$ as follows. Let $C^\infty(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})^\circ$ be the L^2-orthogonal complement of $\mathrm{LC}(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})$ in $C^\infty(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})$, endowed with L^∞-topology. Then $\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{X}_K)$ is the direct sum of $C^\infty(\mathrm{Sh}(\mathbb{V})_{K,E_\infty})^\circ$ and the finite dimensional subspace of cycles with harmonic curvatures at ∞. Endow $\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{X}_K)$ with the direct sum topology. Let $\mathcal{A}\left(\mathbb{G}, \mathfrak{w}, \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{X}_K)\right)$ be the restrictions of $\mathcal{A}\left(G, \mathfrak{w}, \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{X}_K)\right)$ to $\mathbb{G}$.

Theorem 4.2.10 and (4.28) of Theorem 4.4.21 imply the following theorem.

Theorem 4.4.24. *Let $\phi \in \overline{\mathcal{S}}(\mathbb{V})^K_{\mathfrak{Ram}}$, where $K \in \widetilde{K}_\Lambda$. The generating series $z(g,\phi)^{\mathcal{L},\mathrm{Kud}}_{\mathfrak{a}}$ of $\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{X}_K)$-valued functions on $\mathbb{G}$ pointwise converges to an element in $\mathcal{A}\left(\mathbb{G}, \mathfrak{w}, \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{X}_K)\right)$.*

5. Arithmetic mixed Siegel-Weil formula

In this section, we prove our modularity theorem (Theorem 4.4.21) above using an arithmetic analog of the mixed Siegel-Weil formula (3.19). First, we define CM cycles. Then, we state the formula and use this formula to prove Theorem 4.4.21. We end this section by discussing some possible future generalizations and applications of our results. We continue to use notations and assumptions in 4.4.

5.1. CM cycles

In this subsection, we define an orthogonal decomposition

$$\mathbb{V} = \mathbb{W} \oplus V^\sharp(\mathbb{A}_E), \tag{5.1}$$

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where $\mathbb{W}$ is an incoherent hermitian space over $\mathbb{A}_E$ of dimension 1, and $V^\sharp$ is a hermitian space over E of dimension n. Then we define a CM cycle

$$\mathcal{P}_{\mathbb{W},K} \in Z_1(\mathcal{X}_K)_{\mathbb{Q}}$$

associated to the 0-dimensional Shimura variety for $\mathbb{W}$, normalized to be of generic degree 1.

5.1.1. Lattices at unramified places

For a finite place v of F unramified in E, by [Jac62, Section 7], there exists $e_v^{(0)}, \dots, e_v^{(n)} \in \Lambda_v$ of unit norms, such that Λ_v is their orthogonal direct sum.

5.1.2. Lattices at ramified places

Let $\mathfrak{Ram}_{E/F}$ be the set of finite places of F ramified in E. For $v \in \mathfrak{Ram}_{E/F}$, let $\mathbb{M}_v$ be the $\mathcal{O}_{E_v}$-lattice of rank 2 with an isotropic basis $\{X, Y\}$ such that $\langle X, Y\rangle = \varpi_{E_v}$. Then $\mathbb{M}_v$ is a ϖ_{E_v}-modular lattice in $\mathbb{M}_v \otimes E_v$, and its determinant with respect to this basis is $-\mathrm{Nm}(\varpi_{E_v})$. The hermitian space $\mathbb{M}_v \otimes E_v$ has determinant $-1 \in F_v^\times/\mathrm{Nm}(E_v^\times)$. In the other direction, starting with a 2-dimensional hermitian space H over E_v of determinant $-1 \in F_v^\times/\mathrm{Nm}(E_v^\times)$, let $e_v^{(0)}, e_v^{(1)} \in H$ be orthogonal such that $q\left(e_v^{(0)}\right), q\left(e_v^{(1)}\right) \in \mathcal{O}_{F_v}^\times$. Then one can choose $\mathbb{M}_v \subset \mathcal{O}_{E_v} e_v^{(0)} \oplus \mathcal{O}_{E_v} e_v^{(1)}$ to be the preimage of one of the two isotropic lines in $(\mathcal{O}_{E_v} e_v^{(0)} \oplus \mathcal{O}_{E_v} e_v^{(1)})/\varpi_{E_v}$. In particular, one easily sees that

$$\mathbb{M}_v \cap \left(\mathcal{O}_{E_v} e_v^{(0)} \oplus \varpi_{E_v} \mathcal{O}_{E_v} e_v^{(1)}\right) = \varpi_{E_v} \mathcal{O}_{E_v} e_v^{(0)} \oplus \varpi_{E_v} \mathcal{O}_{E_v} e_v^{(1)} \tag{5.2}$$

and

$$\mathbb{M}_v \cap \left(\mathcal{O}_{E_v} e_v^{(0)} \oplus \mathcal{O}_{E_v}^\times e_v^{(1)}\right) \subset \mathcal{O}_{E_v}^\times e_v^{(0)} \oplus \mathcal{O}_{E_v}^\times e_v^{(1)}. \tag{5.3}$$

These two relations will only be used in the proof of Lemma 5.2.3.

Recall that for $v \in \mathfrak{Ram}_{E/F}$, $v \nmid 2$ by Assumption 4.4.1 (1), and Λ_v is (almost) ϖ_{E_v}-modular as in Assumption 4.4.1 (4). Recall that the rank of Λ_v is $n+1$.

Lemma 5.1.1 [Jac62, Section 8]. (1) *If n is odd, then $\Lambda_v \simeq \mathbb{M}_v^{\oplus(n+1)/2}$.*

(2) *If n is even, then Λ_v is the orthogonal direct sum of $n/2$-copies of $\mathbb{M}_v$ and a rank-1 hermitian $\mathcal{O}_{E_v}$-module with determinant in $\mathcal{O}_{F_v}^\times$.*

Remark 5.1.2. Using Lemma 5.1.1 and computing discriminants, we can classify incoherent $\mathbb{V}$ containing a lattice Λ as in Assumption 4.4.1:

(1) If n is odd, then there exist a Λ as in Assumption 4.4.1 if and only if $(n+1)/2$ is odd and $[F : \mathbb{Q}]$ is odd.
(2) If n is even, then there exist a Λ as in Assumption 4.4.1.

5.1.3. CM cycles

For a finite place $v \notin \mathfrak{Ram}_{E/F}$, let $e_v^{(0)}$ be as in 5.1.1. For $v \in \mathfrak{Ram}_{E/F}$, let $\Lambda_{v,1} \subset \Lambda_v$ be a copy of $\mathbb{M}_v$. See Lemma 5.1.1. Let $e_v^{(0)} \in E_v\Lambda_{v,1}$ such that

$$q\left(e_v^{(0)}\right) \in \mathcal{O}_{F_v}^\times,\ q\left(e_v^{(0)}\right) = \det(\mathbb{V}(E_v)) \in F_v^\times/\mathrm{Nm}(E_v^\times).$$

Let $\mathbb{W}$ be the restricted tensor product of $E_v e_v^{(0)}$, for every $v \notin \infty$, and a 1-dimensional subspace of $\mathbb{V}(E_v)$, for every $v \in \infty$. Note that since

$$\det(\mathbb{W}_v) = \det(\mathbb{V}(E_v)) \in F_v^\times/\mathrm{Nm}(E_v^\times),$$

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$\mathbb{W}$ is incoherent. The orthogonal complement of $\mathbb{W}$ in $\mathbb{V}$ is coherent of dimension n. We denote the corresponding hermitian space over E by $V^\sharp$. This gives (5.1).

Let $K_{\mathbb{W}} = K \cap U(\mathbb{W}^\infty)$. The morphism $\mathrm{Sh}(\mathbb{W})_{K_{\mathbb{W}}} \to \mathrm{Sh}(\mathbb{V})_K$, analogous to (4.1), defines a zero cycle on $\mathrm{Sh}(\mathbb{V})_K$. (Indeed, this is a 'simple special zero cycle' compared with 4.1.1.)

Definition 5.1.3. Let $d_{\mathbb{W},K}$ be the degree of this zero cycle. Let $\mathcal{P}_{\mathbb{W},K} \in Z_1(\mathcal{X}_K)_{\mathbb{Q}}$ be $1/d_{\mathbb{W},K}$ times the Zariski closure of this cycle.

Recall that K is normal in K_Λ (Definition 4.4.2), and for $h \in K_\Lambda$, there is the 'right translation by h' automorphism on $\mathcal{X}_K$ (Definition 4.4.9 (3)).

Lemma 5.1.4. *The 'right translation by h' automorphism on $\mathcal{X}_K$ sends $\mathcal{P}_{\mathbb{W},K}$ to $\mathcal{P}_{h\mathbb{W},K}$.*

Proof. When restricted to the generic fiber, the proof goes in the same way as [Kud97a, LEMMA 2.2 (iv)]. Taking Zariski closure, we get the lemma. □

In particular, for $k \in K$, $d_{\mathbb{W},K} = d_{k\mathbb{W},K}$, and $\mathcal{P}_{\mathbb{W},K} = \mathcal{P}_{k\mathbb{W},K}$. Then by the flatness of $\pi_{K,K'}$ (and the commutativity of taking Zariski closure and flat pullback) and [Liu11a, Proposition 3.2],

$$\pi^*_{K,K'}\mathcal{P}_{\mathbb{W},K'} = \sum_{k \in (U(\mathbb{W}) \cap K')\backslash K'/K} \frac{d_{k^{-1}\mathbb{W},K}}{d_{\mathbb{W},K'}} \mathcal{P}_{k^{-1}\mathbb{W},K}, \tag{5.4}$$

where each summand is independent of the choice of the representatives k. We shall later abbreviate $(U(\mathbb{W}) \cap K')\backslash K'/K$ as $U(\mathbb{W})\backslash K'/K$.

5.1.4. Another description

We will give another description of $\mathcal{P}_{\mathbb{W},K}$ that shows the independence of $\mathcal{P}_{\mathbb{W},K_\Lambda}$ on $\mathbb{W}$. (It will also be used in 6.1.2 to compute intersection numbers.) Before that, we introduce new lattices, open compact subgroups and Shimura varieties.

For $v \in \mathfrak{Ram}_{E/F}$, let $e_v^{(1)} \in \Lambda_{v,1}$ be orthogonal to $e_v^{(0)}$ such that $q\left(e_v^{(1)}\right) \in \mathcal{O}_{F_v}^\times$, and let $\Lambda_{v,1}^\perp \subset \Lambda_v$ be the orthogonal complement of $\Lambda_{v,1}$. Then we have

$$\varpi_{E_v}\left(\mathcal{O}_{E_v}e_v^{(0)} \oplus \mathcal{O}_{E_v}e_v^{(1)}\right) \subset \Lambda_{v,1} \subset \mathcal{O}_{E_v}e_v^{(0)} \oplus \mathcal{O}_{E_v}e_v^{(1)},$$

where each inclusion is of colength 1, and $\Lambda_v = \Lambda_{v,1} \oplus \Lambda_{v,1}^\perp$.

Let $K_v^\dagger \subset U(\mathbb{V}_v)$ be the stabilizer of $\mathcal{O}_{E_v}e_v^{(0)} \oplus \mathcal{O}_{E_v}e_v^{(1)} \oplus \Lambda_{v,1}^\perp$, and $K^\dagger = K_v^\dagger \prod_{u \neq v} K_u$.

Lemma 5.1.5. *We have $[K_v^\dagger : K_v^\dagger \cap K_v] = 2$.*

Proof. The index is the cardinality of the isotropic lines in $\left(\mathcal{O}_{E_v}e_v^{(0)} \oplus \mathcal{O}_{E_v}e_v^{(1)}\right)/\varpi_{E_v}$, which is 2. □

Let $K_{\mathbb{W},v}^{(0)} \subset U(\mathbb{W}_v)$ be the stabilizer of $\mathcal{O}_{E_v}e_v^{(0)}$, that is, $K_{\mathbb{W},v}^{(0)} = U(\mathbb{W}_v)$, and $K_{\mathbb{W}}^{(0)} = K_{\mathbb{W},v}^{(0)} \prod_{u \neq v} K_{\mathbb{W},u}$. Then we have a diagram of morphisms of Shimura varieties

$$\mathrm{Sh}(\mathbb{W})_{K_{\mathbb{W}}^{(0)}} \to \mathrm{Sh}(\mathbb{V})_{K^\dagger} \leftarrow \mathrm{Sh}(\mathbb{V})_{K^\dagger \cap K} \to \mathrm{Sh}(\mathbb{V})_K. \tag{5.5}$$

Applying pushfoward, pullback and pushfoward along the diagram (5.5) to the fundamental cycle of $\mathrm{Sh}(\mathbb{W})_{K_{\mathbb{W}}^{(0)}}$, we obtain a zero cycle on $\mathrm{Sh}(\mathbb{V})_K$. Divide it by its degree to obtain a zero cycle of degree 1.

Lemma 5.1.6. *The Zariski closure of this degree 1 zero cycle is $\mathcal{P}_{\mathbb{W},K}$.*

Proof. By Lemma 5.1.5, $[K^\dagger : K^\dagger \cap K] = 2$. Since $K_{\mathbb{W},v}^{(0)} \not\subset K$, $K_{\mathbb{W},v}^{(0)}(K^\dagger \cap K) = K^\dagger$. Then the fiber product of the first two morphisms in (5.5) is $\mathrm{Sh}(\mathbb{W})_{(K^\dagger \cap K) \cap U(\mathbb{W}^\infty)}$ (an analog of (5.4)). The natural morphism $\mathrm{Sh}(\mathbb{W})_{(K^\dagger \cap K) \cap U(\mathbb{W}^\infty)} \to \mathrm{Sh}(\mathbb{V})_K$ factors through $\mathrm{Sh}(\mathbb{W})_{K_{\mathbb{W}}}$. The lemma follows. □

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Remark 5.1.7. One may define $\mathcal{P}_{\mathbb{W},K}$ via a diagram of integral models similar to (5.5), as in [RSZ20, (4.30)].

Lemma 5.1.8. *Let $h \in U(E_v\Lambda_{v,1}) \times \{1_{E_v\Lambda_{v,1}^\perp}\} \subset U(\mathbb{V}(E_v))$ such that $\Lambda_{v,1} \subset h\left(\mathcal{O}_{E_v}e_v^{(0)} \oplus \mathcal{O}_{E_v}e_v^{(1)}\right)$ and is the preimage of one of the two isotropic lines in the reduction modulo ϖ_{E_v}. Then $h \in K_v^\dagger$.*

Proof. Let $K^c = K^\dagger \backslash K$. The two preimages of the two isotropic lines in the reduction of $h\left(\mathcal{O}_{E_v}e_v^{(0)} \oplus \mathcal{O}_{E_v}e_v^{(1)}\right)$ are $h\Lambda_{v,1}$ and $hK^c\Lambda_{v,1}$. Then either $h\Lambda_{v,1} = \Lambda_{v,1}$ or $hK^c\Lambda_{v,1} = \Lambda_{v,1}$. Each implies $h \in K_v^\dagger$. □

Proposition 5.1.9. *The CM cycle $\mathcal{P}_{\mathbb{W},K_\Lambda}$ does not depend on the choice of $\mathbb{W}$.*

Proof. To define $\mathcal{P}_{\mathbb{W},K_\Lambda}$, we specify $e_v^{(0)} \in \mathbb{W}$ as in 5.1.3. For $v \notin \mathfrak{Ram}_{E/F}$, the choices of $e_v^{(0)}$ differ by $K_{\Lambda,v}$ actions, which do not change $\mathcal{P}_{\mathbb{W},K_\Lambda}$ by Lemma 5.1.4. For $v \in \mathfrak{Ram}_{E/F}$, the choices of $\Lambda_{v,1}$ differ by $K_{\Lambda,v}$ actions. See Lemma 5.1.1. By Lemma 5.1.8, we only need to show that $\mathcal{P}_{\mathbb{W},K_\Lambda} = \mathcal{P}_{h\mathbb{W},K_\Lambda}$ for $h \in K_v^\dagger$, where $v \in \mathfrak{Ram}_{E/F}$. We use Lemma 5.1.6. The pushforward of the fundamental cycle of $\mathrm{Sh}(\mathbb{W})_{K_{\mathbb{W}}^{(0)}}$ by the first map in (5.5) is the same as the one obtained by replacing $\mathbb{W}$ by $h\mathbb{W}$, by the analog of Lemma 5.1.4 (on the generic fiber). □

Definition 5.1.10. We denote $\mathcal{P}_{\mathbb{W},K_\Lambda}$ by $\mathcal{P}_{K_\Lambda}$.

This definition is only used in (4.25). Later, we will still use $\mathcal{P}_{\mathbb{W},K_\Lambda}$ for the uniformity of the notation as the level changes. (For $K \neq K_\Lambda$, $\mathcal{P}_{\mathbb{W},K}$ depends on the choice of $\mathbb{W}$.)

5.2. Formula

The arithmetic mixed Siegel-Weil formula compares the generating series of arithmetic intersection numbers between arithmetic special divisors and CM 1-cycles on the integral models with an explicit automorphic form. We use this formula to prove our main theorem Theorem 4.4.21.

5.2.1. Error functions

Both sides of the arithmetic mixed Siegel-Weil formula will have decompositions into local components (we will see in the proof in Section 6). We define some functions measuring the difference between these local components, and they will appear in the explicit automorphic form.

For a place v of F nonsplit in E, let W be the v-nearby hermitian space of $\mathbb{W}$. See 2.2. Define the orthogonal direct sum

$$V = W \oplus V^\sharp.$$

Then we have isomorpisms

$$V(\mathbb{A}_F^v) \simeq \mathbb{V}^{\infty,v},\ U(V(\mathbb{A}_E^v)) \simeq U(\mathbb{V}^v), \tag{5.6}$$

and similar isomorphisms for W and $\mathbb{W}$. Consider

$$V(E_v) - V^\sharp(E_v) = \{(x_1, x_2) \in W(E_v) \oplus V^\sharp(E_v) : x_1 \neq 0\}.$$

Let $\Lambda_v^\sharp \subset \mathbb{W}_v$ be the orthogonal complement of $\mathcal{O}_{E_v}e_v^{(0)}$ in Λ_v. For $x = (x_1, x_2) \in V(E_v) - V^\sharp(E_v)$, let

$$\phi_v'(x) := W\theta_{v,x}'(0,1,\phi_v) - (v(q(x_1)) + 1)1_{\mathcal{O}_{F_v}}(q(x_1))1_{\Lambda_v^\sharp}(x_2)\log q_{F_v}, \tag{5.7}$$

where the smooth function $W\theta_{v,x}'(0,1,\phi_v)$ on $V(E_v) - V^\sharp(E_v)$ is as in (3.21).

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Note that $\Lambda_v = \mathcal{O}_{E_v} e_v^{(0)} \oplus \Lambda_v^\sharp$. So the computation for $\phi'_v(x)$ is only on the component $\mathcal{O}_{E_v} e_v^{(0)}$, and we can apply computations in [YZ18, YZZ13].

Lemma 5.2.1 [YZZ13, Proposition 6.8]. *Assume that $v \notin \mathfrak{Ram}$. Then $\phi'_v = 0$.*

Let $c(g, \phi_v)$ be as below (3.23). Recall that Diff_v is the different of F_v over $\mathbb{Q}_v$.

Lemma 5.2.2 [YZ18, Lemma 9.2]. *Assume that $v \in \mathfrak{Ram} - \mathfrak{Ram}_{E/F}$. Then ϕ'_v extends to a Schwartz function on $V(E_v)$ such that $c(1, \phi_v) - 2\phi'_v(0) = 2\log|\mathrm{Diff}_v|_v$.*

For $v \in \mathfrak{Ram}_{E/F}$, as in 5.1.4, we have

$$\varpi_{E_v}\left(\mathcal{O}_{E_v} e_v^{(0)} \oplus \mathcal{O}_{E_v} e_v^{(1)}\right) \subset \Lambda_{v,1} \subset \mathcal{O}_{E_v} e_v^{(0)} \oplus \mathcal{O}_{E_v} e_v^{(1)},$$

and $\Lambda_v = \Lambda_{v,1} \oplus \Lambda_{v,1}^\perp$. For $x = (x_1, x_2) \in V(E_v) - V^\sharp(E_v)$, let

$$\phi'_v(x) := W\theta'_{v,x}(0,1,\phi_v) - (v(q(x_1)) + 1)1_{\mathcal{O}_{F_v}}(q(x_1))1_{\varpi_{E_v}\mathcal{O}_{E_v} e_v^{(1)} \oplus \Lambda_v^\perp}(x_2)\log q_{F_v}.$$

Lemma 5.2.3. *Assume that $v \in \mathfrak{Ram}_{E/F}$. Then ϕ'_v extends to a Schwartz function on $V(E_v)$ such that $c(1, \phi_v) - 2\phi'_v(0) = 2\log|\mathrm{Diff}_v|_{F_v}$.*

Proof. The computation for $\phi'_v(x)$ is indeed only on the component $\Lambda_{v,1}$ of Λ_v. For simplicity, we assume that $n = 2$ so that we do not have the component $\Lambda_{v,1}^\perp$.

Consider a larger lattice $A = \mathcal{O}_{E_v} e_v^{(0)} \oplus \mathcal{O}_{E_v} e_v^{(1)}$. For $x = (x_1, x_2) \in V(E_v) - V^\sharp(E_v)$, let

$$\varphi(x) = W\theta'_{v,x}(0,1,1_A) - (v(q(x_1)) + 1)1_{\mathcal{O}_{F_v}}(q(x_1))1_{\mathcal{O}_{E_v} e_v^{(1)}}(x_2)\log q_{F_v}.$$

By [YZ18, Lemma 9.2], φ extends to a Schwartz function on $V(E_v)$ such that $c(1, 1_A) - 2\varphi(0) = 2\log|\mathrm{Diff}_v|_{F_v}$. It is enough to extend $\varphi(x)1_{\varpi_{E_v}\mathcal{O}_{E_v} e_v^{(1)}}(x_2) - \phi'_v(x)$ to a Schwartz function on $V(E_v)$, and show that the twice of its value at 0 is $c(1, 1_A) - c(1, \phi_v)$. First,

$$\begin{aligned}
&\varphi(x)1_{\varpi_{E_v}\mathcal{O}_{E_v} e_v^{(1)}}(x_2) - \phi'_v(x)\\
&\quad= W\theta'_{v,x}(0,1,1_A)1_{\varpi_{E_v}\mathcal{O}_{E_v} e_v^{(1)}}(x_2) - W\theta'_{v,x}(0,1,\phi_v)\\
&\quad= W\theta'_{v,x}(0,1,1_A)1_{\varpi_{E_v}\mathcal{O}_{E_v} e_v^{(1)}}(x_2) - W\theta'_{v,x}(0,1,\phi_v)1_{\varpi_{E_v}\mathcal{O}_{E_v} e_v^{(1)}}(x_2)\\
&\qquad - W\theta'_{v,x}(0,1,\phi_v)1_{\mathcal{O}_{E_v}^\times e_v^{(1)}}(x_2).
\end{aligned}$$

By Lemma 3.1.2 and (5.3), $W\theta'_{v,x}(0,1,\phi_v)1_{\mathcal{O}_{E_v}^\times e_v^{(1)}}(x_2)$ extends to a Schwartz function on $V(E_v)$. It is supported on $\{x_2 \in \mathcal{O}_{E_v}^\times e_v^{(1)}\}$ so that its value at 0 is 0. By (3.5), (3.21) and (5.2),

$$\begin{aligned}
&W\theta'_{v,x}(0,1,1_A)1_{\varpi_{E_v}\mathcal{O}_{E_v} e_v^{(1)}}(x_2) - W\theta'_{v,x}(0,1,\phi_v)1_{\varpi_{E_v}\mathcal{O}_{E_v} e_v^{(1)}}(x_2)\\
&\quad= \gamma_{\mathbb{W}_v}^{-1}\frac{1}{\mathrm{Vol}(U(W(E_v)))}W_{v,q(x_1)}{}'(0,1,1_{\mathcal{O}_{E_v}^\times e_v^{(0)}})1_{\varpi_{E_v}\mathcal{O}_{E_v} e_v^{(1)}}(x_2).
\end{aligned} \tag{5.8}$$

Here, we used that $L(s, \eta_v) = 1$ due to the ramification of v in E. By Lemma 3.1.2, (5.8) extends to a Schwartz function on $V(E_v)$.

Second, by a direct computation using (3.6), Lemma 3.1.2 and (5.2), we have

$$c(1, 1_A) - c(1, \phi_v) = \gamma_{\mathbb{W}_v}^{-1}|\mathrm{Diff}_v\mathrm{Disc}_v|_{F_v}^{-1/2}W_{v,0}{}'(0,1,1_{\mathcal{O}_{E_v}^\times e_v^{(0)}}).$$

By [YZZ13, p 23], which says $\mathrm{Vol}(U(W(E_v))) = 2|\mathrm{Diff}_v\mathrm{Disc}_v|_{F_v}^{1/2}$, the lemma follows. □

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5.2.2. Generating series with automorphic Green functions
For $t \in F_{>0}$, let $Z_t(\phi)^{\mathcal{L},\mathrm{aut}} \in \widehat{Z}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K)$ be the admissible extension of $Z_t(\phi)$ that is normalized at all finite places with respect to $\mathcal{L}$ and equals the automorphic Green function (4.10) at all infinite places. Comparing with (4.23), by Lemma 4.2.4 (1),

$$Z_t(\phi)^{\mathcal{L},\mathrm{aut}} \in \widehat{Z}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{X}_K) \subset \widehat{Z}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}})$$

only depends on ϕ, but not on K. For $g \in G(\mathbb{A}_F)$ and $a \in \mathbb{C}$, define $z_t(g,\phi)^{\mathcal{L},\mathrm{aut}}_{\mathfrak{e}}$ (resp. $z(g,\phi)^{\mathcal{L},\mathrm{aut}}_{\mathfrak{e},a}$) by the formula defining $z_t(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{e}}$ (resp. $z(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{e},a}$) in Definition 4.4.13, replacing $Z_t(\phi)^{\overline{\mathcal{L}}}$ by $Z_t(\phi)^{\mathcal{L},\mathrm{aut}}$. By Lemma 4.2.4 (2), (4.26) is equivalent to

$$z(g,\phi)^{\mathcal{L},\mathrm{aut}}_{\mathfrak{e},\mathfrak{a}+\frac{[F:\mathbb{Q}]}{n}} \in \mathcal{A}_{\mathrm{hol}}(G,\mathfrak{w}) \otimes \widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}}).$$

Formally define the 't-th Fourier coefficient' (and compare with Remark 4.4.14 (1))

$$z_t(g,\phi^\infty)^{\mathcal{L},\mathrm{aut}}_{\mathfrak{e}} = \frac{z_t(g,\phi)^{\mathcal{L},\mathrm{aut}}_{\mathfrak{e}}}{W^{\mathfrak{w}}_{\infty,t}(1)}.$$

5.2.3. Arithmetic mixed Siegel-Weil formula
Let $\phi \in \overline{\mathcal{S}}(\mathbb{V})^{\widetilde{K}_\Lambda}$. See Definition 4.4.11. Assume that ϕ is a pure tensor for simplicity. Define the automorphic form to appear in the formula as follows. For $K \in \widetilde{K}_\Lambda$ stabilizing ϕ, and $K' \in \widetilde{K}_\Lambda$ containing K, let

$$\begin{aligned} f^K_{\mathbb{W},K'}(g) = \sum_{k \in U(\mathbb{W})\backslash K'/K} \frac{d_{k^{-1}\mathbb{W},K}}{d_{\mathbb{W},K'}} \Big(& -\theta E'_{\mathrm{chol}}(0,g,\omega(k)\phi) - (2\mathfrak{b}[F:\mathbb{Q}] - \mathfrak{c})E(0,g,\omega(k)\phi) \\ & + C(0,g,\omega(k)\phi) - \sum_{v \in \mathfrak{Ram}} \theta E\big(0,g,\omega(k)\phi^v \otimes \phi'_v\big)\Big). \end{aligned} \tag{5.9}$$

Here, the index set and coefficients are the ones in (5.4). We choose a representative k such that $k_v = 1$ if $K_v = K'_v$. In particular, $\omega(k)(\phi^v \otimes \phi'_v) = (\omega(k)\phi^v) \otimes \phi'_v$ for v nonsplit in E, which is what we wrote $\omega(k)\phi^v \otimes \phi'_v$ for. Inside the bracket, we have 4 automorphic forms in $\mathcal{A}_{\mathrm{hol}}(G,\mathfrak{w})$. Here, we use the orthogonal decomposition $\mathbb{V} = \mathbb{W} \oplus V^\sharp(\mathbb{A}_E)$ to define $\theta E'_{\mathrm{chol}}(\dots)$ and $C(\dots)$, and for a given $v \in \mathfrak{Ram}$, we use the orthogonal decomposition $V = W \oplus V^\sharp$ as in (5.6) to define $\theta E(\dots)$. See 3.2.1, 3.3.1 and 3.3.3 for their definitions. In particular, $f^K_{\mathbb{W},K'} \in \mathcal{A}_{\mathrm{hol}}(G,\mathfrak{w})$. Finally, the constant $\mathfrak{b}$ is as in (3.30), $\mathfrak{c}$ is as in (3.15) and the term $2\mathfrak{b}[F:\mathbb{Q}] - \mathfrak{c}$ appears in both (3.31) and (4.25).

Remark 5.2.4. The bracketed automorphic form indexed by $k \in U(\mathbb{W})\backslash K'/K$ on the right-hand side of (5.9) is independent of the choice of k. Indeed, the 4th automorphic form $\theta E\big(0,g,\omega(k)\phi^v \otimes \phi'_v\big)$ is independent of the choice of k, by the K^v-invariance of ϕ^v and the mixed Siegel-Weil formula (3.19). By (3.31), for $t \in F_{>0}$, the ψ_t-Whittaker function of the rest 3 terms becomes

$$\theta E'_{t,\mathrm{qhol}}(0,g,\omega(k)\phi) - 2\Big(E'_{t,\mathrm{f}}(0,g,\omega(k)\phi) + E_t(0,g,\omega(k)\phi)\log \mathrm{Nm}_{F/\mathbb{Q}}t\Big).$$

Then the independence of the choice of k follows from (3.10), (3.14) and (3.36). Then the independence of the choice of k follows from the automorphy.

Theorem 5.2.5. *Let $\phi \in \overline{\mathcal{S}}(\mathbb{V})^{\widetilde{K}_\Lambda}$ be a pure tensor such that $\phi_v = 1_{\Lambda_v}$ for every finite place v of F nonsplit in E. Let $K \in \widetilde{K}_\Lambda$ stabilize ϕ. For $t \in F_{>0}$ and $g \in P(\mathbb{A}_{F,\mathfrak{Ram}})G\Big(\mathbb{A}_F^{\mathfrak{Ram}\cup\infty}\Big)$, we have*

$$2z_t(g,\phi^\infty)^{\mathcal{L},\mathrm{aut}}_{\mathfrak{e}} \cdot \pi^*_{K,K_\Lambda}\mathcal{P}_{\mathbb{W},K_\Lambda} = f^{K,\infty}_{\mathbb{W},K_\Lambda,t}(g), \tag{5.10}$$

where the arithmetic intersection on the left-hand side is taken on $\mathcal{X}_K$, and $f^{K,\infty}_{\mathbb{W},K_\Lambda,t}$ is the t-th Fourier coefficient of $f^K_{\mathbb{W},K_\Lambda}$. See 2.7.

Theorem 5.2.5 will be proved in (the end of) Section 6.

Remark 5.2.6. (1) By the projection formula, given ϕ and K', the truth of (5.10) does not depend on the choice of K (stabilizing the given ϕ).

(2) We use $\pi^*_{K,K'}\mathcal{P}_{\mathbb{W},K'}$ with $K' = K_\Lambda$, instead of the more natural CM cycle $\mathcal{P}_{\mathbb{W},K}$ (i.e., $K' = K$), in order to apply Corollary A.2.6 to avoid computing normalized admissible extensions. (One may expect to reduce the whole Theorem 5.2.5 to the level K_Λ by the projection formula. However, this is not possible due to Remark 4.1.6.)

(3) We can also consider Theorem 5.2.5 for $K' = K$. Taking advantage of Theorem 4.4.21, the only difficulty in proving Theorem 5.2.5 for $K' = K$ is computing normalized admissible extensions at split place. By considering admissible 1-cycles, the difficulty could be solved as in [YZZ13, 8.5.1]. Note that we do not need Assumption 5.4 in loc. cit. Rather we will get an extra Eisenstein series.

5.2.4. Proof of Theorem 4.4.21

Assuming Theorem 5.2.5, we prove Theorem 4.4.21 as follows. Recall that as we have discussed immediately after Theorem 4.4.21, that (4.28) and (4.29) of Theorem 4.4.21 are equivalent. We will prove (4.28), assuming that ϕ is a pure tensor such that $\phi_v = 1_{\Lambda_v}$ for every finite place v nonsplit in E (so that Theorem 5.2.5 applies to ϕ). So (4.29) holds. It follows from Definition 4.4.18 that (4.29) holds for $\omega(g)\phi$ with $g \in \mathbb{G}^\infty$. Thus, Theorem 4.4.21 holds by the definition of $\overline{\mathcal{S}}(\mathbb{V})^{\widetilde{K}_\Lambda}_{\mathfrak{Ram}}$ above it.

So now we assume that ϕ is a pure tensor such that $\phi_v = 1_{\Lambda_v}$ for every finite place v nonsplit in E and want to prove (4.28); that is,

$$\left(\omega(g)\phi(0)c_1(\overline{\mathcal{L}}^\vee) + \omega(g)\phi(0)\mathfrak{a}\right) + \sum_{t\in F>0} z_t(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{e}} \in \mathcal{A}_{\mathrm{hol}}(\mathbb{G},\mathfrak{w}) \otimes \widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}}). \tag{5.11}$$

Let $K \in \widetilde{K}_\Lambda$ such that ϕ is K-invariant. Let $f^K_{\mathbb{W},K_\Lambda,0}$ be the 0-th Whittaker coefficient of $f^K_{\mathbb{W},K_\Lambda}$. Let $A(\cdot,\phi)_K$ be the $\mathbb{C}$-valued function on $G(\mathbb{A})$, understood as valued $\widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}})$ via $\mathbb{C} \hookrightarrow \widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}})$ as in (4.22), defined by

$$2\left(\omega(g)\phi(0)\left(c_1(\overline{\mathcal{L}}^\vee) + \frac{[F:\mathbb{Q}]}{n}\right) + A(g,\phi)_K\right)\cdot \pi^*_{K,K_\Lambda}\mathcal{P}_{\mathbb{W},K_\Lambda} = f^K_{\mathbb{W},K_\Lambda,0}(g),$$

that is,

$$A(g,\phi)_K = \frac{1}{2\deg \pi_{K,K_\Lambda}}\left(f^K_{\mathbb{W},K_\Lambda,0}(g) - 2\omega(g)\phi(0)\left(c_1(\overline{\mathcal{L}}^\vee) + \frac{[F:\mathbb{Q}]}{n}\right)\cdot \pi^*_{K,K_\Lambda}\mathcal{P}_{\mathbb{W},K_\Lambda}\right). \tag{5.12}$$

By Lemma 4.3.8 (with G and $G(\mathbb{A})$ replaced by $\mathbb{G}$) and Theorem 5.2.5,

$$\left(\omega(g)\phi(0)\left(c_1(\overline{\mathcal{L}}^\vee) + \frac{[F:\mathbb{Q}]}{n}\right) + A(g,\phi)_K\right) + \sum_{t\in F>0} z_t(g,\phi)^{\mathcal{L},\mathrm{aut}}_{\mathfrak{e}} \in \mathcal{A}_{\mathrm{hol}}(\mathbb{G},\mathfrak{w}) \otimes \widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}}).$$

Then by Lemma 4.2.4 (2), we have

$$\left(\omega(g)\phi(0)c_1(\overline{\mathcal{L}}^\vee) + A(g,\phi)_K\right) + \sum_{t\in F>0} z_t(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{e}} \in \mathcal{A}_{\mathrm{hol}}(\mathbb{G},\mathfrak{w}) \otimes \widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}}). \tag{5.13}$$

Thus, (5.11) is reduced to the following lemma, whose proof requires some preparations.

Lemma 5.2.7. *We have $A(g,\phi)_K = \omega(g)\phi(0)\mathfrak{a}$.*

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To determine $A(g,\phi)_K$, one a priori needs to compute $f^K_{\mathbb{W},K_\Lambda,0}(g)$, which could be complicated. Indeed, by (3.20) for $t=0$ and Lemma 3.3.9 (1) (2), we have

$$\begin{aligned} f^K_{\mathbb{W},K_\Lambda,0}(g) = \deg \pi_{K,K\Lambda}\Big(&- (2\mathfrak{b}[F:\mathbb{Q}]-\mathfrak{c})\omega(g)\phi(0) \\ &+ \sum_v c(g_v,\phi_v)\omega(g^v)\phi^v(0) - 2\sum_{v\in\mathfrak{Ram}} \omega(g)\big(\phi^v\otimes\phi'_v\big)(0)\Big). \end{aligned} \tag{5.14}$$

The terms in the second line cause the complicatedness.

We take a different approach. We study the invariance properties of $A(g,\phi)_K$. We need a notation. Let $W_0(\mathbb{G})$ be the space of ψ_0-Whittaker functions on $\mathbb{G}$ (i.e., smooth $\mathbb{C}$-valued functions f such that $f(ng)=f(g)$ for $n\in N(\mathbb{A})$). Then

$$W_0(\mathbb{G})\cap\mathcal{A}(\mathbb{G},\mathfrak{w})=\{0\}. \tag{5.15}$$

By (5.14) and Lemma 3.3.4 (2), we have $f^K_{\mathbb{W},K_\Lambda,0}\in W_0(\mathbb{G})$. Then by (5.12), we have $A(g,\phi)_K\in W_0(\mathbb{G})$. For $K'\in\widetilde{K}_\Lambda$ such that ϕ is K'-invariant, by (5.13), $A(g,\phi)_K - A(g,\phi)_{K'}\in\mathcal{A}_{\mathrm{hol}}(\mathbb{G},\mathfrak{w})$. Thus, by (5.15), $A(g,\phi)_K$ does not depend on K. Let us denote $A(g,\phi)_K$ by $A(g,\phi)$.

Lemma 5.2.8. (1) *For $g'\in\mathbb{G}^\infty$, $A(gg',\phi)=A(g,\omega(g')\phi)$.*

(2) *For $h\in K_\Lambda$, $A(g,\omega(h)\phi)=A(g,\phi)$.*

Proof. (1) Since $A(g,\phi),\omega(g)\phi(0)\in W_0(\mathbb{G})$,

$$B(g,\phi):=A(g,\phi)+\omega(g)\phi(0)\big(-\log\delta_\infty(g_\infty)+[F:\mathbb{Q}](\log\pi-(\log\Gamma)'(n+1))\big)\in W_0(\mathbb{G}).$$

Claim:

$$B(g,\phi)+\sum_{x\in\mathbb{V}^\infty,\ q(x)\in F^\times}\omega(g^\infty)\phi^\infty(x)[Z(x,g_\infty)^{\overline{\mathcal{L}}}_{\mathfrak{t}}]W^{\mathfrak{w}}_{\infty,q(x)}(g_\infty)\in\mathcal{A}(\mathbb{G},\mathfrak{w})\otimes\widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}}). \tag{5.16}$$

Indeed, by Theorem 4.2.7, the difference between the generating series in (5.13) and (5.16) equals

$$E'(0,g,\phi)-E(0,g,\phi)[F:\mathbb{Q}](\log\pi-(\log\Gamma)'(n+1)).$$

(This is the analog of (4.27).) Then the claim follows from (5.13).

Now we prove (1). Apply (5.16) in two ways: first, replace ϕ by $\omega(g')\phi$ and call the resulted series S_1; second, replace g by gg' and call the resulted series S_2. Then $S_1,S_2\in\mathcal{A}(\mathbb{G},\mathfrak{w})\otimes\widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}})$. So

$$B(g,\omega(g')\phi)-B(gg',\phi)=S_1-S_2\in\mathcal{A}(\mathbb{G},\mathfrak{w}).$$

Thus, it must be 0 by (5.15). This gives (1).

(2) By Lemma 4.4.12, after the 'right translation by h', (5.13) becomes

$$\Big(\omega(g)\phi(0)c_1(\overline{\mathcal{L}}^\vee)+A(g,\omega(h)\phi)\Big)+\sum_{t\in F_{>0}} z_t(g,\omega(h)\phi)^{\overline{\mathcal{L}}}_{\mathfrak{e}}\in\mathcal{A}_{\mathrm{hol}}(\mathbb{G},\mathfrak{w})\otimes\widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}}).$$

(This is similar to Lemma 4.4.15.) Since (5.13) holds with ϕ replaced by $\omega(h)\phi$, taking the difference, we have $A(g,\omega(h)\phi)-A(g,\phi)\in\mathcal{A}_{\mathrm{hol}}(\mathbb{G},\mathfrak{w})$. It is 0 by (5.15). □

To prove Lemma 5.2.7, we need a final ingredient, whose proof is easy and left to the reader.

Lemma 5.2.9. *For v split in E, identify $\mathbb{W}_v=E_v$ and $K_{\Lambda,v}\cap U(\mathbb{W}_v)=\mathcal{O}^\times_{E_v}$. Then $\mathcal{S}(\mathbb{W}_v)^{K_{\Lambda,v}\cap U(\mathbb{W}_v)}$ is spanned by f_a's, where $f_a(x):=1_{\mathcal{O}_{E_v}}(xa)$, $a\in E_v^\times$.*

Proof of Lemma 5.2.7. We have some reduction steps about ϕ. By Lemma 5.2.8 (2), we may assume that ϕ is K_Λ-invariant. In particular, for v split in E, ϕ_v is $K_{\Lambda,v} \cap U(\mathbb{W}_v)$-invariant. Identify $\mathbb{W}_v = E_v$ and $q = \mathrm{Nm}$. By Lemma 5.2.8 (1), Lemma 5.2.9 and the first Weil representation formula in 2.8, we may further assume that $\phi_v = \phi_{v,1} \otimes \phi_{v,2}$ where $\phi_{v,1} = 1_{\mathcal{O}_{E_v}}$ and $\phi_{v,2} \in \mathcal{S}\left(V^\sharp(E_v)\right)$ (recall that $V^\sharp(E_v)$ is the orthogonal complement of $\mathbb{W}_v$).

Now we look at $f^K_{\mathbb{W},K_\Lambda,0}(g)$ given by (5.14). By the definition (3.23) and Lemma 3.1.3 (2) (3), $c(g_v, \phi_v) = 0$ unless $v \in \mathfrak{Ram}$ or v split in E (by the same argument as in Lemma 3.3.4). By Lemma 5.2.8 (1), we may assume $g^\infty = 1$. Then by Lemma 3.3.7, Lemma 5.2.2, Lemma 5.2.3,

$$f^K_{\mathbb{W},K_\Lambda,0}(g) = \deg \pi_{K,K_\Lambda}\Big(-(2\mathfrak{b}[F:\mathbb{Q}] - \mathfrak{c})\omega(g)\phi(0) + 2\log|\mathrm{Disc}_F|\omega(g)\phi(0)\Big).$$

Since $c_1(\overline{\mathcal{L}}^\vee_K) = \pi^*_{K,K_\Lambda}\mathcal{L}^\vee_{K_\Lambda}$, by the projection formula,

$$c_1(\overline{\mathcal{L}}^\vee_K) \cdot \pi^*_{K,K_\Lambda}\mathcal{P}_{\mathbb{W},K_\Lambda} = \deg \pi_{K,K_\Lambda} c_1(\overline{\mathcal{L}}^\vee_{K_\Lambda}) \cdot \mathcal{P}_{\mathbb{W},K_\Lambda}.$$

Now by the definition of $A(g,\phi)_K$ in (5.12) and the definition of $\mathfrak{a}$ in the first line of (4.25), the lemma follows. □

5.3. *Generalizations and applications*

5.3.1. Higher codimensions

Based on their modularity result for generating series of special divisors, Yuan, S. Zhang and W. Zhang [YZZ09] proved the modularity for higher-codimensional special cycles inductively, assuming the convergence. One would like to mimic their proof in the arithmetic situation. Then one needs a modularity theorem for divisors with general level structures and test functions, even if the given test function is very good. Thus, the generality of our results is necessary toward modularity in the arithmetic situation in higher codimensions.

In the codimension n case (i.e., for arithmetic 1-cycles), S. Zhang's theory of admissible cycles is unconditional modulo vertical 1-cycles that are numerically trivial [Zha20]. Then the method in the current paper is still applicable to approach the modularity in the arithmetic situation.

5.3.2. Almost-self dual lattice

There is another lattice level structure at a finite place considered in [RSZ20], defined by an almost-self dual lattice. The integral model is not smooth, but is explicitly described in [LTX+22]. We hope to include this level structure in a future work. In fact, if we also use admissible 1-cycles, our approach combined with a recent result of Z. Zhang [Zha21b, Theorem 14.6] is already applicable to prove the analog of Theorem 4.4.21, after replacing normalized admissible extensions of special divisors by the 'admissible projections' of the Kudla-Rapoport divisors at these new places (provided that they can also be suitably defined on our models). However, the difference between two extensions is not clear so far.

5.3.3. Faltings heights of Shimura varieties and Arithmetic Siegel-Weil formula

Following Kudla [Kud03, Kud04], we propose the arithmetic analog of the geometric Siegel Weil formula (4.11).

Problem 5.3.1. Match $c_1(\overline{\mathcal{L}})^n \cdot z(g,\phi)^{\mathcal{L},\mathrm{Kud}}_{\mathfrak{a}}$ with a linear combination of $E(0,g,\phi)$ and $E'(0,g,\phi)$ (possibly up to some error terms).

The modularity of $z(g,\phi)^{\mathcal{L},\mathrm{Kud}}_{\mathfrak{a}}$ helps to attack this problem as follows. The constant term of $c_1(\overline{\mathcal{L}})^n \cdot z(g,\phi)^{\mathcal{L},\mathrm{Kud}}_{\mathfrak{a}}$ is indeed the Faltings height of $\mathcal{X}_K$ itself, while the non-constant terms are given by Faltings heights of Shimura subvarieties with the numbers in Definition 4.2.6. (Here, by the Faltings height of $\mathcal{X}$, we mean $\deg c_1(\overline{\mathcal{L}})^{n+1}$.) There is clearly an inductive scheme to compute the Faltings heights/attack arithmetic Siegel-Weil formula, by applying the modularity of the generating series. Moreover, we only need to compute enough terms. This might enable us to avoid dealing with Shimura subvarieties of general level structures from the inductive steps. We can also use $z(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{k},\mathfrak{a}}$ to attack Problem 5.3.1, since by a direct computation,

$$c_1(\overline{\mathcal{L}})^n \cdot z(g,\phi)^{\mathcal{L},\mathrm{Kud}}_{\mathfrak{a}} = c_1(\overline{\mathcal{L}})^n \cdot z(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{k},\mathfrak{a}}. \tag{5.17}$$

For quaternionic Shimura curves, Faltings heights are computed in [KRY06] [Yua22]. For unitary Shimura varieties, in the case $F = \mathbb{Q}$, the Faltings height of $\mathcal{X}_{K_\Lambda}$ (for a different lattice Λ) was computed in [BH21] using Borcherds' theory. See [BH21] for other related results.

5.3.4. Arithmetic theta lifting and Gross-Zagier type formula

Consider the Petersson inner product between the modular generating series of special divisors on the generic fiber and a cusp form f of G [Kud03, Kud04]. When $n = 1$, it is cohomological trivial and its Beilinson-Bloch height was studied in [Liu11a, Liu11b]. However, when $n > 1$, the Picard group of $\mathrm{Sh}(\mathbb{V})_K$ is CM by [MR92], so that the inner product is 0 in most cases after cohomological trivialization. Thus, it is necessary to consider arithmetic intersection pairing on an integral model (without cohomological trivialization).

The arithmetic intersection pairing with our CM 1-cycle as in Theorem 5.2.5 is simply the Petersson inner product between (5.9) and f. Such a pairing appeared in the work of Gross and Zagier [GZ86] and leads to their celebrated formula. We hope to get a Gross-Zagier type formula. In the case $F = \mathbb{Q}$ and $\phi^\infty = 1_\Lambda$ (for a different Λ), a Gross-Zagier type formula was obtained in [BHY15] and [BHK+20b]. See also [BY09]. For general F and test functions as in our case, a general theory of Shimura-type integrals is to be developed.

6. Intersections

In this section, we prove the arithmetic mixed Siegel-Weil formula (Theorem 5.2.5). First, we prove *local* analogs of the formula, under some regularity assumption which forces improper intersections to disappear. Then, to prove Theorem 5.2.5, we use a *global* argument involving admissibility of our arithmetic divisors and modularity on the generic fiber (more precisely, Lemma 4.3.6).

6.1. Proper intersections

In order to state the local analogs of the Arithmetic mixed Siegel-Weil formula (Proposition 6.1.4), we need some preliminaries.

We use the CM cycles in 5.1, as well as the notations there – in particular, the orthogonal decomposition $\mathbb{V} = \mathbb{W} \oplus V^\sharp(\mathbb{A}_E)$.

Definition 6.1.1. For a finite place v of F, a Schwartz function on $\mathbb{V}(E_v)$ is $\mathbb{W}_v$-regular if it is supported outside the orthogonal complement $\mathbb{W}_v^\perp = V^\sharp(E_v)$.

Let $\phi \in \overline{\mathcal{S}}(\mathbb{V})$ be a pure tensor such that $\phi_v = 1_{\Lambda_v}$ for every finite place v nonsplit in E. Let $K \in \widetilde{K}_\Lambda$ such that ϕ is K-invariant.

We assume the following assumption throughout this subsection.

Assumption 6.1.2. There is a nonempty set R of places of F such that ϕ_v is $\mathbb{W}_v$-regular for $v \in R$.

Since $\phi_v = 1_{\Lambda_v}$ for every finite place v nonsplit in E, R necessarily contains only places split in E.

As we have seen in the proof of Lemma 4.4.3, for a finite place v of E, there exists a finite unramified extension N/E_v such that $\mathcal{X}_{K,\mathcal{O}_N}$ is (part of) a PEL moduli space, and the supersingular locus is thus defined and obviously independent of the choice of N. Let $t \in F_{>0}$.

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Lemma 6.1.3. *For $g \in P(\mathbb{A}_{F,R})G(\mathbb{A}_F^R)$, the supports of the Zariski closure of $Z_t(\omega(g)\phi)^{\mathrm{zar}}$ and $\mathcal{P}_{\mathbb{W},K}$ on $\mathcal{X}_K$ only intersect on the supersingular loci at finite places of E nonsplit over F.*

Proof. The regularity of ϕ at R is preserved by the action of $P(\mathbb{A}_{F,R})$ on ϕ. By the regularity of $\omega(g)\phi$, the lemma follows from the same proof as [KR14, Lemma 2.21]. Or one may reduce the lemma to (the version over a general CM field of) [KR14, Lemma 2.21] as follows. Choose n vectors $x_1, \dots, x_n$ spanning $V^\sharp$. Then $\mathcal{P}_{\mathbb{W},K,E} \subset Z(x_1) \cap \dots \cap Z(x_n)$. Since $Z_t(\omega(g)\phi)$ is a finite sum of $Z(x)$'s with x outside $V^\sharp(\mathbb{A}_E)$, [KR14, Lemma 2.21] applies. □

For $k \in U(\mathbb{V}^{\infty\cup R})$, that is, $k_v = 1$ for $v \in R$. Then $\mathbb{W}_v = (k\mathbb{W})_v$ and ϕ_v is $(k\mathbb{W})_v$-regular. By Lemma 6.1.3, the intersection number $\left(Z_t(\omega(g)\phi)^{\mathrm{zar}} \cdot \mathcal{P}_{k^{-1}\mathbb{W},K}\right)_{\mathcal{X}_{K,\mathcal{O}_{E_v}}}$ of the restrictions of the cycles to $\mathcal{X}_{K,\mathcal{O}_{E_v}}$ is well defined as in (A.1).

Proposition 6.1.4. *Recall the set $\mathfrak{Ram}$ of finite places of F nonsplit in E, and ramified in E or over $\mathbb{Q}$. Let $g \in P(\mathbb{A}_{F,\mathfrak{Ram}\cup R})G\left(\mathbb{A}_F^{\mathfrak{Ram}\cup R}\right)$. Let v be a place of F nonsplit in E and $k \in U(\mathbb{V}^{\infty\cup R\cup\{v\}})$. For $v \notin \mathfrak{Ram} \cup \infty$, resp. $v \in \mathfrak{Ram}$, resp. $v \in \infty$, we respectively have*

$$2\left(Z_t(\omega(g)\phi)^{\mathrm{zar}} \cdot \mathcal{P}_{k^{-1}\mathbb{W},K}\right)_{\mathcal{X}_{K,\mathcal{O}_{E_v}}} \log q_{E_v} = \theta E_t'(0, g, \omega(k)\phi)(v), \tag{6.1}$$

$$2\left(Z_t(\omega(g)\phi)^{\mathrm{zar}} \cdot \mathcal{P}_{k^{-1}\mathbb{W},K}\right)_{\mathcal{X}_{K,\mathcal{O}_{E_v}}} \log q_{E_v} = \theta E_t'(0, g, \omega(k)\phi)(v) - \theta E\left(0, g, \omega(k)\phi^v \otimes \phi_v'\right), \tag{6.2}$$

$$2\int_{\left(\mathcal{P}_{k^{-1}\mathbb{W},K}\right)_{E_v}} \mathcal{G}^{\mathrm{aut}}_{Z_t(\omega(g)\phi)_{E_v}} = \widetilde{\lim_{s\to 0}} \theta E_{t,s}'(0, g, \omega(k)\phi)(v). \tag{6.3}$$

We will prove Proposition 6.1.4 for $k = 1$, in Proposition 6.1.6 ($v \notin \mathfrak{Ram} \cup \infty$), Proposition 6.1.7 and Proposition 6.1.9 ($v \in \mathfrak{Ram}$), Proposition 6.1.11 ($v \in \infty$). The proof for the general k is the same, except one needs to keep track of k. For simplicity, let

$$\mathcal{P} = \mathcal{P}_{\mathbb{W},K}.$$

We prepare more notations for later computations. For a finite place v of F nonsplit in E, let E_v^{ur} be the complete maximal unramified extension of E_v. Let $\mathbb{E}$ be the unique formal $\mathcal{O}_{F_v}$-module of relative height 2 and dimension 1 over $\operatorname{Spec} E_v^{\mathrm{ur}}/\varpi_{E_v}$. The endomorphism ring of $\mathbb{E}$ is the maximal order of the unique division quaternion algebra B over F_v. Fixing an embedding $\iota : E_v \hookrightarrow B$ such that $\iota(\mathcal{O}_{E_v})$ is in the maximal order of B. Then $\mathbb{E}$ becomes a formal $\mathcal{O}_{E_v}$-module of relative height 1 and dimension 1, which we still donote by $\mathbb{E}$. Let $\bar{\iota}$ be ι precomposed with the nontrivial $\mathrm{Gal}(E_v/F_v)$-conjugation. It produces another $\mathcal{O}_{E_v}$-module $\overline{\mathbb{E}}$. Fix an $\mathcal{O}_{F_v}$-linear principal polarization $\lambda_{\mathbb{E}}$ on $\mathbb{E}$. Let $\mathcal{E}$ and $\overline{\mathcal{E}}$ be the canonical liftings of $\mathbb{E}$ and $\overline{\mathbb{E}}$ respectively, as $\mathcal{O}_{E_v}$-modules. They are isomorphic as formal $\mathcal{O}_{F_v}$-modules, and equipped with a unique $\mathcal{O}_{F_v}$-linear principal polarization $\lambda_{\mathcal{E}}$ lifting $\lambda_{\mathbb{E}}$.

6.1.1. Finite places of F inert in E

For such a v, Λ_v is self-dual.

Before we can compute the intersection number, we need to uniformize the integral model, CM cycle and special divisors using Rapoport-Zink spaces.

For a nonnegative integer m, let $\mathcal{N}_m$ be the unramified relative unitary Rapoport-Zink space of signature $(m,1)$ [KR11] [LZ21, 2.1] over $\operatorname{Spf} \mathcal{O}_{E_v^{\mathrm{ur}}}$. It is the deformation space of the polarized hermitian $\mathcal{O}_{E_v}$-module $\mathbb{X}_m := \overline{\mathbb{E}} \times \mathbb{E}^m$ with the product polarization λ_m. It is formally smooth of relative dimension m. The space $\operatorname{Hom}_{\mathcal{O}_{E_v}}(\mathbb{E}, \mathbb{X}_m)_{\mathbb{Q}}$ carries a natural hermitian pairing

$$(x, y) \mapsto \lambda_{\mathbb{E}}^{-1} \circ x^\vee \circ \lambda_m \circ y \in \operatorname{Hom}_{\mathcal{O}_{E_v}}(\mathbb{E}, \mathbb{E})_{\mathbb{Q}} \simeq E_v. \tag{6.4}$$

For $m = n$, we let $\mathcal{N} = \mathcal{N}_n$. By [LZ21, 2.2], we have $\operatorname{Hom}_{\mathcal{O}_{E_v}}(\mathbb{E}, \mathbb{X}_n)_{\mathbb{Q}} \simeq V(E_v)$. And $U(V(E_v))$ is isomorphic to the group of $\mathcal{O}_{E_v}$-linear self-quasi-isogenies of $\mathbb{X}_n$ preserving λ_n [RSZ18, (4.3)].

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In particular, $U(V(E_v))$ acts on $\mathcal{N}$. For every $x \in V(E_v) - \{0\}$, we have the Kudla-Rapoport divisor $\mathcal{Z}(x)$ on $\mathcal{N}$[KR11] [LZ21, 2.3] that is the locus where x lifts to a quasi-isogeny. It is a (possibly empty) relative Cartier divisor. See [KR11, Proposition 3.5], which is only stated for $F_v = \mathbb{Q}_p$ but holds in the general case.

Let $\widehat{\mathcal{X}^{ss}_{K,\mathcal{O}_{E_v^{\mathrm{ur}}}}}$ be the formal completion of $\mathcal{X}_{K,\mathcal{O}_{E_v^{\mathrm{ur}}}}$ along the supersingular locus. Then we have the following formal uniformization [LZ21, 13.1]:

$$\widehat{\mathcal{X}^{ss}_{K,\mathcal{O}_{E_v^{\mathrm{ur}}}}} \simeq U(V)\backslash(\mathcal{N} \times U(\mathbb{V}^{\infty,v})/K^v). \tag{6.5}$$

For $x \in V(E_v) - \{0\}$ and $h \in U(\mathbb{V}^{\infty,v})$, we have a relative cartier divisor $[\mathcal{Z}(x), h]$ of $\widehat{\mathcal{X}^{ss}_{K,\mathcal{O}_{E_v^{\mathrm{ur}}}}}$.

The Rapoport-Zink space $\mathcal{N}_0$ is naturally a closed formal subscheme of $\mathcal{N}$ by adding canonical liftings; that is, the morphism $\mathcal{N}_0 \to \mathcal{N}$ is given by $X \mapsto X \times \mathcal{E}^n$. The subspace

$$\mathrm{Hom}_{\mathcal{O}_{E_v}}(\mathbb{E}, \overline{\mathbb{E}})_{\mathbb{Q}} \subset \mathrm{Hom}_{\mathcal{O}_{E_v}}(\mathbb{E}, \mathbb{X}_n)_{\mathbb{Q}} \simeq V(E_v)$$

becomes the subspace $W(E_v)$ of $V(E_v)$, and the subgroup $U(W(E_v))$ stabilizes $\mathcal{N}_0$. We have

$$\mathcal{P}_{\mathcal{O}_{E_v^{\mathrm{ur}}}} = \frac{1}{d_{\mathbb{W},K}} U(W)\backslash(\mathcal{N}_0 \times U(\mathbb{W}^{\infty,v})/K^v_{\mathbb{W}}), \tag{6.6}$$

where $d_{\mathbb{W},K}$ is the degree of the fundamental cycle of $\mathrm{Sh}(\mathbb{W})_{K_{\mathbb{W}}}$. See Definition 5.1.3), and the right-hand side is defined using the formal uniformization (6.5) of $\widehat{\mathcal{X}^{ss}_{K,\mathcal{O}_{E_v^{\mathrm{ur}}}}}$ with $\mathcal{N}_0$ understood as a 1-cycle on $\mathcal{N}$.

Recall that V^t is the subset of $V(E)$ of elements of norm t.

Proposition 6.1.5. *Under the formal uniformization (6.5) of $\widehat{\mathcal{X}^{ss}_{K,\mathcal{O}_{E_v^{\mathrm{ur}}}}}$, for $g \in G(\mathbb{A}_F^v)$, we have*

$$Z_t(\omega(g)\phi)^{\mathrm{zar}}|_{\widehat{\mathcal{X}^{ss}_{K,\mathcal{O}_{E_v^{\mathrm{ur}}}}}} = \sum_{x \in U(V)\backslash V^t} \sum_{h \in U_x^v \backslash U(\mathbb{V}^{\infty,v})/K^v} \omega(g)\phi^v(h^{-1}x)[\mathcal{Z}(x), h]. \tag{6.7}$$

Proof. This follows from [LL21, (8.3)] and the flatness of $\mathcal{Z}(x)$. □

Let $(\mathcal{Z}(x) \cdot \mathcal{N}_0)_{\mathcal{N}}$ be the Euler-Poincaré characteristic of the derived tensor product $\mathcal{O}_{\mathcal{Z}(x)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{N}_0}$. Since $h \in U(W(E_v))$ stabilizes $\mathcal{N}_0$, $(\mathcal{Z}(x) \cdot \mathcal{N}_0)_{\mathcal{N}} = (\mathcal{Z}(hx) \cdot \mathcal{N}_0)_{\mathcal{N}}$. By Lemma 6.1.3, (6.6), Proposition 6.1.5 and a direct computation, for $g \in P(\mathbb{A}_{F,R\cup\{v\}})G(\mathbb{A}_F^{R\cup\{v\}})$,

$$(Z_t(\omega(g)\phi)^{\mathrm{zar}} \cdot \mathcal{P})_{\mathcal{X}_{K,\mathcal{O}_{E_v}}} = \frac{1}{\mathrm{Vol}([U(W)])} \int_{h \in [U(W)]} \sum_{x \in V^t - V^\sharp} \left(\mathcal{Z}(h_v^{-1}x) \cdot \mathcal{N}_0\right)_{\mathcal{N}} \omega(g)\phi^v(h^{-1}x)dh \tag{6.8}$$

Proposition 6.1.6. *If v is unramified over $\mathbb{Q}$, for $g \in P(\mathbb{A}_{F,R})G(\mathbb{A}_F^R)$ and $k = 1$, (6.1) holds.*

Proof. First, we compute $(\mathcal{Z}(x) \cdot \mathcal{N}_0)_{\mathcal{N}}$ (i.e., we need to compute the length of the locus on $\mathcal{N}_0$ to which x lifts). Recall that $\Lambda_v^\sharp \subset \mathbb{V}(E_v)$ is the orthogonal complement of $\mathcal{O}_{E_v} e_v^{(0)}$ in Λ_v. Under the isomorphism $\mathrm{Hom}_{\mathcal{O}_{E_v}}(\mathbb{E}, \mathbb{X}_n)_{\mathbb{Q}} \simeq V(E_v)$, the image of $\mathrm{Hom}_{\mathcal{O}_{E_v}}(\mathbb{E}, \mathbb{E}^n)$ is $\Lambda_v^\sharp$. Then for $x = (x_1, x_2) \in W(E_v) \oplus V^\sharp(E_v)$ with $x_1 \neq 0$, if it lifts, then $x_2 \in \Lambda_v^\sharp$. Moreover, by Gross' result on canonical lifting [Gro86, Proposition3.3], we have

$$(\mathcal{Z}(x) \cdot \mathcal{N}_0)_{\mathcal{N}} = \frac{v(q(x_1)) + 1}{2} 1_{\mathcal{O}_{F_v}}(q(x_1)) 1_{\Lambda_v^\sharp}(x_2). \tag{6.9}$$

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Second, if $g_v = 1$, express the left-hand side of (6.1) by (6.8) and (6.9). Compare it with the expression of the right-hand side of (6.1) given by (3.22). By Lemma 5.2.1, (6.1) follows.

Finally, we reduce the general case to the case $g_v = 1$ in two reduction steps. (I) We claim the following: replacing g_v by $g_v n(b_v)$ for $b_v \in F_v$ or by $g_v k_v$ for $k_v \in K_v^{\max}$ (see 2.5), both sides of (6.1) are multiplied by the same constant. Indeed, for the left-hand side of (6.1), we directly use the definition of the Weil representation. For the right-hand side, besides the definition, we further need (3.2) and Lemma 3.2.2. (II), By Corollary 3.2.3 and Lemma 4.1.3, we can replace g by $m(a)g$ for some $a \in E^\times$. By the Iwasawa decomposition, the fact that $E_v^\times = E^\times \mathcal{O}_{E_v}^\times$, and the claim in the first reduction step, we may assume that $g_v = 1$. The proposition is proved. □

Proposition 6.1.7. *If v is ramified over $\mathbb{Q}$, for $g \in P(\mathbb{A}_{F,R\cup\{v\}})G(\mathbb{A}_F^{R\cup\{v\}})$, (6.2) holds.*

Proof. The proof is the same with the proof of Proposition 6.1.6 with the following exceptions:

- Lemma 5.2.1 should be replaced by Lemma 5.2.2;
- in the claim in the reduction step (I), remove 'or by $g_v k_v$ for $k_v \in K_v^{\max}$';
- in the reduction step (II), 'Corollary 3.2.3' should be replaced by 'Corollary 3.2.3 and that $\theta E\big(0, m(a)g, \phi^v \otimes \phi_v'\big) = \theta E\big(0, g, \phi^v \otimes \phi_v'\big)$ for $a \in E^\times$'. □

6.1.2. Finite places of F ramified in E

For such a v, Λ_v is a π_v-modular or almost π_v-modular lattice. This case is more complicated.

We still need formal uniformization using Rapoport-Zink spaces. For a nonnegative integer m, let $\mathcal{N}_m$ be the exotic smooth relative unitary Rapoport-Zink space of signature $(m, 1)$ over $\operatorname{Spf} \mathcal{O}_{E_v^{\mathrm{ur}}}$. See [RSZ18, Section 6,7], [RSZ17, 3.5] and [LL22, 2.1]. It will also be specified below. It is formally smooth over $\operatorname{Spf} \mathcal{O}_{E_v^{\mathrm{ur}}}$ of relative dimension m. (Note that the case $m = 0$ is not covered in either [RSZ17] or [LL22], but is specifically indicated in [RSZ18, Example 7.2].) Let $\mathcal{N} = \mathcal{N}_n$. We will use $\mathcal{N}$ for the formal uniformization of $\mathcal{X}_K$. The analog of the formal uniformization (6.6) of $\mathcal{P}$ using $\mathcal{N}_0$ is more subtle: we will use $\mathcal{N}_1$ to define morphisms $\mathcal{N}_0 \to \mathcal{N}_1 \to \mathcal{N}$, which will lead us to the formal uniformization (6.11) of $\mathcal{P}$.

Remark 6.1.8. The reason for this subtlety might be explained as follows. Recall that the construction of $\mathcal{P}$ requires an additional rank 2 sub-lattice $\Lambda_{v,1}$ as in 5.1.3 at each finite place, which is a direct summand of Λ_v. And $E_v\Lambda_{v,1}$ contains a distinguished vector $e_v^{(0)}$ of unit norm. One might consider $\mathcal{O}_{E_v} e_v^{(0)} \subset E_v\Lambda_{v,1}$ and $\Lambda_{v,1} \subset \Lambda_v$ as being parallel to the morphisms $\mathcal{N}_0 \to \mathcal{N}_1$ and $\mathcal{N}_1 \to \mathcal{N}$. Note that $\mathcal{O}_{E_v} e_v^{(0)}$ is not contained in $\Lambda_{v,1}$ (or Λ_v). This makes it nontrivial to define a morphism $\mathcal{N}_0 \to \mathcal{N}_1$ or $\mathcal{N}_0 \to \mathcal{N}$. See also [RSZ18, Remark 12.3]. The morphism $\mathcal{N}_0 \to \mathcal{N}_1$ we use is given by [RSZ18, Section 12].

We have 6 steps before the main result Proposition 6.1.9 of this 6.1.2.

First, we specify $\mathcal{N}_1$, $\mathcal{N}$ and $\mathcal{N}_1 \to \mathcal{N}$. Assume that $\varpi_{E_v}^2 = \varpi_{F_v}$. The framing object $\mathbb{X}_1$ for the deformation space $\mathcal{N}_1$ is the Serre tensor $\mathcal{O}_{E_v} \otimes_{\mathcal{O}_{F_v}} \overline{\mathbb{E}}$, which is the conjugate of the framing object [RSZ17, (3.5)], with the polarization conjugate to the one in [RSZ17, (3.6)]. In the case that n is odd (the case of Lemma 5.1.1 (1)), the framing object for $\mathcal{N}$ is $\mathbb{X}_n := \mathbb{X}_1 \times (\mathbb{E}^2)^{(n-1)/2}$ with the product polarization λ_n, where the polarization on $\mathbb{E}^2$ is given by

$$\lambda = \begin{bmatrix} 0 & \lambda_{\mathbb{E}}\iota(\varpi_{E_v}) \\ -\lambda_{\mathbb{E}}\iota(\varpi_{E_v}) & 0 \end{bmatrix}. \tag{6.10}$$

In the same way, we have a polarization $\widetilde{\lambda}$ on $\mathcal{E}^2$ using $\lambda_{\mathcal{E}}$. This gives us a morphism $\mathcal{N}_1 \to \mathcal{N}$ by $X \mapsto X \times (\mathcal{E}^2)^{(n-1)/2}$ with the polarization $\widetilde{\lambda}$ on each of $\mathcal{E}^2$. In the case that n is even (the case of Lemma 5.1.1 (2)), the framing object for $\mathcal{N}$ is $\mathbb{X}_n := \mathbb{X}_1 \times (\mathbb{E}^2)^{(n-2)/2} \times \mathbb{E}$ where the polarization $\lambda_{\mathbb{E}}'$ on the last component is a multiple of $\lambda_{\mathbb{E}}$ so that the induced hermitian pairing on $\operatorname{Hom}(\mathbb{E}, \mathbb{E})_{\mathbb{Q}}$ (defined as in (6.4)) has determinant $q(e_v)$ as in Lemma 5.1.1 (2). This gives us a morphism $\mathcal{N}_1 \to \mathcal{N}$ by $X \mapsto X \times (\mathcal{E}^2)^{(n-1)/2} \times \mathcal{E}$ with the unique lifting of $\lambda_{\mathbb{E}}'$ on the last component $\mathcal{E}$.

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Second, the uniformizations of $\widehat{\mathcal{X}^{ss}_{K,\mathcal{O}_{E_v^{\mathrm{ur}}}}}$ and $Z_t(\phi)^{\mathrm{zar}}$ are as follows. By [RSZ17, (3.10)], $V(E_v) \simeq \mathrm{Hom}_{\mathcal{O}_{E_v}}(\mathbb{E}, \mathbb{X}_n)_{\mathbb{Q}}$ and $U(V(E_v))$ is isomorphic to the group of $\mathcal{O}_{E_v}$-linear self-quasi-isogenies of $\mathbb{X}_n$ preserving λ_n. In particular, $U(V)$ acts on $\mathcal{N}$. The analog of the formal uniformization (6.5) of $\widehat{\mathcal{X}^{ss}_{K,\mathcal{O}_{E_v^{\mathrm{ur}}}}}$ holds by [LL22, (4.9)]. For every $x \in V(E_v) - \{0\}$, we have the Kudla-Rapoport divisor $\mathcal{Z}(x)$ on $\mathcal{N}$, which is a (possibly empty) relative Cartier divisor [LL22, Lemma 2.41]. The analog of Proposition 6.1.5 holds by [LL22, Proposition 4.29] combined with the argument in the proof of Proposition 6.1.5. Though [LL22] only uses even dimensional hermitian spaces, the specific results that we cite hold in the general case by the same proof.

Third, we recall the morphisms $\mathcal{N}_0 \to \mathcal{N}_1$ defined in [RSZ18, Section 12]. This is rather complicated for general $\mathcal{N}_{2m} \to \mathcal{N}_{2m+1}$. Fortunately, in our case, we have the following convenient description. By [RSZ18, Example 12.2], $\mathcal{N}_1$ is isomorphic to the disjoint union of two copies of the Lubin-Tate deformation space for the formal $\mathcal{O}_{F_v}$-module $\mathbb{E}$. We write $\mathcal{N}_1 = \mathcal{N}_1^+ \coprod \mathcal{N}_1^-$. Recall that B is the unique division quaternion algebra over F_v, and its maximal order $\mathcal{O}_B$ is the endomorphism ring of $\mathbb{E}$. For $c \in B^\times$, we have two closed embeddings (moduli of the canonical lifting) $\mathcal{N}_0 \to \mathcal{N}_1^\pm$ associated to $c\iota c^{-1} : E_v \hookrightarrow B$. Let $\mathcal{N}_0^{c,\pm}$ be the union of the images.

Fourth, we need to specify c so that we can use $\mathcal{N}_0^{c,\pm}$ to uniformize $\mathcal{P}$. See (6.11) below. Let $e_v^{(1)}$ be as in 5.1.4. Then

$$B = \mathrm{Hom}_{\mathcal{O}_{F_v}}(\mathbb{E}, \overline{\mathbb{E}})_{\mathbb{Q}} \simeq \mathrm{Hom}_{\mathcal{O}_{E_v}}(\mathbb{E}, \mathbb{X}_1)_{\mathbb{Q}} \simeq W(E_v) \oplus E_v e_v^{(1)},$$

where the middle is the adjunction isomorphism, and the last isomorphism is compatible with $\mathrm{Hom}_{\mathcal{O}_{E_v}}(\mathbb{E}, \mathbb{X}_n)_{\mathbb{Q}} \simeq V(E_v)$. And the hermitian form on $W(E_v) \oplus E_v e_v^{(1)}$ corresponds to $-2\varpi_{F_v}\mathrm{Nm}_B$ (see the proof of [RSZ17, Lemma 3.5]), where Nm_B is the reduced norm on B. Let c correspond to $\varpi_{E_v} e_v^{(1)}$. Since $q\left(e_v^{(1)}\right) \in \mathcal{O}_{F_v}^\times$ (see 5.1.2), $c \in \mathcal{O}_B^\times$ (note that $v \nmid 2$ here).

Fifth, we uniformize $\mathcal{P}$. By Lemma 5.1.6, we have another description of $\mathcal{P}$ via the diagram (5.5) of morphisms of Shimura varieties. See also Remark 5.1.7. Comparing it with the moduli interpretation of $\mathcal{N}_0 \to \mathcal{N}_1$ in [RSZ18, Proposition 12.1], we have the following analog of (6.6):

$$\mathcal{P}_{\mathcal{O}_{E_v^{\mathrm{ur}}}} = \frac{1}{2d_{K_{\mathbb{W}}^{(0)}}} U(W)\backslash\left(\mathcal{N}^{c,\pm}_{0,\mathcal{O}_{E_v^{\mathrm{ur}}}} \times U(\mathbb{W}^{\infty,v})/K_{\mathbb{W}}^v\right), \tag{6.11}$$

where $d_{K_{\mathbb{W}}^{(0)}}$ is the degree of the fundamental cycle of $\mathrm{Sh}(\mathbb{W})_{K_{\mathbb{W}}^{(0)}}$. See (5.5). Here, the extra factor 2 comes from Lemma 5.1.5.

Finally, we compute $\left(\mathcal{Z}(x) \cdot \mathcal{N}_0^{c,\pm}\right)_{\mathcal{N}}$. Since $c \in \mathcal{O}_B^\times$, by [RSZ17, Lemma 6.5, Proposition 7.1],

$$\mathcal{N}_0^{c,\pm} = \mathcal{Y}\left(\varpi_{E_v} e_v^{(1)}\right). \tag{6.12}$$

Here, $\mathcal{Y}\left(\varpi_{E_v} e_v^{(1)}\right)$ is the Kudla-Rapoport divisor on $\mathcal{N}_1$, where $\varpi_{E_v} e_v^{(1)}$ lifts. Let $\mathbb{X}^\perp$ be the direct complement of $\mathbb{X}_1$ in $\mathbb{X}_n$, that is, $\mathbb{X}^\perp = (\mathbb{E}^2)^{(n-1)/2}$ if n is odd, and $\mathbb{X}^\perp = (\mathbb{E}^2)^{(n-2)/2} \times \mathbb{E}$ is n is even. Let $\Lambda_{v,1}^\perp$ be as in 5.1.4. By Lemma 5.1.1 and (6.10),

$$\mathrm{Hom}_{\mathcal{O}_{E_v}}(\mathbb{E}, \mathbb{X}^\perp) \subset \mathrm{Hom}_{\mathcal{O}_{E_v}}(\mathbb{E}, \mathbb{X}_n)_{\mathbb{Q}} \simeq V(E_v)$$

corresponds to $\Lambda_v^\perp$. Then (similar to the deduction of (6.9)) by (6.12) and Gross' result on canonical lifting [Gro86, Proposition 3.3], we have

$$\left(\mathcal{Z}(x) \cdot \mathcal{N}_0^{c,\pm}\right)_{\mathcal{N}} = 2(v(q(x_1)) + 1)1_{\mathcal{O}_{F_v}}(q(x_1))1_{\varpi_{E_v}\mathcal{O}_{E_v}e_v^{(1)} \oplus \Lambda_v^\perp}(x_2) \tag{6.13}$$

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for $x = (x_1, x_2) \in W(E_v) \oplus V^\sharp(E_v)$ with $x_1 \neq 0$. Here, the extra factor 2 comes from that $\mathcal{N}_0^{c,\pm}$ has 2 components.

Proposition 6.1.9. *Assume that $g \in P(\mathbb{A}_{F,R\cup\{v\}})G(\mathbb{A}_F^{R\cup\{v\}})$ and $k = 1$. Then (6.2) holds.*

Proof. As an analog of (6.8), we have

$$(Z_t(\omega(g)\phi)^{\mathrm{zar}} \cdot \mathcal{P})_{\mathcal{X}_{K,\mathcal{O}_{E_v}}} = \frac{1}{2\mathrm{Vol}([U(W)])} \int_{h\in[U(W)]} \sum_{x\in V^t} \left(\mathcal{Z}(h_v^{-1}x) \cdot \mathcal{N}_0\right)_{\mathcal{N}} \omega(g)\phi^v(h^{-1}x)dh.$$

The rest of the proof is the same as the proof of Proposition 6.1.7, after replacing Lemma 5.2.2 by Lemma 5.2.3. □

6.1.3. Infinite places of F

Let $v \in \infty$. Under the complex uniformization (4.4) of $\mathrm{Sh}(\mathbb{V})_{K,E_v}$,

$$\mathcal{P}_{E_v} = \frac{1}{d_{\mathbb{W},K}} U(W)\backslash(\{o\} \times U(\mathbb{W}^\infty)/K_{\mathbb{W}}),$$

where $o := [0, \dots, 0] \in \mathbb{B}_n$, and $d_{\mathbb{W},K}$ is the degree of the fundamental cycle of $\mathrm{Sh}(\mathbb{W})_{K_{\mathbb{W}}}$. For $g \in P(\mathbb{A}_{F,R})G(\mathbb{A}_F^R)$, by the definition of $\mathcal{G}_{Z_t(\cdot)_{E_v},s}$ (above (4.10)), a direct computation gives

$$\int_{\mathcal{P}_{E_v}} \mathcal{G}_{Z_t(\omega(g)\phi)_{E_v},s} = \frac{W_{v,t}^{\mathfrak{w}}(g_v)}{\mathrm{Vol}([U(W)])} \int_{[U(W)]} \sum_{x\in V^t - V^\sharp} G_{h_v^{-1}x,s}(o)\omega(g^v)\phi^v(h^{v,-1}x)dh. \tag{6.14}$$

Now we compare the inner sums in (3.33) and (6.14). Recall the involved functions P_s and Q_s. See (3.32) and (4.7), respectively. From (3.32), we have

$$P_s(u) = \frac{1}{(s+n)u^{s+n}} F\left(s+n, s+n, s+n+1, \frac{-1}{u}\right),$$

where F is the hypergeometric function. In particular,

$$P_s(u) = \frac{1}{(s+n)u^{s+n}} + O\left(\frac{1}{u^{s+n+1}}\right), \ u \to \infty, \tag{6.15}$$

where the constant for $O(\cdot)$ is uniform near $s = 0$. We also have

$$P_0(u) = \log(1+u) - \log u - \sum_{i=1}^{n-1} \frac{1}{i(1+u)^i}, \ u > 0. \tag{6.16}$$

From (4.7), we have

$$Q_s(1+u) = \frac{\Gamma(s+n)\Gamma(s+1)}{\Gamma(2s+n+1)u^{s+n}} + O\left(\frac{1}{u^{s+n+1}}\right), \ u \to \infty, \tag{6.17}$$

where the constant for $O(\cdot)$ is uniform near $s = 0$. We also have

$$Q_0(1+u) = \log(1+u) - \log u - \sum_{i=1}^{n-1} \frac{1}{i(1+u)^i}, \ u > 0. \tag{6.18}$$

Lemma 6.1.10. *For $s_0 > -1$, on $\{s \in \mathbb{C}, \mathrm{Re} s > s_0\}$, the sum*

$$\sum_{x \in V^t - V^\sharp} \left(\left(\frac{\Gamma(s+n+1)\Gamma(s+1)}{\Gamma(2s+n+1)} \right)^{-1} G_{h_v^{-1}x,s}(o) - \left(\frac{\Gamma(s+n)}{\Gamma(n)(4\pi)^s} \right)^{-1} \widetilde{W\theta}_s(h_v^{-1}x) \right) \omega(g^v)\phi^v\left(h^{v,-1}x\right)$$

converges uniformly and absolutely. And its value at $s = 0$ is 0.

Proof. We compare the sum in the lemma with the sum in (6.14), which is absolutely convergent by Lemma 4.2.1. By (6.15) and (6.17), the sum in the lemma for s is dominated by a multiple of the sum in (6.14) with s replaced by $s + 1$. The convergence in the lemma follows. By (6.16) and (6.18), the value at $s = 0$ is 0. □

By Theorem 4.2.2, the integration on the right-hand side of (6.14) admits a meromorphic continuation to s around 0 with a simple pole at $s = 0$.

Proof of Lemma 3.3.12. For this moment, we consider a general $\phi \in \overline{\mathcal{S}}(\mathbb{V})$ (without any regularity assumption). The above discussion still holds replacing $Z_t(\phi)$ by

$$Z_t(\phi) - \sum_{x \in K \backslash K V^\sharp,\ q(x)=t} \phi(x) Z(x).$$

Then Lemma 3.3.12 follows from (6.14) and the first part of Lemma 6.1.10. □

Recall the definition of $\mathcal{G}^{\mathrm{aut}}_{Z_t(\cdot)_{E_v}}$ in (4.10). Then

$$\begin{aligned} 2\int_{\mathcal{P}_{E_v}} \mathcal{G}^{\mathrm{aut}}_{Z_t(\omega(g)\phi)_{E_v}} &= 2\widetilde{\lim_{s\to 0}} \left(\frac{\Gamma(s+n+1)\Gamma(s+1)}{\Gamma(2s+n+1)} \right)^{-1} \int_{\mathcal{P}_{E_v}} \mathcal{G}_{Z_t(\phi)_{E_v},s} \\ &= \widetilde{\lim_{s\to 0}} \left(\frac{\Gamma(s+n)}{\Gamma(n)(4\pi)^s} \right)^{-1} \theta E'_{t,s}(0, g, \phi)(v) = \widetilde{\lim_{s\to 0}} \theta E'_{t,s}(0, g, \phi)(v), \end{aligned}$$

where both multipliers $(\cdot)^{-1}$ go to 1 as $s \to 0$ (this gives the first and third '='), and the second '=' follows from Lemma 6.1.10. Thus, we have proved the following proposition.

Proposition 6.1.11. *For $g \in P(\mathbb{A}_{F,R})G(\mathbb{A}_F^R)$, (6.3) holds for v.*

6.2. Improper intersections

In this subsection, we prove the arithmetic mixed Siegel-Weil formula (Theorem 5.2.5). The proof starts in 6.2.1. Before that, let us discuss the strategy.

Lemma 6.2.1. *Let $Y = X \oplus X'$ be the orthogonal direct sum of two non-degenerate quadratic spaces over a non-archimedean local field of characteristic $\neq 2$. Let $\widehat{\mathcal{S}}(Y - X')_{\overline{\mathbb{Q}}}$ be the space of the Fourier transforms of functions in the space $\mathcal{S}(Y - X')_{\overline{\mathbb{Q}}}$ of $\overline{\mathbb{Q}}$-valued Schwartz functions on Y supported on $Y - X'$ (the Fourier transforms are clearly also $\overline{\mathbb{Q}}$-valued). Then*

$$\mathcal{S}(Y)_{\overline{\mathbb{Q}}} = \mathcal{S}(Y - X')_{\overline{\mathbb{Q}}} + \widehat{\mathcal{S}}(Y - X')_{\overline{\mathbb{Q}}}.$$

Proof. Since Fourier transform respects orthogonal direct sum, one may assume that $X' = \{0\}$. Then the lemma is well known and also easy to check directly. □

We will use the following notation. For a finite set S of finite places of F, let

$$\overline{\mathbb{Q}} \log S := \overline{\mathbb{Q}}\{\log p : v | p \text{ for some } v \in S\} \subset \mathbb{C}. \tag{6.19}$$

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Table 1.

		CM cycle	
		$\mathbb{W}$	$\mathbb{W}'$
Regularity	$\mathbb{W}$	L	
	$\mathbb{W}'$	B	L′
	No	C	

The table displays the mod $\overline{\mathbb{Q}}\log S$-version of Theorem 5.2.5 under different conditions. The second row indicates that we consider the CM cycle associated to $\mathbb{W}$ or $\mathbb{W}'$. The second column indicates that we impose the regularity assumption on ϕ associated to $\mathbb{W}$ or $\mathbb{W}'$, or no regularity assumption. Then a cell indexed by them indicates the mod $\overline{\mathbb{Q}}\log S$-version of Theorem 5.2.5 for the corresponding CM cycle under the corresponding regularity assumption on ϕ.

Table 2.

		CM cycle		
		$\mathbb{W}$	$\mathbb{W}'$	Difference
Regularity	$\mathbb{W}$	L		
	$\mathbb{W}'$	L′+P/$\mathbb{W}'$ $\Rightarrow$ B	L′	P/$\mathbb{W}'$
	No	C		P

The main difficulty in proving Theorem 5.2.5 is from improper-intersections. However, if we choose a $\overline{\mathbb{Q}}$-valued pure tensor ϕ that is $\mathbb{W}_v$-regular at some places v in S, we can prove a mod $\overline{\mathbb{Q}}\log S$-version of Theorem 5.2.5 for ϕ, see Lemma 6.2.3. Note that both the CM cycle and the regularity assumption are associated to $\mathbb{W}$. The same result holds if we replace $\mathbb{W}$ by some $\mathbb{W}'$ (and use the corresponding CM cycle and regularity assumption). Accordingly, we make Table 1.

Lemma 6.2.3 gives the cell L of Table 1. Replacing $\mathbb{W}$ by $\mathbb{W}'$, we get L′.

We want to arrive at the cell C (proved in Corollary 6.2.9), the mod $\overline{\mathbb{Q}}\log S$-version of Theorem 5.2.5 for the general ϕ. We use the cell B the bridge from L to C. The relation between this cell B and L′ is the 'switch CM cycles' indicated in the last paragraph above 1.3. The relation between B and L could be considered as 'switch regularity assumptions'.

Instead of considering this cell B directly, we consider the generating series of arithmetic intersection numbers with the difference of the two CM cycles, which is modular by Lemma 4.3.6. Then using Lemma 6.2.1, we prove the mod $\overline{\mathbb{Q}}\log S$-version of Theorem 5.2.5 for the general ϕ, after replacing the CM cycle by the difference. See Proposition 6.2.5. This gives P of Table 2. Then the combination of L′ and P/$\mathbb{W}'$ proves the cell B. Here, P/$\mathbb{W}'$ is the special case of P under the extra regularity assumption associated to $\mathbb{W}'$.

We need to remove '(mod $\overline{\mathbb{Q}}\log S$)'. We will use the following theorem. It is a corollary of Baker's celebrated theorem on transcendence of logarithms of algebraic numbers (see [Wal03, Theorem 1.1]), and the fact that logarithms of prime numbers are $\mathbb{Q}$-linearly independent.

Theorem 6.2.2. *Let $p_1, \ldots, p_m$ be distinct prime numbers, then $\log p_1, \ldots, \log p_m$ are $\overline{\mathbb{Q}}$-linearly independent.*

6.2.1. Set-up

We need the following convenient notation. For a set T of finite places of F, let

$$\mathbb{G}_T = P(\mathbb{A}_{F,\mathfrak{Ram}\cup T})G\left(\mathbb{A}_F^{\mathfrak{Ram}\cup T\cup\infty}\right).$$

For example, $\mathbb{G}_\emptyset = G\left(\mathbb{A}_F^{\mathfrak{Ram}\cup\infty}\right)P(\mathbb{A}_{F,\mathfrak{Ram}})$ is the group appearing in Theorem 5.2.5.

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Below, let $\phi \in \overline{\mathcal{S}}(\mathbb{V})^{\widetilde{K}_\Lambda}$ be a pure tensor such that ϕ^∞ is $\overline{\mathbb{Q}}$-valued and $\phi_v = 1_{\Lambda_v}$ for every finite place v of F nonsplit in E. Let $K \in \widetilde{K}_\Lambda$ stabilize ϕ. Let S be a set of finite places of F, and $K' = K_S K_\Lambda^S$. Let

$$\mathcal{P}_{\mathbb{W}} = \pi_{K,K'}^* \mathcal{P}_{\mathbb{W},K'}.$$

Let $f_{\mathbb{W}} = f_{\mathbb{W},K'}^K$ be defined as in (5.9). We remind the reader that we will use other incoherent hermitian spaces $\mathbb{W}'$ over $\mathbb{A}_E$ of dimension 1. These notations apply to $\mathbb{W}'$ in the same way.

Let $t \in F_{>0}$. Our goal is to prove that for $g \in \mathbb{G}_\emptyset$ and a suitable set S of finite places of F,

$$2z_t(g, \phi^\infty)_{\mathfrak{e}}^{\mathcal{L},\mathrm{aut}} \cdot \mathcal{P}_{\mathbb{W}} \bmod \overline{\mathbb{Q}} \log S = f_{\mathbb{W},t}^\infty(g) \bmod \overline{\mathbb{Q}} \log S. \tag{6.20}$$

We also introduce an equation equivalent to (6.20), and both will play roles in the proof of Theorem 5.2.5. The ψ_t-Whittaker function of the right-hand side of (5.9) (which is the definition of $f_{\mathbb{W}}$), coincides with the right-hand side of (3.31) up to the last term. Comparing the definition of $z_t(g,\phi)_{\mathfrak{e}}^{\mathcal{L},\mathrm{aut}}$ with the left-hand side of (3.31), (6.20) is equivalent to the following equation:

$$\begin{aligned} &\frac{2[Z_t(\omega(g)\phi)^{\mathcal{L},\mathrm{aut}}] \cdot \mathcal{P}_{\mathbb{W}}}{W_{\infty,t}^{\mathfrak{w}}(1)} \bmod \overline{\mathbb{Q}} \log S = \\ &\sum_{k \in U(\mathbb{W}) \backslash K'/K} \left(-\frac{\theta E'_{t,\mathrm{qhol}}(0,g,\omega(k)\phi)}{W_{\infty,t}^{\mathfrak{w}}(1)} - \sum_{v \in \mathfrak{Ram}} \frac{\theta E_t(0,g,\omega(k)\phi^v \otimes \phi'_v)}{W_{\infty,t}^{\mathfrak{w}}(1)} \right) \bmod \overline{\mathbb{Q}} \log S. \end{aligned} \tag{6.21}$$

6.2.2. Regular test functions

We use Assumption 6.1.2 on regularity here.

Lemma 6.2.3. *Assume that S contains a nonempty subset R such that ϕ_v is $\mathbb{W}_v$-regular for $v \in R$ (i.e., Assumption 6.1.2 holds). Then for $g \in \mathbb{G}_R$, (6.20) holds. Equivalently, (6.21) holds.*

Remark 6.2.4. The statement in Lemma 6.2.3 becomes more transparent if $S = R$. However, we need the flexibility to vary such R in S later.

Proof. Recall (5.4),

$$\pi_{K,K'}^* \mathcal{P}_{\mathbb{W},K'} = \sum_{k \in (U(\mathbb{W}) \cap K') \backslash K'/K} \frac{d_{k^{-1}\mathbb{W},K}}{d_{\mathbb{W},K'}} \mathcal{P}_{k^{-1}\mathbb{W},K}.$$

Here, we choose k such that $k_v = 1$ for $v \in S$ or nonsplit in E. (This is possible since $K' = K_S K_\Lambda^S$ and $K_v = K_{\Lambda,v}$ for v nonsplit in E.) Then Assumption 6.1.2 still holds with $\mathbb{W}$ replaced by $k^{-1}\mathbb{W}$. So by Lemma 6.1.3, $Z_t(\omega(g)\phi^\infty)$ and $\mathcal{P}_{k^{-1}\mathbb{W},K}$ do not meet on the generic fiber. Then by Lemma 4.4.6, we can apply Corollary A.2.6 at finite places $v \notin S$ (nothing happens if $K_v = K_{\Lambda,v}$). Then we have

$$\begin{aligned} &[Z_t(\omega(g)\phi^\infty)^{\mathcal{L},\mathrm{aut}}] \cdot \mathcal{P}_{\mathbb{W}} \bmod \overline{\mathbb{Q}} \log S = \\ &\left(\sum_{\substack{v \notin S \cup \infty, \\ \text{nonsplit in } E}} \left(Z_t(\omega(g)\phi^\infty)^{\mathrm{zar}} \cdot \mathcal{P}_{k^{-1}\mathbb{W},K}\right)_{\mathcal{X}_{K,\mathcal{O}_{E_v}}} \log q_{E_v} + \sum_{v \in \infty} \int_{\left(\mathcal{P}_{k^{-1}\mathbb{W},K}\right)_{E_v}} \mathcal{G}^{\mathrm{aut}}_{Z_t(\omega(g)\phi^\infty)_{E_v}} \right) \bmod \overline{\mathbb{Q}} \log S. \end{aligned}$$

Comparing this equation with (3.35), (6.21) is implied by Proposition 6.1.4. ◻

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6.2.3. CM cycles of degree 0

Let $\mathbb{W}'$ be another incoherent hermitian subspace of $\mathbb{W}$ and $\mathcal{P}_{\mathbb{W}'}$ the CM cycle defined accordingly as in 5.1. Since the automorphic Green function is admissible and $\mathcal{P}_{\mathbb{W},E} - \mathcal{P}_{\mathbb{W}',E}$ has degree 0, by Lemma 4.3.6,

$$z(\cdot,\phi)^{\mathcal{L},\mathrm{aut}}_{\mathfrak{e},a} \cdot (\mathcal{P}_{\mathbb{W}} - \mathcal{P}_{\mathbb{W}'}) \in \mathcal{A}_{\mathrm{hol}}(G,\mathfrak{w}). \tag{6.22}$$

Moreover, (6.22) is independent of the choice of a. We abbreviate $z(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{e},a}$ to $z(g,\phi)^{\overline{\mathcal{L}}}_{\mathfrak{e}}$. The 0-th Fourier coefficient of (6.22) is 0. Indeed, by Lemma 4.4.12, the action of K_Λ on $\widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\widetilde{\mathcal{X}})$ fixes $c_1(\overline{\mathcal{L}}^\vee_K)$. The vanishing of the 0-th Fourier coefficient follows from Lemma 5.1.4.

Proposition 6.2.5. *Assume that the cardinality of S is at least 2. Given ϕ as in 6.2.1, if K_S is small enough (depending on ϕ_S), then for all $g \in G(\mathbb{A}_F^\infty)$,*

$$2z_t(g,\phi^\infty)^{\mathcal{L},\mathrm{aut}}_{\mathfrak{e}} \cdot (\mathcal{P}_{\mathbb{W}} - \mathcal{P}_{\mathbb{W}'}) \bmod \overline{\mathbb{Q}}\log S = \left(f^\infty_{\mathbb{W},t}(g) - f^\infty_{\mathbb{W}',t}(g)\right) \bmod \overline{\mathbb{Q}}\log S. \tag{6.23}$$

Proof. For $G^{\mathrm{der}} = SU(1,1)$, $G(F_v) = G^{\mathrm{der}}(F_v)K_v^{\max}$. By Lemma 3.3.8, it is enough to prove (6.23) for $g \in G^{\mathrm{der}}(\mathbb{A}_F)$.

We need a lemma whose statement requires some more notations. Since $G^{\mathrm{der}} \simeq \mathrm{SL}_{2,F}$, by the q-expansion principle for $\mathrm{SL}_{2,F}$ [Cha90], we have

$$\mathcal{A}_{\mathrm{hol}}(G^{\mathrm{der}},\mathfrak{w}) = \mathcal{A}_{\mathrm{hol}}(G^{\mathrm{der}},\mathfrak{w})_{\overline{\mathbb{Q}}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}.$$

Here, $\mathcal{A}_{\mathrm{hol}}(G^{\mathrm{der}},\mathfrak{w})_{\overline{\mathbb{Q}}}$ is as in 2.7 with G replaced by G^{der}, and $\mathfrak{w}$ is understood as the restriction of $\mathfrak{w}$ to $G^{\mathrm{der}}(F_\infty) \cap K_v^{\max}$ for $v \in \infty$. Thus, we have Fourier coefficients as in 2.7. For $f \in \mathcal{A}_{\mathrm{hol}}(G^{\mathrm{der}},\mathfrak{w})$, let $[f]$ be its image in $\mathcal{A}_{\mathrm{hol}}(G^{\mathrm{der}},\mathfrak{w})_{\overline{\mathbb{Q}}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}/\overline{\mathbb{Q}}\log S$. Then the $\mathbb{C}/\overline{\mathbb{Q}}\log S$-valued locally constant function $f_t^\infty \bmod \overline{\mathbb{Q}}\log S$ on $G^{\mathrm{der}}(\mathbb{A}_F^\infty)$ coincides with the t-th Fourier coefficient of $[f]$.

Lemma 6.2.6. *Assume that ϕ_R is $\mathbb{W}_R$-regular, where $R \subset S$ consists of a single element, and $\phi_{R'}$ is $\mathbb{W}'_{R'}$-regular, where $R' \subset S\backslash R$ consists of a single element. Then we have the following equality in $\mathcal{A}_{\mathrm{hol}}(G^{\mathrm{der}},\mathfrak{w})_{\overline{\mathbb{Q}}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}/\overline{\mathbb{Q}}\log S$ after restriction from G to G^{der}:*

$$\left[2z(\cdot,\phi)^{\mathcal{L},\mathrm{aut}}_{\mathfrak{e}} \cdot (\mathcal{P}_{\mathbb{W}} - \mathcal{P}_{\mathbb{W}'})\right] = [f_{\mathbb{W}} - f_{\mathbb{W}'}]. \tag{6.24}$$

Proof. Consider the difference

$$f = 2z(\cdot,\phi)^{\mathcal{L},\mathrm{aut}}_{\mathfrak{e}} \cdot (\mathcal{P}_{\mathbb{W}} - \mathcal{P}_{\mathbb{W}'}) - (f_{\mathbb{W}} - f_{\mathbb{W}'}) \in \mathcal{A}_{\mathrm{hol}}(G,\mathfrak{w})$$

of the two sides of (6.24), before passing to $\mathcal{A}_{\mathrm{hol}}(G^{\mathrm{der}},\mathfrak{w})_{\overline{\mathbb{Q}}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}/\overline{\mathbb{Q}}\log S$. By the cuspidality of (6.22) and Lemma 3.3.9 (3), the 0-th Fourier coefficient $f_0^\infty(g) = 0$ for $g \in \mathbb{G}_{\{v_1,v_2\}}$. Write

$$[f|_{G^{\mathrm{der}}(\mathbb{A}_F)}] = \sum_i f_i \otimes a_i \tag{6.25}$$

as a finite sum, where $f_i \in \mathcal{A}_{\mathrm{hol}}(G^{\mathrm{der}},\mathfrak{w})_{\overline{\mathbb{Q}}}$ and $a_i \in \mathbb{C}/\overline{\mathbb{Q}}\log S$ are $\overline{\mathbb{Q}}$-linearly independent. Then for $t \in F_{>0} \cup \{0\}$, the t-th Fourier coefficient of (6.25) is $\sum_i f_{i,t}^\infty a_i$. For $g \in \mathbb{G}_{R\cup R'} \cap G^{\mathrm{der}}(\mathbb{A}_F)$, $\sum_i f_{i,t}^\infty(g)a_i = 0$ for $t \in F_{>0}$ by Lemma 6.2.3 (applied to $\mathbb{W},\mathbb{W}'$ respectively), and also for $t = 0$ by the above discussion for the constant term. Thus, $f_{i,t}^\infty(g) = 0$ by the $\overline{\mathbb{Q}}$-linear independence of a_i's. So $f_i(g) = 0$. By the density of $\mathbb{G}_{R\cup R'} \cap G^{\mathrm{der}}(\mathbb{A}_F)$ in $G^{\mathrm{der}}(F)\backslash G^{\mathrm{der}}(\mathbb{A}_F)$, $f_i(g) = 0$ for $G^{\mathrm{der}}(\mathbb{A}_F)$. So (6.24) holds. □

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Remark 6.2.7. The density argument can not be applied directly to $[f|_{G^{\mathrm{der}}(\mathbb{A}_F)}]$.

Now we continue the proof of the proposition. Recall that $w_v \in G^{\mathrm{der}}(F_v) \subset G(F_v)$ as in 2.5 acts on $\mathcal{S}(\mathbb{V}_v)$ by Fourier transform (multiplied by the Weil index) via the Weil representation ω. See 2.8. By Lemma 6.2.1, for a finite place v of F, there exists a $\mathbb{W}_v$-regular Schwartz function Φ_v on $\mathbb{V}(E_v)$ such that $\phi_v = \Phi_v + \omega(w_v)\Phi_v$. Choose $K_R, K_{R'}$ small enough to stabilize $\Phi_R, \Phi_{R'}$. By Lemma 6.2.6, (6.23) with $\phi_R, \phi_{R'}$ replaced by $\Phi_R, \Phi_{R'}$ holds for $g \in G^{\mathrm{der}}(\mathbb{A}_F)$, and thus, it holds for $gw_R, gw_{R'}, gw_Rw_{R'} \in G^{\mathrm{der}}(\mathbb{A}_F)$ replacing g as well. Then by Lemma 3.3.8, (6.23) with one or both of $\phi_R, \phi_{R'}$ replaced by $\omega(w_R)\Phi_R, \omega(w_{R'})\Phi_{R'}$ respectively holds for $g \in G^{\mathrm{der}}(\mathbb{A}_F)$. Thus, including (6.23), we have four equations in total. Taking their sum, we have (6.23) for the original ϕ and $g \in G^{\mathrm{der}}(\mathbb{A}_F)$. □

6.2.4. Remove regularity and log S

Lemma 6.2.8. *Assume that the cardinality of S is at least 2. Assume that $R \subset S$ consists of a single element and ϕ_R is $\mathbb{W}'_R$-regular. If K_S is small enough (depending on ϕ_S), then for $g \in \mathbb{G}_R$, (6.20) holds (literally, for $\mathbb{W}$ rather than $\mathbb{W}'$). Equivalently, (6.21) holds.*

Proof. By Lemma 6.2.3 (with $\mathbb{W}$ replaced by $\mathbb{W}'$), for $g \in \mathbb{G}_R$, we have

$$2z_t(g, \phi^\infty)_{\mathrm{e}}^{\mathcal{L},\mathrm{aut}} \cdot \mathcal{P}_{\mathbb{W}'} \bmod \overline{\mathbb{Q}} \log S = f^\infty_{\mathbb{W}',t}(g) \bmod \overline{\mathbb{Q}} \log S. \tag{6.26}$$

Taking the difference between (6.23) and (6.26), (6.20) follows for $g \in \mathbb{G}_R$. □

Corollary 6.2.9. *If K is small enough, then for $g \in \mathbb{G}_\emptyset$, (6.20) holds.*

Proof. We prove (6.21), which is equivalent to (6.20). Let $R \subset S$ consist of a single element. By Lemma 3.3.8 and the Iwasawa decomposition, it is enough to prove (6.21) for $g \in \mathbb{G}_R$. Then by Lemma 3.2.1, Lemma 3.3.14 and Lemma 4.1.4, we may assume that $\phi_R(0) = 0$. Such a ϕ_R can be written as a sum of $\mathbb{W}'_R$-regular functions for finitely many $\mathbb{W}'$'s (in fact, only depending on $\mathbb{W}'_R$). Since (6.21) is linear on ϕ_R, the corollary follows from Lemma 6.2.8 with $\mathbb{W}'^R = \mathbb{W}^R$ and $\mathbb{W}'_R$ varying. □

Proof of Theorem 5.2.5. We may assume that ϕ^∞ is $\overline{\mathbb{Q}}$-valued. It is enough to prove (5.10) modulo $\overline{\mathbb{Q}} \log S$. Indeed, choosing another set S' of 4 places split in E modulo $\overline{\mathbb{Q}} \log S'$ and requiring S and S' to have no same residue characteristics, then (5.10) follows from Theorem 6.2.2 (i.e., the $\overline{\mathbb{Q}}$-linear independence of $\log p$'s).

Now we prove (5.10) modulo $\overline{\mathbb{Q}} \log S$ by decomposing it into equations established in Corollary 6.2.9 for $l^{-1}\mathbb{W}$'s where $l \in U(\mathbb{W})\backslash K_\Lambda/K'$ (instead of a single $\mathbb{W}$, and the double coset is clarified below (5.4)). Note that by Remark 5.2.6, we may shrink K freely. For the left-hand side of (5.10), that is, $2z_t(g, \phi^\infty)_{\mathrm{e}}^{\mathcal{L},\mathrm{aut}} \cdot \pi^*_{K,K_\Lambda}\mathcal{P}_{\mathbb{W},K_\Lambda}$, by (5.4), we have

$$\begin{aligned}\pi^*_{K,K_\Lambda}\mathcal{P}_{\mathbb{W},K_\Lambda} &= \pi^*_{K,K'}\pi^*_{K',K_\Lambda}\mathcal{P}_{\mathbb{W},K_\Lambda} \\ &= \sum_{l \in U(\mathbb{W})\backslash K_\Lambda/K'} \frac{d_{l^{-1}\mathbb{W},K'}}{d_{\mathbb{W},K_\Lambda}} \pi^*_{K,K'}\mathcal{P}_{l^{-1}\mathbb{W},K'}.\end{aligned} \tag{6.27}$$

Now we consider the right-hand side of (5.10); that is, $f^{K,\infty}_{\mathbb{W},K_\Lambda,t}(g)$. We choose l such that $l_v = 1$ for $v \notin S$ (this is possible since $K' = K_S K^S_\Lambda$). In particular, $(l\mathbb{W})_v = \mathbb{W}_v$ for $v \in \mathfrak{Ram}$ so that ϕ'_v defined in (5.7) in terms of $\mathbb{W}_v$ is the same as that defined in (5.7) in terms of $(l\mathbb{W})_v$. By the coset decomposition

$$U(\mathbb{W})\backslash K_\Lambda/K = \coprod_{l \in U(\mathbb{W})\backslash K_\Lambda/K'} l\Big(U(l^{-1}\mathbb{W})\backslash K'\Big)/K,$$

and

$$\frac{d_{k^{-1}l^{-1}\mathbb{W},K}}{d_{\mathbb{W},K_\Lambda}} = \frac{d_{k^{-1}l^{-1}\mathbb{W},K}}{d_{l^{-1}\mathbb{W},K'}} \frac{d_{l^{-1}\mathbb{W},K'}}{d_{\mathbb{W},K_\Lambda}},\ k \in U(l^{-1}\mathbb{W})\backslash K',$$

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we deduce from the definition (5.9) of $f^K_{\mathbb{W},K_\Lambda}$ that

$$f^K_{\mathbb{W},K_\Lambda} = \sum_{l \in U(\mathbb{W})\backslash K_\Lambda/K'} \frac{d_{l^{-1}\mathbb{W},K'}}{d_{\mathbb{W},K_\Lambda}} f^K_{l^{-1}\mathbb{W},K'}. \tag{6.28}$$

By Corollary 6.2.9, (6.27) and (6.28) imply (5.10) modulo $\overline{\mathbb{Q}} \log S$. □

A. Admissible divisors

We recall S. Zhang's theory of admissible cycles on a polarized arithmetic variety [Zha20]. They are cycles with 'harmonic curvatures'. We only consider admissible divisors [Zha20, 2.5, Admissible cycles]. In particular, for a divisor on the generic fiber, we have its admissible extensions. With an extra local condition, we have the normalized admissible extension. The functoriality of (normalized) admissibile cycles under flat morphisms is important for us.

It is worth mentioning that while the normalized admissible extension is defined purely locally and at the level of divisors, a global lifting of a divisors class on the generic fiber is defined (and it is called L-lifting) in [Zha20, Corollary 2.5.7]. It is an admissible extension [Zha20, Corollary 2.5.7 (1)] with vanishing Faltings height [Zha20, Corollary 2.5.7 (2)]. They will not be further discussed in this appendix, and are not needed in this paper.

A.1. Deligne-Mumford stacks over a Dedekind domain

Let $\mathcal{O}$ be a Dedekind domain. Let $\mathcal{M}$ be a connected regular Deligne-Mumford stack proper flat over $\operatorname{Spec}\mathcal{O}$ of relative dimension n. Let M be its generic fiber. Definitions of cycles, rational equivalence, proper pushforward and flat pullback for (Chow) cycles are applicable to Deligne-Mumford stacks over $\operatorname{Spec}\mathcal{O}$. See [Gil09]. Let $Z^*(M)$ (resp. $Z^*(\mathcal{M})$) be the graded $\mathbb{Q}$-vector space of cycles on M (resp. $\mathcal{M}$) with $\mathbb{Q}$-coefficients. Let $\mathrm{Ch}^*(M)$ and $\mathrm{Ch}^*(\mathcal{M})$ be the $\mathbb{Q}$-vector spaces of Chow cycles.

We shall only work under the following convenient assumption, which simplifies the local intersection theory. For every closed point $s \in \operatorname{Spec}\mathcal{O}$, let $\mathcal{O}_s$ be the completed local ring.

Assumption A.1.1. (1) There is a finite subset $S \subset \operatorname{Spec}\mathcal{O}$, a regular scheme $\widetilde{\mathcal{M}}$ proper flat over $\operatorname{Spec}\mathcal{O} - S$ and a finite étale morphism $\pi : \widetilde{\mathcal{M}} \to \mathcal{M}|_{\operatorname{Spec}\mathcal{O}-S}$ over $\operatorname{Spec}\mathcal{O} - S$.

(2) For every $s \in S$, there is a regular scheme $\widetilde{\mathcal{M}}$ proper flat over $\operatorname{Spec}\mathcal{O}_s$ and a finite étale morphism $\pi : \widetilde{\mathcal{M}} \to \mathcal{M}_{\operatorname{Spec}\mathcal{O}_s}$ over $\operatorname{Spec}\mathcal{O}_s$.

In either case (1) or (2), we call $\widetilde{\mathcal{M}}$ a covering of $\mathcal{M}|_{\operatorname{Spec}\mathcal{O}-S}$ or $\mathcal{M}_{\operatorname{Spec}\mathcal{O}_s}$. For another covering $\widetilde{\mathcal{M}}'$, the fiber product is regular and proper flat over $\operatorname{Spec}\mathcal{O} - S$ or $\operatorname{Spec}\mathcal{O}_s$, making a third covering.

A line bundle $\mathcal{L}$ on $\mathcal{M}$ is ample if in both cases (1) and (2), its pullback to some covering is ample. And it is relatively positive if $\deg \mathcal{L}|_C > 0$ for every closed curve (1-dimensional closed substack) C in every special fiber of $\mathcal{M}$. It is routine to check that the definition does not depend on the choice of covering by making a third covering. The following notions, which are used in the whole paper, are also defined via coverings: intersection number, Chern class and Zariski closure.

A.2. Local cycles

Assume that $\mathcal{O}$ is a completed local ring (so a DVR). Let s be the unique closed point of $\operatorname{Spec}\mathcal{O}$. We will use two intersection pairings. First, for $X \in Z^i(\mathcal{M})$ and $Y \in Z^{n+1-i}(\mathcal{M})$ with disjoint supports on M, define

$$X \cdot Y = \frac{1}{\deg \pi} \pi^*(X) \cdot \pi^*(Y) \in \mathbb{Q}, \tag{A.1}$$

where π is a covering morphism and the latter intersection number is a usual one, defined either using Serre's Tor-formula (equivalently rephrased as the Euler-Poincaré characteristic of the derived tensor product $\mathcal{O}_{\pi^*(X)} \bigotimes^{\mathbb{L}} \mathcal{O}_{\pi^*(Y)}$ ([GS90, 4.3.8 (iv)]), or as a cohomological pairing.

Second, let $\mathcal{M}_s$ be the special fiber of $\mathcal{M}$ and $Z^1_s(\mathcal{M}) \subset Z^1(\mathcal{M})$ the subspace of divisors supported on $\mathcal{M}_s$. We use the intersection pairing between $Z^1_s(\mathcal{M})$ and an n-tuple of $\mathbb{Q}$-Cartier divisors as in [Ful84, Example 6.5.1] (defined using a covering as in (A.1)):

$$Z^1_s(\mathcal{M}) \times Z^1(\mathcal{M})^n \to \mathbb{Q}.$$

It only depends on the rational equivalence classes of the Cartier divisors. (In particular, by fixing $n-1$ rational equivalence classes of Cartier divisors, we get a pairing between $Z^1_s(\mathcal{M})$ and $Z^1(\mathcal{M})$. This view point might be helpful.) In this subsection, we will use this second intersection pairing until Corollary A.2.6.

Let $\mathcal{L}$ be a line bundle on $\mathcal{M}$. Let $B^1_{\mathcal{L}}(\mathcal{M}) \subset Z^1_s(\mathcal{M}$ be the kernel of the linear form $Z^1_s(\mathcal{M}) \to \mathbb{Q}$ defined by intersection with $c_1(\mathcal{L})^n$. Assume that the generic fiber of $\mathcal{L}$ is ample and $\mathcal{L}$ is relatively positive. The local index theorem [Zha20, Lemma 2.5.1] (see also [YZ17]) implies the following lemma.

Lemma A.2.1. *The pairing $(X, Y) \mapsto X \cdot c_1(\mathcal{L})^{n-1} \cdot Y$ on $B^1_{\mathcal{L}}(\mathcal{M})$ is negative definite.*

Let $Z^1_{\mathcal{L}}(\mathcal{M})$ be the orthogonal complement of $B^1_{\mathcal{L}}(\mathcal{M})$ under the pairing $X \cdot c_1(\mathcal{L})^{n-1} \cdot Y$; that is,

$$Z^1_{\mathcal{L}}(\mathcal{M}) = \{Y \in Z^1(\mathcal{M}) : X \cdot c_1(\mathcal{L})^{n-1} \cdot Y = 0 \text{ for every } X \in B^1_{\mathcal{L}}(\mathcal{M})\}.$$

Then by definition, we have a decomposition

$$Z^1(\mathcal{M}) = Z^1_{\mathcal{L}}(\mathcal{M}) \oplus B^1_{\mathcal{L}}(\mathcal{M}), \tag{A.2}$$

and an exact sequence

$$0 \to \mathbb{Q}\mathcal{M}_s \to Z^1_{\mathcal{L}}(\mathcal{M}) \to Z^1(M) \to 0. \tag{A.3}$$

For a prime cycle X on M, let X^{zar} be its Zariski closure on $\mathcal{M}$. Extend the definition by linearity.

Definition A.2.2. (1) We call $Z^1_{\mathcal{L}}(\mathcal{M})$ the space of admissible divisors with respect to $\mathcal{L}$.

(2) For $X \in Z^1(M)$, an admissible extension with respect to $\mathcal{L}$ is an element in its preimage by $Z^1_{\mathcal{L}}(\mathcal{M}) \to Z^1(M)$. Define the normalized admissible extension $X^{\mathcal{L}}$ of X with respect to $\mathcal{L}$ to be the projection of X^{zar} to $Z^1_{\mathcal{L}}(\mathcal{M})$ in (A.2).

Remark A.2.3. In terms of [Zha20, Corollary 2.5.7 (1)], $Z^1_{\mathcal{L}}(\mathcal{M}) \subset Z^1(\mathcal{M})$ is the subspace of cycles X with 'harmonic curvatures' (compare with Remark A.3.2 (3)); that is, the element in $\mathrm{Hom}_{\mathbb{Q}}\big(Z^1_s(\mathcal{M}), \mathbb{Q}\big)$ defined by intersection with $X \cdot c_1(\mathcal{L})^{n-1}$ is a multiple of the one defined by intersection with $c_1(\mathcal{L})^n$.

Then by definition, we have the following lemma.

Lemma A.2.4. *Assume that $\mathcal{M}$ is smooth over* Spec $\mathcal{O}$. *Then $B^1_{\mathcal{L}}(\mathcal{M}) = 0$. In particular, for $X \in Z^1(M)$, X^{zar} is the normalized admissible extension.*

By the projection formula (and the commutativity of taking Zariski closure and closed pushforward/flat pullback), we easily deduce the following lemma.

Lemma A.2.5. *Let $\mathcal{M}'$ be a regular Deligne-Mumford stack and $f : \mathcal{M}' \to \mathcal{M}$ a finite flat morphism. Let $\mathcal{L}'$ be the pullback of $\mathcal{L}$ to $\mathcal{M}'$. Consider $\mathcal{M}'$ as a Deligne-Mumford stack over* Spec $\mathcal{O}$ *via f and $\mathcal{M}$. Then $f^*\big(B^1_{\mathcal{L}}(\mathcal{M})\big) \subset B^1_{\mathcal{L}'}(\mathcal{M}')$, $f_*\big(B^1_{\mathcal{L}'}(\mathcal{M}')\big) = B^1_{\mathcal{L}}(\mathcal{M})$, $f^*\big(Z^1_{\mathcal{L}}(\mathcal{M})\big) \subset Z^1_{\mathcal{L}'}(\mathcal{M}')$ and $f_*\big(Z^1_{\mathcal{L}'}(\mathcal{M}')\big) = Z^1_{\mathcal{L}}(\mathcal{M})$. In particular, the decomposition (A.2) and the formation of normalized admissible extension are preserved under pullback and pushforward by f.*

Corollary A.2.6. *In Lemma A.2.5 with f finite, let M' be the generic fiber of $\mathcal{M}'$ and assume that $\mathcal{M} \to \operatorname{Spec} \mathcal{O}$ is smooth. Let $X \in Z^1(M')$ and $Y \in Z^n(\mathcal{M})$ such that X and $f^*(Y)$ have disjoint supports. Then we have $X^{\mathcal{L}} \cdot f^*(Y) = X^{\mathrm{zar}} \cdot f^*(Y)$.*

Proof. By Lemma A.2.4 and Lemma A.2.5, we have $f_*(X^{\mathcal{L}}) = (f|_{M',*}X)^{\mathcal{L}} = (f|_{M',*}X)^{\mathrm{zar}} = f_*(X^{\mathrm{zar}})$. The corollary follows from the projection formula. □

A.3. Admissible arithmetic Chow group of divisors

Now let $\mathcal{O}$ be the ring of integers of a number field. In particular, $\mathcal{M}_{\mathbb{C}} := \mathcal{M} \otimes_{\mathbb{Z}} \mathbb{C} = M \otimes_{\mathbb{Q}} \mathbb{C}$ is a complex orbifold. Let $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ be a hermitian line bundle on $\mathcal{M}$ such that the generic fiber of $\mathcal{L}$ is ample, $\mathcal{L}$ is relatively positive, and the hermitian metric $\|\cdot\|$ is invariant under the involution induced by complex conjugation. See [GS90, 3.1.2]. In particular, endowing $\mathcal{M}_{\mathbb{C}}$ with the Chern curvature form $\operatorname{curv}\left(\overline{\mathcal{L}}_{\mathbb{C}}\right)$, it is a smooth Kähler orbifold.

Definition A.3.1. (1) The group $\widehat{Z}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{M})$ of admissible (with respect to $\overline{\mathcal{L}}$) arithmetic divisors on $\mathcal{M}$ with $\mathbb{C}$-coefficients is the $\mathbb{C}$-vector space of pairs (X, g) where

- ∘ $X \in Z^1(\mathcal{M})_{\mathbb{C}}$ such that for every closed point $s \in \operatorname{Spec} \mathcal{O}$, the restriction of X to $\mathcal{M}_{\mathcal{O}_s}$ is contained in $Z^1_{\mathcal{L}_{\mathcal{O}_s}}(\mathcal{M}_{\mathcal{O}_s})_{\mathbb{C}}$,
- ∘ g is a Green function for $X_{\mathbb{C}}$ on $\mathcal{M}_{\mathbb{C}}$, admissible with respect to $\overline{\mathcal{L}}_{\mathbb{C}}$, and invariant under the involution induced by complex conjugation. Here, admissibility means that the curvature form $\delta_X + \frac{i}{2\pi}\partial\bar{\partial} g$ is harmonic.

(2) For $X \in Z^1(M)_{\mathbb{C}}$, an admissible extension of X with respect to $\overline{\mathcal{L}}$ is an element in the preimage of X by the natural surjection $\widehat{Z}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{M}) \to Z^1(M)_{\mathbb{C}}$.

Remark A.3.2. (1) A Green current on Deligne-Mumford stacks is defined in [Gil09, Section 1]. In our situation, a Green function is simply an orbifold function whose pullback to the finite étale cover by a smooth variety is a usual Green function.

(2) Admissible Green functions for $X_{\mathbb{C}}$ always exists and are the same modulo locally constant functions, and (A.3) is the non-archimedean analog of this fact.

(3) By [Zha20, 2.2], a closed $(1,1)$-form α is harmonic if and and only if on each connected component of $\mathcal{M}_{\mathbb{C}}$, $\alpha \wedge \operatorname{curv}\left(\overline{\mathcal{L}}_{\mathbb{C}}\right)^{n-1}$ is a constant multiple of $\operatorname{curv}\left(\overline{\mathcal{L}}_{\mathbb{C}}\right)^{n}$.

Definition A.3.3. (1) An admissible Green function is normalized with respect to $\overline{\mathcal{L}}_{\mathbb{C}}$ if it has vanishing harmonic projection; that is, on each connected component of $\mathcal{M}_{\mathbb{C}}$, its integration against $\operatorname{curv}(\overline{\mathcal{L}}_{\mathbb{C}})^n$ is 0.

(2) An element in $\widehat{Z}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{M})$ is normalized with respect to $\overline{\mathcal{L}}$ if it is normalized at every place. For $X \in Z^1(M)_{\mathbb{C}}$, let $X^{\overline{\mathcal{L}}} \in \widehat{Z}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{M})$ be its normalized admissible extension with respect to $\overline{\mathcal{L}}$.

Then the normalized admissible extension of a divisor on M exists and is unique.

For every nonzero rational function f on M, $(\operatorname{div}(f), -\log|f|^2)$ is contained in $\widehat{Z}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{M})$.

Definition A.3.4. (1) Let $\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{M})$ be the quotient of the space of arithmetic divisors with $\mathbb{C}$-coefficients by the $\mathbb{C}$-span of $(\operatorname{div}(f), -\log|f|^2)$'s for all nonzero rational functions.

(2) Let $\widehat{\mathrm{Ch}}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{M})$ be the quotient of $\widehat{Z}^1_{\overline{\mathcal{L}},\mathbb{C}}(\mathcal{M})$ by the $\mathbb{C}$-span of $(\operatorname{div}(f), -\log|f|^2)$'s.

Remark A.3.5. Let $\widehat{\mathrm{Ch}}^1(\mathcal{M})$ be the Chow group of arithmetic divisors with $\mathbb{Z}$-coefficients defined by Gillet and Soulé [GS90] for schemes, which is extended to the stacky case in [Gil09]. Then $\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{M})$

is the quotient of $\widehat{\mathrm{Ch}}^1(\mathcal{M})_{\mathbb{C}}$ by the pullback of the kernel of the degree map $\widehat{\mathrm{Ch}}^1(\operatorname{Spec}\mathcal{O})_{\mathbb{C}} \to \mathbb{C}$. In particular, we have an isomorphism $\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\operatorname{Spec}\mathcal{O}) \simeq \mathbb{C}$ by taking degrees.

Lemma A.3.6. *(1) The natural map $\widehat{\mathrm{Ch}}^1_{\mathcal{L},\mathbb{C}}(\mathcal{M}) \to \mathrm{Ch}^1(M)_{\mathbb{C}}$ is surjective. Its kernel is generated by connected components of special fibers of $\mathcal{M}$ at all finite places and locally constant functions on $\mathcal{M}_{\mathbb{C}}$.*

(2) Assume that $\mathcal{M}$ is connected. Then the kernel of $\widehat{\mathrm{Ch}}^1_{\mathcal{L},\mathbb{C}}(\mathcal{M}) \to \mathrm{Ch}^1(M)_{\mathbb{C}}$ is 1-dimensional, and is the pullback of $\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\operatorname{Spec}\mathcal{O})$.

Proof. By (A.3) and Remark A.3.2 (2), (1) holds. If $\mathcal{M}$ is connected, then $E_1 := \mathcal{O}_M(M)$ is a finite field extension of the fraction field of $\mathcal{O}$. Then $\mathcal{M}$ over $\operatorname{Spec}\mathcal{O}_{E_1}$ has geometrically connected fibers by Stein factorization. By (1), the kernel is the pullback of $\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\operatorname{Spec}\mathcal{O}_{E_1})$, which is 1-dimensional by the finiteness of the class number of E_1 and Dirichlet's unit theorem. See [GS90, 3.4.3]. And it equals the pullback of $\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\operatorname{Spec}\mathcal{O})$. □

Example A.3.7. The arithmetic first Chern class $c_1(\overline{\mathcal{L}})$ of $\overline{\mathcal{L}}$ is the class of $(\operatorname{div}(s), -\log\|s\|^2)$ for a nonzero rational section s. By Remark A.2.3 (or one may follow our definition), one immediately sees that $\operatorname{div}(s)$ has 'harmonic curvature' at every finite place. The curvature form of $-\log\|s\|^2$ is by definition the Kähler form. So $c_1(\overline{\mathcal{L}}) \in \widehat{\mathrm{Ch}}^1_{\mathbb{C},\overline{\mathcal{L}}}(\mathcal{M})$.

Now we consider the functoriality. By Lemma A.2.5, we have the following proposition.

Proposition A.3.8. *Let $\mathcal{M}'$ be a regular Deligne-Mumford stack and $f : \mathcal{M}' \to \mathcal{M}$ a finite flat morphism over* Spec $\mathcal{O}$*, such that the restriction of f to the generic fibers is finite étale. Let $\mathcal{L}'$ be the pullback of $\mathcal{L}$ to $\mathcal{M}'$. Consider $\mathcal{M}'$ as a Deligne-Mumford stack over* Spec $\mathcal{O}$ *via f and $\mathcal{M}$. Then the formation of $\widehat{Z}^1_{\mathbb{C},\overline{\mathcal{L}}}(\mathcal{M})$, $\widehat{\mathrm{Ch}}^1_{\mathbb{C},\overline{\mathcal{L}}}(\mathcal{M})$ and normalized admissible extension with respect to $\overline{\mathcal{L}}$ is preserved under pullback and pushforward by f.*

A.4. Arithmetic intersection pairing

Let $Z_1(\mathcal{M})_{\mathbb{C}}$ be the group of 1-cycles on $\mathcal{M}$. We define an arithmetic intersection pairing following [BGS94, 2.3.1]:

$$\widehat{\mathrm{Ch}}^1(\mathcal{M})_{\mathbb{C}} \times Z_1(\mathcal{M})_{\mathbb{C}} \to \mathbb{C}, \ (\widehat{x}, Y) \mapsto \widehat{x}\cdot Y.$$

We reduce the pairing to the arithmetic intersection pairing between $\widehat{\mathrm{Ch}}^1(\mathcal{M})_{\mathbb{C}}$ and $\widehat{\mathrm{Ch}}^n(\mathcal{M})_{\mathbb{C}}$, which is defined in [Gil09] for general Deligne-Mumford stacks (without Assumption A.1.1). Let $\omega(\widehat{x})$ be the curvature form of $\widehat{x}$, which is a smooth $(1,1)$ form on the orbifold $M_{\mathbb{C}}$ independent of the choice of a representative of $\widehat{x}$. Choose $\widehat{y} = (Y, g_Y) \in \widehat{\mathrm{Ch}}^n_{\mathbb{C}}(\mathcal{M})$. Then $\widehat{x}\cdot Y$ is the arithmetic intersection number $\widehat{x}\cdot\widehat{y}$ minus $\int_{\mathcal{M}_{\mathbb{C}}} \omega(\widehat{x}) g_Y$.

Now assume Assumption A.1.1 and that (X, g_X) is a representative of $\widehat{x}$ such that $X \cap Y$ is empty on the generic fiber of $\mathcal{M}$. Then $\widehat{x}\cdot Y$ is the sum of the intersection numbers of the restrictions of X and Y to $\mathcal{M}_{\mathcal{O}_s}$ over all closed points $s \in \operatorname{Spec}\mathcal{O}$ defined in (A.1), and $\int_{Y_{\mathbb{C}}} g_X$.

The pullback of the kernel of the degree map $\widehat{\mathrm{Ch}}^1(\operatorname{Spec}\mathcal{O})_{\mathbb{C}} \to \mathbb{C}$ to $\widehat{\mathrm{Ch}}^1(\mathcal{M})_{\mathbb{C}}$ is annihilated by the arithmetic intersection pairing with $Z_1(\mathcal{M})_{\mathbb{C}}$. By Remark A.3.5, the above arithmetic intersection pairing factors through an arithmetic intersection pairing

$$\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{M}) \times Z_1(\mathcal{M})_{\mathbb{C}} \to \mathbb{C}, \ (\widehat{z}, Y) \mapsto \widehat{z}\cdot Y.$$

Similar to Corollary A.2.6, Proposition A.3.8 implies the following result.

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Corollary A.4.1. *In Proposition A.3.8, let M' be the generic fiber of $\mathcal{M}'$ and assume that $\mathcal{M} \to \operatorname{Spec} \mathcal{O}_E$ is smooth. Then for $X \in Z^1(M')_{\mathbb{C}}$ and $Y \in Z^n_{\mathbb{C}}(\mathcal{M})$, we have*

$$[X^{\overline{\mathcal{L}}}] \cdot f^*(Y) = [(X^{\mathrm{zar}}, g_X^{\overline{\mathcal{L}}})] \cdot f^*(Y),$$

where $g_X^{\overline{\mathcal{L}}}$ is the normalized admissible Green function for X.

B. A comparison of the 'closure' model with Rapoport–Smithling–Zhang model (appendix by Yujie Xu)

B.1. Preliminaries

B.1.1.

Let F be a CM field and F_0 its maximal totally real subfield of index 2. Let $a \mapsto \overline{a}$ be the nontrivial automorphism of F/F_0. We fix a presentation $F = F_0(\sqrt{\Delta})$ for some totally negative element $\Delta \in F_0$. Let Φ denote the CM type for F determined by $\sqrt{\Delta}$; that is,

$$\Phi := \{\varphi : F \to \mathbb{C} | \varphi(\sqrt{\Delta}) \in \mathbb{R}_{>0} \cdot \sqrt{-1}\}. \tag{B.1.2}$$

Let W be a non-degenerate F/F_0-hermitian space of dimension $n \geq 2$. Let

$$G := \operatorname{Res}_{F_0/\mathbb{Q}} U(W). \tag{B.1.3}$$

As in [RSZ20, §2.1], we use the symbol c to denote the similitude factor of a point on a unitary similitude group. We consider the following algebraic groups over $\mathbb{Q}$:

$$Z^{\mathbb{Q}} := \{z \in \operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_m | \operatorname{Nm}_{F/F_0}(z) \in \mathbb{G}_m\} \tag{B.1.4}$$

$$G^{\mathbb{Q}} := \{g \in \operatorname{Res}_{F_0/\mathbb{Q}} \mathrm{GU}(W) | c(g) \in \mathbb{G}_m\} \tag{B.1.5}$$

$$\widetilde{G} := Z^{\mathbb{Q}} \times_{\mathbb{G}_m} G^{\mathbb{Q}} = \{(z, g) \in Z^{\mathbb{Q}} \times G^{\mathbb{Q}} | \operatorname{Nm}_{F/F_0}(z) = c(g)\}. \tag{B.1.6}$$

Note that $Z^{\mathbb{Q}}$ is naturally a central subgroup of $G^{\mathbb{Q}}$, and we have the following product decompositions

$$\widetilde{G} \longrightarrow \sim Z^{\mathbb{Q}} \times G, \ (z, g) \longmapsto (z, z^{-1} g). \tag{B.1.7}$$

B.1.8.

From now on, we assume, moreover, that the hermitian space W has the following signatures at the archimedean places of F_0: for a distinguished element $\varphi_0 \in \Phi$, the signature of W_{φ_0} is $(1, n-1)$; and for all other $\varphi \in \Phi$, the signature of W_φ is $(0, n)$. In order to define a Shimura datum $(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\})$, by the canonical inclusions $G^{\mathbb{Q}}_{\mathbb{R}} \subset \prod_{\varphi \in \Phi} \mathrm{GU}(W_\varphi)$, it suffices to define the components $h_{G^{\mathbb{Q}}, \varphi}$ of $h_{G^{\mathbb{Q}}}$. Consider the matrices

$$J_\varphi := \begin{cases} \operatorname{diag}(1, (-1)^{(n-1)}), & \varphi = \varphi_0, \\ \operatorname{diag}(-1, -1, \cdots, -1), & \varphi \in \Phi \setminus \{\varphi_0\}. \end{cases} \tag{B.1.9}$$

We also choose bases $W_\varphi \simeq \mathbb{C}^n$ such that the hermitian form on W_φ is given by J_φ. Consider the $\mathbb{R}$-algebra homomorphisms

$$\mathbb{C} \longrightarrow \operatorname{End}(W_\varphi), \ \sqrt{-1} \longmapsto \sqrt{-1} J_\varphi, \tag{B.1.10}$$

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which induce our desired component maps $h_{G^{\mathbb{Q}},\varphi} : \mathbb{C}^\times \to \mathrm{GU}(W_\varphi)(\mathbb{R})$. This gives us our desired Shimura datum $(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\})$.

B.1.11.
For the group $Z^{\mathbb{Q}}$ defined in B.1.4, the CM type Φ induces an identification

$$Z^{\mathbb{Q}}(\mathbb{R}) \simeq \left\{(z_\varphi) \in (\mathbb{C}^\times)^\Phi \middle| |z_\varphi| = |z_{\varphi'}| \text{ for all } \varphi, \varphi' \in \Phi\right\}, \tag{B.1.12}$$

which allows us to define $h_{Z^{\mathbb{Q}}} : \mathbb{C}^\times \to Z^{\mathbb{Q}}(\mathbb{R})$ as the diagonal embedding (via the identification B.1.12) precomposed with complex conjugation. This gives us a Shimura datum $(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})$ with reflex field

$$E(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\}) = E_\Phi, \tag{B.1.13}$$

which is the reflex field for the CM type Φ. Recall that this can be computed as the fixed field in $\mathbb{C}$ of the group $\{\sigma \in \mathrm{Aut}(\mathbb{C}) = |\sigma \circ \Phi = \Phi\}$.

For the group $\widetilde{G}$ defined in B.1.6, we consider the map

$$h_{\widetilde{G}} : \mathbb{C}^\times \xrightarrow{(h_{Z^{\mathbb{Q}}}, h_{G^{\mathbb{Q}}})} \widetilde{G}(\mathbb{R}), \tag{B.1.14}$$

which gives us a Shimura datum $(\widetilde{G}, \{h_{\widetilde{G}}\})$. Let

$$E := E(\widetilde{G}, \{h_{\widetilde{G}}\}) \tag{B.1.15}$$

be the reflex field for $(\widetilde{G}, \{h_{\widetilde{G}}\})$, by definition it is computed via

$$\mathrm{Aut}(\mathbb{C}/E) = \{\sigma \in \mathrm{Aut}(\mathbb{C}) | \sigma \circ \Phi = \Phi, \sigma \circ \varphi_0 = \varphi_0\}. \tag{B.1.16}$$

Note that E is the compositum of E_Φ (as in B.1.13) and F. By [Del71], we have canonical models $\mathrm{Sh}_{K_{\widetilde{G}}}(\widetilde{G}, \{h_{\widetilde{G}}\})$ over E, for compact open subgroups $K_{\widetilde{G}} \subset \widetilde{G}(\mathbb{A}_f)$. By [Kis10] (resp. [KP18] depending on the level structure), we have integral models $\mathcal{S}_{K_{\widetilde{G}}}(\widetilde{G}, \{h_{\widetilde{G}}\})$ over $\mathcal{O}_{E,(v)}$.

B.1.17.
We introduce a Shimura datum $(G, \{h_G\})$. Let h_G be the map

$$h_G : \mathbb{C}^\times \xrightarrow{h_{\widetilde{G}}} \widetilde{G}(\mathbb{R}) \xrightarrow{B.1.7} G(\mathbb{R}) \tag{B.1.18}$$

defined by composing $h_{\widetilde{G}}$ (from B.1.14) with the projection onto the second factor in the map B.1.7. The reflex field for $(G, \{h_G\})$ is F, embedded into $\mathbb{C}$ via φ_0.

Moreover, the decomposition in B.1.7 induces a decomposition of Shimura data

$$(\widetilde{G}, \{h_{\widetilde{G}}\}) = (Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\}) \times (G, \{h_G\}). \tag{B.1.19}$$

Let $K_{\widetilde{G}}$ be decomposed via B.1.7 into

$$K_{\widetilde{G}} = K_{Z^{\mathbb{Q}}} \times K_G. \tag{B.1.20}$$

The natural projections in B.1.19 then induce morphisms of Shimura varieties

$$\mathrm{Sh}_{K_{\widetilde{G}}}(\widetilde{G}, \{h_{\widetilde{G}}\}) \to \mathrm{Sh}_{K_{Z^{\mathbb{Q}}}}(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})_E, \tag{B.1.21}$$

$$\mathrm{Sh}_{K_{\widetilde{G}}}(\widetilde{G}, \{h_{\widetilde{G}}\}) \to \mathrm{Sh}_{K_G}(G, \{h_G\})_E. \tag{B.1.22}$$

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Note that the Shimura variety $\mathrm{Sh}_{K_G}(G,\{h_G\})$, which originally appeared in [GGP12], is not of PEL type. However, it is of abelian type, and we have an integral model $\mathcal{S}_{K_G}(G,\{h_G\})$ defined over $\mathcal{O}_{F,(\nu)}$ by [Kis10] (resp. [KP18] depending on the level structure[4]).

B.1.23.
Let $\nu|p$ be a place of E. Let $\mathcal{M}_0$ be the moduli functor which associates to each locally Noetherian $\mathcal{O}_{E,\nu}$-scheme S the groupoid of tuples $\mathcal{M}_0(S) := (A_0, \lambda_0, \iota_0)$ where

(i) A_0 is an abelian scheme over S;
(ii) $\iota_0 : \mathcal{O}_F \to \mathrm{End}_S(A_0)$ is an $\mathcal{O}_F$-endomorphism structure on A_0 satisfying the Kottwitz condition of signature $((0,1)_{\varphi\in\Phi})$; that is,

$$\mathrm{Char}(\iota_0(a)|\mathrm{Lie}A_0) = \prod_{\varphi\in\Phi}(T - \overline{\varphi}(a)) \qquad \text{for all } a \in \mathcal{O}_F;$$

(iii) λ_0 is a principal polarization of A_0 such that the associated Rosati involution induces the nontrivial Galois automorphism of F/F_0 on $\mathcal{O}_F$ via ι_0.

Then $\mathcal{M}_0$ is representable by a Deligne-Mumford stack $\mathcal{M}_0$ finite étale over $\mathrm{Spec}\,\mathcal{O}_{E,\nu}$. Moreover, we assume K^p is small enough, so that $\mathcal{M}_0$ is nonempty. We shall assume throughout the rest of this appendix that $\mathcal{M}_0$ is nonempty.

Lemma B.1.24 [RSZ20, Lemma 3.4]. *The stack $\mathcal{M}_0$ admits the following decompositon into open and closed substacks*[5]*:*

$$\mathcal{M}_0 = \coprod_{\xi\in\mathcal{L}_\Phi/\sim} \mathcal{M}_0^\xi \tag{B.1.25}$$

such that the generic fiber of $\mathcal{M}_0^\xi$ is canonically isomorphic to $\mathrm{Sh}_{K_{Z^{\mathbb{Q}}}}(Z^{\mathbb{Q}},\{h_{Z^{\mathbb{Q}}}\})_E$.

B.1.26.
Let $F_{0,\nu}$ be the ν-adic completion of F_0, and we set $F_\nu := F \otimes_{F_0} F_{0,\nu}$. Suppose for now that the place ν_0 of F_0 is unramified over p, and that ν_0 either splits in F or is inert in F. Suppose, moreover, that the hermitian space W_{ν_0} is split. If there exists a prime ν of F_0 above p that is non-split in F, we assume additionally that $p \neq 2$. We choose a vertex lattice Λ_ν in the $F_\nu/F_{0,\nu}$-hermitian space W_ν. For now, we assume that Λ_{ν_0} is self-dual. We recall that an $\mathcal{O}_{F,\nu}$-lattice Λ in an $F_\nu/F_{0,\nu}$-hermitian space is called a *vertex lattice of type r* if $\Lambda \subset^r \Lambda^* \subset \pi_\nu^{-1}\Lambda$.[6] An $\mathcal{O}_{F,\nu}$-lattice Λ in an $F_\nu/F_{0,\nu}$-hermitian space is called a *vertex lattice* if it is a vertex lattice of type r for some r. Here, π_ν is a uniformizer in $F_\nu := F \otimes_F F_{0,\nu}$, where $F_{0,\nu}$ is the ν-adic completion of F_0 for a place ν of F_0. In particular, a self-dual lattice is simply a vertex lattice of type 0. Assume that $K_{G,\nu} = \mathrm{Stab}(\Lambda_\nu) \subset G(F_{0,\nu})$.

B.1.27.
Let $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ be the moduli functor which associates to each locally Noetherian $\mathcal{O}_{E,(\nu)}$-scheme S the groupoid of triples $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})(S) := (A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}^p)$, where

- $(A_0, \iota_0, \lambda_0) \in \mathcal{M}_0^\xi(S)$ as is defined in B.1.25;
- (A,ι) is an abelian scheme over S, equipped with an $\mathcal{O}_F \otimes \mathbb{Z}_{(p)}$-endomorphism structure ι satisfying

[4]Note that the construction of [KP18] assumes that $p > 2$, and that G splits over a tamely ramified extension of $\mathbb{Q}_p$, and that p does not divide the order $|\pi_1(G^{\mathrm{der}})|$ of the algebraic fundamental group of the derived group G^{der} over $\overline{\mathbb{Q}}_p$. We expect that the condition 'G splits over a tamely ramified extension of $\mathbb{Q}_p$' can certainly be relaxed using [KZ21].
[5]Here the index set $\mathcal{L}_\Phi/\sim$ need not be specified for our purposes, for more details see [RSZ20]
[6]Here, the notation $\Lambda \subset^r \Lambda^*$ means that Λ is an R-submodule of Λ^* of finite colength r

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the Kottwitz condition of signature $((1,n-1)_{\varphi_0},(0,n)_{\varphi\in\Phi\setminus\{\varphi_0\}})$; that is,

$$\mathrm{Char}(\iota(a)|\mathrm{Lie}A)=(T-\varphi_0(a))(T-\varphi_0(\overline{a}))^{n-1}\prod_{\varphi\in\Phi\setminus\{\varphi_0\}}(T-\varphi(\overline{a}))^n \quad \text{for all } a\in F;$$

- λ is a polarization of A such that the associated Rosati involution induces the nontrivial Galois automorphism of F/F_0 on $\mathcal{O}_F\otimes\mathbb{Z}_{(p)}$ via ι, and such that the following additional assumption in [RSZ20, (4.2)] is also satisfied: the action of $\mathcal{O}_{F_0}\otimes\mathbb{Z}_p\simeq\prod_{v|p}\mathcal{O}_{F_0,v}$ on the p-divisible group $A[p^\infty]$ induces a decomposition $A[p^\infty]=\prod_{v|p}A[v^\infty]$, where v ranges over the places of F_0 above p; the polarization λ then induces a polarization
$$\lambda_v: A[v^\infty]\to A^\vee[v^\infty]\simeq A[v^\vee]^\vee \tag{B.1.28}$$
for each v; we require $\ker\lambda_v$ to be contained in $A[\iota(\pi_v)]$ of rank $\#(\Lambda_v^*/\Lambda_v)$ for each place v of F_0 above p;
- $\overline{\eta}^p$ is a K_G^p-orbit of the $\mathbb{A}_{F,f}^p$-linear isometry
$$\mathrm{Hom}_F(\widehat{V}^p(A_0),\widehat{V}^p(A))\ \simeq\ -W\otimes_F\mathbb{A}_{F,f}^p, \tag{B.1.29}$$
where the hermitian form on the left-hand side is $(x,y)\mapsto\lambda_0^{-1}\circ y^\vee\circ\lambda\circ x$.
- For each $v\neq v_0$ over p, we impose the *sign condition* and *Eisenstein condition* at v [RSZ20, (4.4), (4.10)].

By [RSZ20, Theorem 4.1], the forgetful map $(A_0,\iota_0,\lambda_0,A,\iota,\lambda,\overline{\eta}^p)\mapsto(A_0,\iota_0,\lambda_0)$ is representable and induces a morphism of $\mathcal{O}_{E,(\nu)}$-schemes

$$\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G},\{h_{\widetilde{G}}\})\to\mathcal{M}_0^\xi:=\mathcal{M}_{K_{Z^\mathbb{Q}}}(Z^\mathbb{Q})\simeq\mathcal{S}_{K_{Z^\mathbb{Q}}}(Z^\mathbb{Q},\{h_{Z^\mathbb{Q}}\}). \tag{B.1.30}$$

On the level of generic fibres, (B.1.30) recovers the map (B.1.21).

B.2. Comparison of integral models

B.2.1.
Let $\mathcal{S}_{K_{\widetilde{G}}}(\widetilde{G},\{h_{\widetilde{G}}\})$ be the integral model defined over $\mathcal{O}_{E,(\nu)}$ for $\mathrm{Sh}_{K_{\widetilde{G}}}(\widetilde{G},\{h_{\widetilde{G}}\})$ as constructed in [KP18, §4.6]. Recall that the abelian type integral model $\mathcal{S}_{K_{\widetilde{G}}}(\widetilde{G},\{h_{\widetilde{G}}\})$ is built out of the integral model $\mathcal{S}(G^\mathbb{Q},\{h_{G^\mathbb{Q}}\})$ for a corresponding Hodge type Shimura variety associated to the abelian type $\mathrm{Sh}_{K_{\widetilde{G}}}(\widetilde{G},\{h_{\widetilde{G}}\})$, which, as in [KP18, 4.6.21] (see also [Kis10, 3.4.13] and [Del79] for more details), is $\mathrm{Sh}_{K_{G^\mathbb{Q}}}(G^\mathbb{Q},\{h_{G^\mathbb{Q}}\})$. Recall that $\mathcal{S}(G^\mathbb{Q},\{h_{G^\mathbb{Q}}\})$ is the $\mathcal{O}_{F,(\nu)}$-scheme constructed by taking the flat closure $\mathcal{S}^-(G^\mathbb{Q},\{h_{G^\mathbb{Q}}\})$ of the generic fibre $\mathrm{Sh}(G^\mathbb{Q},\{h_{G^\mathbb{Q}}\})$ inside some suitable Siegel integral model $\mathcal{S}_{K'}(\mathrm{GSp},S^\pm)_{\mathcal{O}_{F,(\nu)}}$ for some prime ν of F above a fixed prime p. For convenience of expositions, we shall fix a symplectic embedding $i:(G^\mathbb{Q},\{h_{G^\mathbb{Q}}\})\hookrightarrow(\mathrm{GSp}(V,\psi),S^\pm)$.[7] By [Xu21, Xu25],[8] this flat closure $\mathcal{S}^-(G^\mathbb{Q},\{h_{G^\mathbb{Q}}\})\simeq\mathcal{S}(G^\mathbb{Q},\{h_{G^\mathbb{Q}}\})$ is the desired integral model.[9] We shall use the model for $\mathrm{Sh}(G^\mathbb{Q},\{h_{G^\mathbb{Q}}\})$ as building blocks for integral models for $\mathrm{Sh}_{K_{\widetilde{G}}}(\widetilde{G},\{h_{\widetilde{G}}\})$ and $\mathrm{Sh}_{K_G}(G,\{h_G\})$.

[7]Note that the independence on symplectic embeddings of Hodge type integral models was first proven in [Kis10, Theorem 2.3.8]. For the parahoric integral models constructed in [KP18], it was later proven in [Pap21, Theorem 8.1.6] that the Kisin-Pappas models (constructed under the assumption that the group G in (G,X) splits over a tamely ramified extension of $\mathbb{Q}_p$) are independent of the choice of a symplectic embedding. In our case, we use the same symplectic space (V,ψ) to construct the right-hand side of Lemmas B.2.5 and B.2.12 as the one used on the left-hand side compatible with W from (B.1.3).
[8]which reference to use depends on the specific level structure at p
[9]and it is moreover normal when in the setting of [Xu21].

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Fix a connected component $\{h_{G^{\mathbb{Q}}}\}^+ \subset \{h_{G^{\mathbb{Q}}}\}$, and let $\mathrm{Sh}(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\})^+ \subset \mathrm{Sh}(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\})$ be the geometrically connected component which is the image of $\{h_{G^{\mathbb{Q}}}\}^+ \times 1$. Let $F^p \subset \overline{F}$ be the maximal extension of F that is unramified at primes dividing p. By [Del79, Theorem 2.6.3], the action of $\mathrm{Gal}(\overline{F}/F)$ on $\mathrm{Sh}_{K_{G^{\mathbb{Q}},p}}(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\})^+$ factors through $\mathrm{Gal}(\overline{F}^p/F)$. We abuse the notation and still denote $\mathrm{Sh}_{K_{G^{\mathbb{Q}},p}}(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\})^+$ as the F^p-scheme obtained via descent. Let $\mathscr{S}_{K_{G^{\mathbb{Q}},p}}(G^{\mathbb{Q}})^+$ be the closure of $\mathrm{Sh}_{K_{G^{\mathbb{Q}},p}}(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\})^+$ in $\mathscr{S}^-_{K_{G^{\mathbb{Q}},p}}(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\}) \otimes_{\mathcal{O}_{F,(v)}} \mathcal{O}_{F^p,(v)}$. Here, the notation $\mathcal{O}_{F^p,(v)}$ denotes the ring of integers of F^p localized at (p).[10]

Let $\mathscr{A}(\widetilde{G}_{\mathbb{Z}_{(p)}})$ (resp. $\mathscr{A}(G^{\mathbb{Q}}_{\mathbb{Z}_{(p)}})^\circ$) be the group defined in [KP18, 4.6.8] for $\widetilde{G}$ (resp. $G^{\mathbb{Q}}$), which was originally defined in [Del79]. We recall that

$$\mathscr{A}(\widetilde{G}_{\mathbb{Z}_{(p)}}) := \widetilde{G}(\mathbb{A}_f^p)/Z_{\widetilde{G}}(\mathbb{Z}_{(p)})^- *_{\widetilde{G}^\circ(\mathbb{Z}_{(p)})_+/Z_{\widetilde{G}}(\mathbb{Z}_{(p)})} (G^{\mathbb{Q}})^{\mathrm{ad}\circ}(\mathbb{Z}_{(p)})^+, \tag{B.2.2}$$

and $\mathscr{A}(G^{\mathbb{Q}}_{\mathbb{Z}_{(p)}})^\circ := (G^{\mathbb{Q}})^\circ(\mathbb{Z}_{(p)})_+^-/Z(\mathbb{Z}_{(p)})^- *_{(G^{\mathbb{Q}})^\circ(\mathbb{Z}_{(p)})_+/Z(\mathbb{Z}_{(p)})} (G^{\mathbb{Q}})^{\mathrm{ad}\circ}(\mathbb{Z}_{(p)})^+$, where $(G^{\mathbb{Q}})^\circ(\mathbb{Z}_{(p)})_+^-$ is the closure of $(G^{\mathbb{Q}})^\circ(\mathbb{Z}_{(p)})_+$ in $(G^{\mathbb{Q}})(\mathbb{A}_f^p)$. By [KP18, Lemma 4.6.10], we have an inclusion

$$\mathscr{A}(G^{\mathbb{Q}}_{\mathbb{Z}_{(p)}})^\circ \backslash \mathscr{A}(\widetilde{G}_{\mathbb{Z}_{(p)}}) \hookrightarrow \mathscr{A}(G^{\mathbb{Q}})^\circ \backslash \mathscr{A}(\widetilde{G})/K_{\widetilde{G},p}. \tag{B.2.3}$$

Here, $\mathscr{A}(\widetilde{G}) := \widetilde{G}(\mathbb{A}_f)/\widetilde{Z}(\mathbb{Q})^- *_{\widetilde{G}(\mathbb{Q})_+/\widetilde{Z}(\mathbb{Q})} \widetilde{G}^{\mathrm{ad}}(\mathbb{Q})^+$, where $\widetilde{Z}(\mathbb{Q})^-$ denotes the closure of $Z_{\widetilde{G}}(\mathbb{Q})$ in $\widetilde{G}(\mathbb{A}_f)$, and

$$\mathscr{A}(G^{\mathbb{Q}})^\circ := G^{\mathbb{Q}}(\mathbb{Q})_+^-/Z(\mathbb{Q})^- *_{G^{\mathbb{Q}}_+/Z(\mathbb{Q})} (G^{\mathbb{Q}})^{\mathrm{ad}}(\mathbb{Q})^+,$$

where $G^{\mathbb{Q}}(\mathbb{Q})_+^-$ denotes the closure of $G^{\mathbb{Q}}(\mathbb{Q})_+$ in $G^{\mathbb{Q}}(\mathbb{A}_f)$. Let $\widetilde{J} \subset \widetilde{G}(\mathbb{Q}_p)$ denote a set which maps bijectively to a set of coset representatives for the image of $\mathscr{A}(\widetilde{G}_{\mathbb{Z}_{(p)}})$ in $\mathscr{A}(G^{\mathbb{Q}})^\circ \backslash \mathscr{A}(\widetilde{G})/K_{\widetilde{G},p}$ under (B.2.3). Recall from [KP18, 4.6.15], we have

$$\mathscr{S}_{K_{\widetilde{G},p}}(\widetilde{G}, \{h_{\widetilde{G}}\}) = \left[[\mathscr{A}(\widetilde{G}_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{G^{\mathbb{Q}},p}}(G^{\mathbb{Q}})^+]/\mathscr{A}(G^{\mathbb{Q}}_{\mathbb{Z}_{(p)}})^\circ\right]^{|\widetilde{J}|}. \tag{B.2.4}$$

Note that by analogous arguments as *loc.cit.*, the right-hand side of (B.2.4) has a natural structure of a $\mathcal{O}_{E,(v)} := \mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_{F,(v)}$-scheme with $\widetilde{G}(\mathbb{A}_f^p)$-action and is a model for $\mathrm{Sh}_{K_{\widetilde{G},p}}(\widetilde{G}, \{h_{\widetilde{G}}\})$. Moreover, for sufficiently small $K^p_{\widetilde{G}}$, the quotient $\mathscr{S}_{K_{\widetilde{G},p}}(\widetilde{G}, \{h_{\widetilde{G}}\})/K^p_{\widetilde{G}} := \mathscr{S}_{K_{\widetilde{G}}}(\widetilde{G}, \{h_{\widetilde{G}}\})$ is a finite type $\mathcal{O}_{E,(v)}$-scheme extending $\mathrm{Sh}_{K_{\widetilde{G}}}(\widetilde{G}, \{h_{\widetilde{G}}\})$.

Lemma B.2.5. *$\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G}, \{h_{\widetilde{G}}\}) \simeq \mathscr{S}_{K_{\widetilde{G}}}(\widetilde{G}, \{h_{\widetilde{G}}\})$ as* $\mathrm{Spec}\, \mathcal{O}_{E,(v)}$*-schemes.*

Proof. The moduli description for $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G}, \{h_{\widetilde{G}}\})$ in B.1.26 induces a natural map

$$\mathcal{M}_{K_{\widetilde{G},p}}(\widetilde{G}, \{h_{\widetilde{G}}\}) \to \mathscr{S}_{K_{G^{\mathbb{Q}},p}}(G^{\mathbb{Q}}) \tag{B.2.6}$$

$$(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}^p) \mapsto (A, \iota, \lambda) \tag{B.2.7}$$

by simply forgetting the component $(A_0, \iota_0, \lambda_0, \overline{\eta}^p)$ in the tuple. Note that the map (B.2.6) is proper, and in particular closed. We fix an arbitrary $(A^\star, \iota^\star, \lambda^\star) \in \mathscr{S}_{K_{G^{\mathbb{Q}},p}}(G^{\mathbb{Q}})^+$, and suppose

$$(A_0^\star, \iota_0^\star, \lambda_0^\star, A^\star, \iota^\star, \lambda^\star, \eta^\star) \mapsto (A^\star, \iota^\star, \lambda^\star)$$

[10]Here, we are abusing the notation v to always denote the prime above p in the relevant fields.

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under the map (B.2.6). Take any $(h, \gamma^{-1}) \in \mathscr{A}(\widetilde{G}_{\mathbb{Z}_{(p)}})$. As in [KP18, 4.5.3], let $\widetilde{\mathcal{P}}_\gamma \subset \widetilde{G}$ be the torsor given by the fibre over $\gamma \in \widetilde{G}^{\mathrm{ad}}(\mathbb{Z}_{(p)})$. First we check that

$$(h, \gamma^{-1}) \cdot (A_0^\star, \lambda_0^\star, \iota_0^\star, \eta^\star) = ((A_0^\star)^{\widetilde{\mathcal{P}}_\gamma}, (\lambda_0^\star)^{\widetilde{\mathcal{P}}_\gamma}, (\iota_0^\star)^{\widetilde{\mathcal{P}}_\gamma}, (\eta^\star)^{\widetilde{\mathcal{P}}_\gamma})$$

gives another point in the fibre over $(A^\star, \iota^\star, \lambda^\star)$ under the map (B.2.6). This is clear as we only need to check that $(\eta^\star)^{\widetilde{\mathcal{P}}_\gamma}$ are $\mathbb{A}_{F,f}$-linear isometries

$$(\eta^\star)^{\widetilde{\mathcal{P}}_\gamma} : \widehat{V}\Big((A_0^\star)^{\widetilde{\mathcal{P}}_\gamma}, A^\star\Big) \simeq -W \otimes_F \mathbb{A}_{F,f}, \tag{B.2.8}$$

but this is simply given by the composite $\widetilde{\gamma}^{-1} \circ \eta^\star \circ \iota_{\widetilde{\gamma}}^{-1}$, where $\iota_{\widetilde{\gamma}}$ is as defined in [KP18, 4.5.3].

It then remains to check that $\ker(\mathscr{A}(G^{\mathbb{Q}}_{\mathbb{Z}_{(p)}})^\circ \to \mathscr{A}(\widetilde{G}_{\mathbb{Z}_{(p)}}))$ acts freely on $\mathscr{S}_{K_{G^{\mathbb{Q}},p}}(G^{\mathbb{Q}})^+$, and this follows from [KP18, 4.6.17] and the fact that $\ker(\mathscr{A}(G^{\mathbb{Q}}_{\mathbb{Z}_{(p)}})^\circ \to \mathscr{A}(\widetilde{G}_{\mathbb{Z}_{(p)}}))$ is a subgroup of $\Delta(G^{\mathbb{Q}}, (G^{\mathbb{Q}})^{\mathrm{ad}}) := \ker(\mathscr{A}(G^{\mathbb{Q}}_{\mathbb{Z}_{(p)}}) \to \mathscr{A}(G^{\mathbb{Q}\mathrm{ad}}_{\mathbb{Z}_{(p)}}))$. Thus,

$$\mathcal{M}_{K_{\widetilde{G},p}}(\widetilde{G}, \{h_{\widetilde{G}}\}) \simeq \Big[[\mathscr{A}(\widetilde{G}_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{G^{\mathbb{Q}},p}}(G^{\mathbb{Q}})^+]/\mathscr{A}(G^{\mathbb{Q}}_{\mathbb{Z}_{(p)}})^\circ\Big]^{|\widetilde{J}|} \simeq \mathscr{S}_{K_{\widetilde{G},p}}(\widetilde{G}, \{h_{\widetilde{G}}\}). \tag{B.2.9}$$

In particular, $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G}, \{h_{\widetilde{G}}\}) \simeq \mathscr{S}_{K_{\widetilde{G}}}(\widetilde{G}, \{h_{\widetilde{G}}\})$. □

B.2.10.
For an arbitrary extension L/E, taking the fibre in (B.1.30) over a fixed $\mathcal{O}_{L,(v)}$-point $(A_0^\star, \iota_0^\star, \lambda_0^\star)$ of $\mathcal{M}_0^{\mathcal{O}_F,\xi}$ gives a flat integral model $\mathcal{M}^\star_{K_G}(G, \{h_G\})$ over $\mathcal{O}_L$. Here, we use the upper script $\star$ to emphasize that the model $\mathcal{M}^\star_{K_G}(G, \{h_G\})$ thus obtained a priori depends on the choice of a base point $(A_0^\star, \iota_0^\star, \lambda_0^\star)$. However, recall from B.1.17 that the reflex field for $(G, \{h_G\})$ is F, by [KP18] we also have a normal integral model $\mathscr{S}_{K_G}(G, \{h_G\})$ over $\mathrm{Spec}\,\mathcal{O}_{F,(v)}$, which is given by

$$\mathscr{S}_{K_{G,p}}(G, \{h_G\}) := \Big[[\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{G^{\mathbb{Q}},p}}(G^{\mathbb{Q}})^+_{\mathcal{O}_{L,(v)}}]/\mathscr{A}(G^{\mathbb{Q}}_{\mathbb{Z}_{(p)}})^\circ\Big]^{|J|}. \tag{B.2.11}$$

Here, $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$ is the analogous group for G as defined in (B.2.2), and $J \subset G(\mathbb{Q}_p)$ denotes the set analogous to $\widetilde{J}$ defined above (B.2.4), using the analogous map for G as in (B.2.3).

Lemma B.2.12. *$\mathcal{M}_{K_G}(G, \{h_G\}) \simeq \mathscr{S}_{K_G}(G, \{h_G\})_{\mathcal{O}_{L,(v)}}$ as* Spec $\mathcal{O}_{L,(v)}$*-schemes.*

Proof. Consider the map

$$\mathcal{M}^\star_{K_{G,p}}(G, \{h_G\}) \to \mathscr{S}_{K_{G^{\mathbb{Q}},p}}(G^{\mathbb{Q}})_{\mathcal{O}_{L,(v)}} \tag{B.2.13}$$

$$(A_0^\star, \iota_0^\star, \lambda_0^\star, A, \iota, \lambda, \overline{\eta}^p) \mapsto (A, \iota, \lambda) \tag{B.2.14}$$

given by forgetting the component $(A_0^\star, \iota_0^\star, \lambda_0^\star)$ in the tuple. Let $\mathscr{S}^\star_{K_{G^{\mathbb{Q}},p}}(G^{\mathbb{Q}})_{\mathcal{O}_{L,(v)}}$ denote the image of the map (B.2.13). We take an arbitrary $(A, \iota, \lambda) \in \mathscr{S}^\star_{K_{G^{\mathbb{Q}},p}}(G^{\mathbb{Q}})^+_{\mathcal{O}_{L,(v)}}$, and thus by construction of $\mathcal{M}^\star_{K_G}(G, \{h_G\})$, we clearly have

$$(A_0^\star, \iota_0^\star, \lambda_0^\star, A, \iota, \lambda, \overline{\eta}^p) \in \mathcal{M}_{K_{\widetilde{G},p}}(\widetilde{G}). \tag{B.2.15}$$

Take any $(h, \gamma^{-1}) \in \mathscr{A}(G_{\mathbb{Z}_{(p)}})$. In particular, $\gamma \in G^{\mathrm{ad}} \simeq (G^{\mathbb{Q}})^{\mathrm{ad}}$. Again as in [KP18, 4.5.3], let $\mathcal{P}_\gamma \subset G^{\mathbb{Q}}$ be the torsor given by the fibre over $\gamma \in (G^{\mathbb{Q}})^{\mathrm{ad}}(\mathbb{Z}_{(p)})$. By the same reasoning as in the

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proof of Lemma B.2.5, we also have

$$(h,\gamma^{-1})\cdot(A_0^\star,\iota_0^\star,\lambda_0^\star,A,\iota,\lambda,\overline{\eta}^p)=(A_0^\star,\iota_0^\star,\lambda_0^\star,A^{\mathcal{P}_\gamma},\lambda^{\mathcal{P}_\gamma},\iota^{\mathcal{P}_\gamma},\eta^{\mathcal{P}_\gamma})\in\mathcal{M}_{K_{\widetilde{G},p}}(\widetilde{G})$$

gives another point in the fibre over $(A_0^\star,\iota_0^\star,\lambda_0^\star)$ under the map (B.1.30). The rest of the argument proceeds similarly as in the proof of Lemma B.2.5 – that is, the kernel $\ker(\mathscr{A}(G^{\mathbb{Q}}_{\mathbb{Z}_{(p)}})^\circ\to\mathscr{A}(G_{\mathbb{Z}_{(p)}}))$ acts freely on $\mathcal{S}_{K_{G^{\mathbb{Q}},p}}(G^{\mathbb{Q}})^+$. In particular, we have

$$\mathcal{M}^\star_{K_{G,p}}(G)\simeq\left[[\mathscr{A}(G_{\mathbb{Z}_{(p)}})\times\mathcal{S}^\star_{K_{G^{\mathbb{Q}},p}}(G^{\mathbb{Q}})^+_{\mathcal{O}_{L,(\nu)}}]/\mathscr{A}(G^{\mathbb{Q}}_{\mathbb{Z}_{(p)}})^\circ\right]^{|J|},\tag{B.2.16}$$

and thus, $\mathcal{M}^\star_{K_G}(G)\simeq\mathcal{S}_{K_G}(G)_{\mathcal{O}_{L,(\nu)}}$. (Since the choice of base point $\star$ does not affect the proof, we may drop the upper script $\star$ from our notations.) □

B.2.17.
We consider the Drinfeld level structure integral models analogous to those in [RSZ20, § 4.3]. Consider the embedding $\widetilde{\nu}:\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}}_p$, which identifies

$$\mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}})\ \simeq\ \mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}}_p).\tag{B.2.18}$$

The above identification (B.2.18) then gives an identification

$$\{\varphi\in\mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}})|w_\varphi=w\}\ \simeq\ \mathrm{Hom}_{\mathbb{Q}_p}(F_w,\overline{\mathbb{Q}}_p),\tag{B.2.19}$$

where w_φ denotes the p-adic place in F induced by $\widetilde{\nu}\circ\varphi$.

We fix a place v_0 of F over p that is split in F (and possibly ramified over p) into w_0 and another place $\overline{w}_0$ in F. We require, moreover, that the CM type Φ considered in (B.1.2) and the chosen place ν of E above p satisfy the following *matching condition*:

$$\{\varphi\in\mathrm{Hom}(F,\overline{\mathbb{Q}})|w_\varphi=w_0\}\subset\Phi.\tag{B.2.20}$$

This condition (B.2.20) only depends on the place ν of E induced by $\widetilde{\nu}$.

Now we introduce a *Drinfeld level structure* at v_0. Recall the level structure subgroup K_G from (B.1.20). We define a variant compact open subgroup $K_G^m\subset G(\mathbb{A}_{F_0,f})$ in exactly the same way as K_G, except that, in the v_0-factor, we require $K^m_{G,v_0}\subset G(F_{0,v_0})$ to be the principal congruence subgroup modulo $\mathfrak{p}^m_{v_0}$ inside K_{G,v_0}. Clearly, $K_G=K_G^{m=0}$. As in (B.1.20), we define $K^m_{\widetilde{G}}=K_{Z^{\mathbb{Q}}}\times K^m_G$.

Let $\Lambda_{v_0}=\Lambda_{w_0}\oplus\Lambda_{\overline{w}_0}$ denote the natural decomposition of the lattice Λ_{v_0} attached to the split place v_0. For a point $(A_0,\iota_0,\lambda_0,A,\iota,\lambda,\overline{\eta}^p)\in\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})(S)$, we have a decomposition of p-divisible groups

$$A[p^\infty]=\prod_{w|p}A[w^\infty],\tag{B.2.21}$$

where w ranges over the places of F lying over p. Moreover, we further decompose the v_0-term in (B.2.21) and consider

$$A[v_0^\infty] = A[w_0^\infty] \times A[\overline{w}_0^\infty], \tag{B.2.22}$$

where, when p is locally nilpotent on S, the p-divisible group $A[w_0^\infty]$ satisfies the Kottwitz condition of type $r|_{w_0}$ for the action of $\mathcal{O}_{F,w_0}$ on its Lie algebra, in the sense of [RZ17, §8]. Here, $r|_{w_0}$ denotes the restriction of the function r on $\mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}})$ to $\mathrm{Hom}_{\mathbb{Q}_p}(F_{w_0}, \overline{\mathbb{Q}}_p)$ under (B.2.19).

Likewise, we have the same decomposition as (B.2.22) for A_0; that is, we have

$$A_0[v_0^\infty] = A_0[w_0^\infty] \times A_0[\overline{w}_0^\infty]. \tag{B.2.23}$$

Let π_{w_0} be a uniformizer of F_{0,w_0}. In addition to the moduli functor $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ which classifies tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}^p)$, we impose the following additional *Drinfeld level structure* as in [HT01, § II.2]; that is,

- an $\mathcal{O}_{F,w_0}$-linear homomorphism of finite flat group schemes

$$\eta : \pi_{w_0}^{-m}\Lambda_{w_0}/\Lambda_{w_0} \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{O}_{F,w_0}}(A_0[w_0^m], A[w_0^m]). \tag{B.2.24}$$

We denote the resulting moduli problem by $\mathcal{M}_{K_{\widetilde{G}}^m}(\widetilde{G})$, which is relatively representable by a finite flat morphism to $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$. In fact, $\mathcal{M}_{K_{\widetilde{G}}^m}(\widetilde{G})$ is regular and flat over $\mathrm{Spec}\,\mathcal{O}_{E,(\nu)}$ by [HT01, Lemma III.4.1].

B.2.25.
Recall the integral model $\mathcal{S}_{K_{\widetilde{G}}}(\widetilde{G}, \{h_{\widetilde{G}}\})$ (resp. $\mathcal{S}_{K_G}(G, \{h_G\})$) defined in (B.2.4) (resp. (B.2.11)). We define $\mathcal{S}_{K_{\widetilde{G}}^m}(\widetilde{G}, \{h_{\widetilde{G}}\})$ (resp. $\mathcal{S}_{K_G^m}(G, \{h_G\})_{\mathcal{O}_{L,(\nu)}}$) as the normalization of $\mathcal{S}_{K_{\widetilde{G}}}(\widetilde{G}, \{h_{\widetilde{G}}\})$ (resp. $\mathcal{S}_{K_G}(G, \{h_G\})_{\mathcal{O}_{L,(\nu)}}$) inside $\mathrm{Sh}_{K_{\widetilde{G}}^m}(\widetilde{G}, \{h_{\widetilde{G}}\}) \simeq M_{K_{\widetilde{G}}^m}(\widetilde{G})$ (resp. $\mathrm{Sh}_{K_G^m}(G, \{h_G\})_L \simeq M_{K_G^m}(G)_L$).

Corollary B.2.26. $\mathcal{S}_{K_{\widetilde{G}}^m}(\widetilde{G}, \{h_{\widetilde{G}}\}) \simeq \mathcal{M}_{K_{\widetilde{G}}^m}(\widetilde{G})$ *as* $\mathrm{Spec}\,\mathcal{O}_{E,(\nu)}$*-schemes, and*

$$\mathcal{M}_{K_G^m}(G, \{h_G\}) \simeq \mathcal{S}_{K_G^m}(G, \{h_G\})_{\mathcal{O}_{L,(\nu)}} \tag{B.2.27}$$

as $\mathrm{Spec}\,\mathcal{O}_{L,(\nu)}$*-schemes.*

Proof. By Lemma B.2.5 (resp. B.2.12), $\mathcal{S}_{K_{\widetilde{G}}^m}(\widetilde{G}, \{h_{\widetilde{G}}\})$ (resp. $\mathcal{S}_{K_G^m}(G, \{h_G\})_{\mathcal{O}_L}$) is the normalization of $\mathcal{S}_{K_{\widetilde{G}}}(\widetilde{G}, \{h_{\widetilde{G}}\}) \simeq \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G}, \{h_{\widetilde{G}}\})$ (resp. $\mathcal{S}_{K_G}(G, \{h_G\})_{\mathcal{O}_L} \simeq \mathcal{M}_{K_G}(G, \{h_G\})$) inside $\mathrm{Sh}_{K_{\widetilde{G}}^m}(\widetilde{G}, \{h_{\widetilde{G}}\}) \simeq M_{K_{\widetilde{G}}^m}(\widetilde{G})$ (resp. $\mathrm{Sh}_{K_G^m}(G)_L \simeq M_{K_G^m}(G)_L$). Since $\mathcal{M}_{K_{\widetilde{G}}^m}(\widetilde{G})$ is regular and flat, in particular it is normal. Thus, by [Gro67, IV-2, 6.14.1], $\mathcal{M}_{K_G^m}(G)$ is normal (even though it may not necessarily be regular). By [Sta18, 035I] applied to the scheme $\mathcal{M}_{K_{\widetilde{G}}^m}(\widetilde{G})$ (resp. $\mathcal{M}_{K_G^m}(G)$), there exists a unique morphism $\mathcal{S}_{K_{\widetilde{G}}^m}(\widetilde{G}, \{h_{\widetilde{G}}\}) \to \mathcal{M}_{K_{\widetilde{G}}^m}(\widetilde{G})$ (resp. $\mathcal{S}_{K_G^m}(G, \{h_G\}) \to \mathcal{M}_{K_G^m}(G)$), which is the normalization of $\mathcal{M}_{K_{\widetilde{G}}^m}(\widetilde{G})$ (resp. $\mathcal{M}_{K_G^m}(G)$) in $\mathrm{Sh}_{K_{\widetilde{G}}^m}(\widetilde{G}, \{h_{\widetilde{G}}\})$ (resp. $\mathrm{Sh}_{K_G^m}(G, \{h_G\})$). Since $\mathcal{M}_{K_{\widetilde{G}}^m}(\widetilde{G})$ (resp. $\mathcal{M}_{K_G^m}(G)$) is already normal, we have an isomorphism $\mathcal{S}_{K_{\widetilde{G}}^m}(\widetilde{G}, \{h_{\widetilde{G}}\}) \simeq \mathcal{M}_{K_{\widetilde{G}}^m}(\widetilde{G})$ (resp. $\mathcal{M}_{K_G^m}(G, \{h_G\}) \simeq \mathcal{S}_{K_G^m}(G, \{h_G\})_{\mathcal{O}_{L,(\nu)}}$). □

B.2.28.
In this last section, we recall the construction of semi-global integral models with *AT parahoric level* as in [RSZ20, § 4.4]. Recall the notion of *vertex lattice* from §B.1.26. We say that a vertex lattice Λ is *almost self-dual* if it is a vertex lattice of type 1. We say that a vertex lattice Λ is π_ν*-modular* (resp. *almost* π_ν*-modular*) if $\Lambda^* = \pi_\nu^{-1}\Lambda$ (resp. $\Lambda \subset \Lambda^* \subset^1 \pi_\nu^{-1}\Lambda$).

Suppose $p \neq 2$ and v_0 is unramified over p. As in §B.1.26, we take a vertex lattice $\Lambda_v \subset W_v$ for each prime v of F_0 above p. Unlike in §B.1.26, let (v_0, Λ_{v_0}) be of one of the following types:

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1. v_0 is inert in F and Λ_{v_0} is almost self-dual as an $\mathcal{O}_{F,v_0}$-lattice;
2. n is even, v_0 ramifies in F and Λ_{v_0} is π_{v_0}-modular;
3. n is odd, v_0 ramifies in F and Λ_{v_0} is almost π_{v_0}-modular;
4. $n = 2$, v_0 ramifies in F and Λ_{v_0} is self-dual.

To the moduli functor $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ which classifies tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}^p)$ as in §B.1.26 (except that the condition on (v_0, Λ_{v_0}) is different), we impose the following additional condition:

- When the pair (v_0, Λ_{v_0}) is of AT type (2), (3) or (4), we impose the *Eisenstein condition* on the summand $\mathrm{Lie}_{\psi} A[v_0^\infty]$ [RSZ20, 4.10];
- When the pair (v_0, Λ_{v_0}) is of AT type (2), we impose additionally the *wedge condition* [RSZ20, 4.27] and the *spin condition* [RSZ20, 4.28]
- When the pair (v_0, Λ_{v_0}) is of AT type (3), we impose additionally the *refined spin condition* [RSZ18, (7.9)] on $\mathrm{Lie}_{\psi_0} A[v_0^\infty]$.

By [RSZ20, Theorem 4.7], the moduli functor above is representable by a Deligne-Mumford stack flat over Spec $\mathcal{O}_{E,(\nu)}$ and relatively representable over $\mathcal{M}_0^{\mathcal{O}_F,\xi}$, i.e. (B.1.30) still holds in this case. To see that $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G}, \{h_{\widetilde{G}}\}) \simeq \mathcal{S}_{K_{\widetilde{G}}}(\widetilde{G}, \{h_{\widetilde{G}}\})$ as Spec $\mathcal{O}_{E,(\nu)}$-schemes, one simply proceeds as in Lemma B.2.5.

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