Can. J. Math., Vol. XXXV, No. 4, 1983, pp. 577-588

ERGODIC PROPERTIES OF LAMPERTI OPERATORS, II

CHARN-HUEN KAN

1. Introduction. For T in our main Theorem 5, T^* is called *Lamperti* in [**11**], whose terminology and notation we shall follow in the sequel. To avoid longish expressions, we shall also say that T^* here is *disjunctive* and, dually, $T = (T^*)^*$ is *codisjunctive*. The present work grows out of an attempt to establish a DEE for the general power bounded positive operator on L_p, in view of the success in the contraction case [**1**, **11**], and forms a continuation of [**11**]. (In passing, we note that Calderon's technique [**2**] mentioned in [**11**] was anticipated in 1938 by M. Fukamiya [7], though in a variant form and for a more classical case, namely that of a positive L_p isometry induced by an invertible, measure preserving transformation on a totally finite measure space. Calderon's case does not assume invertibility nor total finiteness.) In the course of proving our main result, we establish (in Theorem 2) a DEE for positive L_1 contractions which are simultaneously L_∞ power bounded. A vector-valued version of this will appear in a separate paper. We also note that if T is disjunctive, codisjunctive and L_p power bounded by K, then it has a DEE with constant $Kp/(p - 1)$, [**11**, Theorem 5.2]. Can this sharper constant be retained without disjunctiveness?

Lastly, a point on technicality. The measure space, if not σ-finite, can be replaced by a direct sum of σ-finite ones without altering L_p, $1 \leqq p < \infty$. This direct sum is not hard to readjust so that the induced l_p-direct sum decomposition of L_p reduces the operator considered in each case hereinafter. This done, the extension of S in Section 2 to act on L_∞, or of $\tilde{T}$ in Section 3 to act on nonnegative measurable functions, can be achieved as in the σ-finite case.

2. L_∞ power bounded L_1 contractions. Let S be a positive L_1 contraction which has bounded L_∞ operator norm also (when restricted to $L_1 \cap L_\infty$). Then it can be extended to a bounded operator on L_p, for any $1 \leqq p \leqq \infty$. Let

Received June 30, 1980 and in revised form March 4, 1983. This research was partially supported by NSERC grant no. A3974.

https://doi.org/10.4153/CJM-1983-033-1 Published online by Cambridge University Press

$$S_m = m^{-1} \sum_{i=0}^{m-1} S^i, \quad (m \geqq 1).$$

We define the *truncated weighted maximal operators* $\widetilde{M}_n \equiv \widetilde{M}_n(S)$ as follows: for $f \in L_p$, $1 \leqq p < \infty$,

$$\widetilde{M}_n f(x) = \sup_{1 \leqq m \leqq n} |S_m f|(x)/S_m 1(x), \quad (1 \leqq n < \infty).$$

The usual *truncated maximal operators* $M_n \equiv M_n(S)$ are defined as in [**11**], by replacing $S_m 1$ in the above expression by 1. The *weighted maximal operator* $\widetilde{M} \equiv \widetilde{M}(S)$ and the *maximal operator* $M \equiv M(S)$ are the monotone limits of $\widetilde{M}_n$ and M_n respectively i.e., $\widetilde{M}f(x) = \lim \widetilde{M}_n f(x)$, etc.

We have the following inequality which is reminiscent of the Hopf Maximal Ergodic Inequality (MEI).

THEOREM 1. *Let S be a positive L_1 contraction which is also L_∞ bounded. Then for any real $f \in L_p$ $(1 \leqq p < \infty)$, and for any $\lambda > 0$ and $1 \leqq n < \infty$, we have, writing E^n for*

$$E^n_\lambda(f) \equiv \bigcup_{m=1}^{n} \{S_m f > \lambda S_m 1\},$$

(a) $\mu E^n < \infty$,

(b) $\displaystyle\int_{E^n} (f - \lambda) d\mu \geqq 0.$

Proof. By the Riesz convexity theorem, each S_m is a bounded operator on L_p. Moreover, $S_m 1 \geqq 1/m$ a.e. Hence,

$$\mu E^n \leqq \sum_{m=1}^{n} \mu\{\,|S_m f| > \lambda/m\} \leqq \lambda^{-p} \sum_{m=1}^{n} m^p \|S_m f\|_p^{\ p} < \infty,$$

by Chebyshev's inequality. This proves (a).

Let $R_0 = 0$, $R_m = I + \ldots + S^{m-1} = mS_m$, $(m \geqq 1)$, and

$$\bar{R}_n(f - \lambda) = \max_{0 \leqq m \leqq n} R_m(f - \lambda) = \max_{0 \leqq m \leqq n} (R_m f - \lambda R_m 1), \quad (n \geqq 1).$$

Then

$$E^n = \{\bar{R}_n(f - \lambda) > 0\}.$$

Since $I = R_m - SR_{m-1}$, $(m \geqq 1)$, we have

$$(f - \lambda) = R_m(f - \lambda) - SR_{m-1}(f - \lambda)$$
$$\geqq R_m(f - \lambda) - S\bar{R}_n(f - \lambda), \ (1 \leqq m \leqq n).$$

Hence, noting that

$$\max \{R_n(f - \lambda):1 \leqq m \leqq n\} = \bar{R}_n(f - \lambda) \quad \text{in } E^n,$$

we get, upon multiplying by the indicator function of E^n and taking this maximum,

$$(1) \quad 1_{E^n} \cdot (f - \lambda) \geqq \bar{R}_n(f - \lambda) - S\bar{R}_n(f - \lambda).$$

Now $\bar{R}_n(f - \lambda) \in L_1^+$ since, E^n being the support of $\bar{R}_n(f - \lambda)$, we have

$$\bar{R}_n(f - \lambda) \leqq \sum_{m=1}^{n} (1_{E^n} \cdot |R_m f| + \lambda 1_{E^n} \cdot R_m 1)$$

and each term on the right-hand side is integrable, by (a). Similarly

$$1_{E^n} \cdot (f - \lambda) \in L_1.$$

Integrating (1), we immediately get (b).

COROLLARY. *Let S be a positive L_1 contraction which is also L_∞ bounded. Then for any (real or complex) $f \in L_p$, $(1 \leqq p < \infty)$, and for any $\lambda > 0$ and $1 \leqq n < \infty$, we have, writing F^n for $F_\lambda^n(f) \equiv \{\widetilde{M}_n f > \lambda\}$,*

(a′) $\mu F^n < \infty$,

(b′) $\displaystyle\int_{F^n} (|f| - \lambda)d\mu \geqq 0.$

Proof. Since $|S_m f| \leqq S_m|f|$, $(1 \leqq m \leqq n)$, we have, with notation of Theorem 1,

$$E_\lambda^n(|f|) \supset F_\lambda^n(f).$$

(a′) follows. Further, $|f| - \lambda \leqq 0$ outside of $F_\lambda^n(f) \supset F_\lambda^1(f) = \{|f| > \lambda\}$. Hence

$$\int_{F_\lambda^n(f)} (|f| - \lambda)d\mu \geqq \int_{E_\lambda^n(|f|)} (|f| - \lambda)d\mu.$$

Application of Theorem 1(b) to $|f|$ then gives (b′).

https://doi.org/10.4153/CJM-1983-033-1 Published online by Cambridge University Press

Remarks. (1). Theorem 1(b) will become the Hopf MEI, which is for S a positive L_1 contraction and f a real-valued L_1 function, if we change λ to 0. The former can be construed as a particular case of the latter in the following way. Fix $n \geqq 1$ and $\lambda > 0$, and let $h = f - \lambda$ on $E_\lambda^n(f)$, 0 elsewhere. So $h \in L_1$ by Theorem 1(a). $f - \lambda \leqq h$. Hence

$$S_m(f - \lambda) \leqq S_m h, \quad (1 \leqq m \leqq n),$$

and so $E_\lambda^n(f) \subset E_0^n(h)$. It follows that

$$\int_{E_\lambda^n(f)} (f - \lambda) = \int_{E_\lambda^n(f)} h = \int_{E_0^n(h)} h \geqq 0$$

by the Hopf MEI. In fact, Garsia's proof of the Hopf MEI (see [**8**] or [**9**, Theorem 2.2.1]) is similar to that of Theorem 1(b). (2). If S is an L_∞ contraction, the argument in the proof of the Corollary shows that $S_m 1$ in Theorem 1 can be replaced by 1. This becomes Theorem 2.2.2 in [**9**].(3). Analogous to inequality (1), we can show that under conditions in the corollary,

$$1_{F^n} \cdot (|f| - \lambda) \geqq \bar{R}_n(f;\lambda) - S\bar{R}_n(f;\lambda),$$

where

$$\bar{R}_n(f;\lambda) = \max\{|R_m f| - \lambda R_m 1 : 0 \leqq m \leqq n\}.$$

Integration of this gives a direct proof of (b′). (a′), like (a), follows from Chebyshev's inequality.

THEOREM 2. *Let S be a positive L_1 contraction which is simultaneously L_∞ bounded. Then for any $f \in L_p$, $(1 < p < \infty)$,*

$$\text{(a)} \quad \|\tilde{M}(S)f\|_p \leqq \frac{p}{p-1} \|f\|_p.$$

If, further, $\sup \|S_m\|_\infty = H$, *or more strongly if* $\sup \|S^m\|_\infty = H$, *then*

$$\text{(b)} \quad \|M(S)f\|_p \leqq H \frac{p}{p-1} \|f\|_p.$$

Proof. (b) follows from (a) by the fact that

$$Mf \leqq (\sup \|S_m 1\|_\infty)\tilde{M}f = (\sup \|S_m\|_\infty)\tilde{M}f \leqq H\tilde{M}f.$$

To prove (a), we need only prove it first for $\tilde{M}_n f$ instead of $\tilde{M}f$, for all $n \geqq 1$. This can be achieved by invoking the corollary to Theorem 1 and applying the Strong Estimate Theorem 2.2.3 in [**9**] (or Theorem 3.4′, Chapter VII, in [**5**]), which says that (a′) and (b′) imply (a) for $\tilde{M}_n f$.

https://doi.org/10.4153/CJM-1983-033-1 Published online by Cambridge University Press

3. Operator-modulated backward shifts. Suppose that T is a bounded operator on $L_p \equiv L_p(X, \mathcal{F}, \mu)$, $1 \leqq p < \infty$. Let $\tilde{L}_p$ be the l_p-direct sum of countably many copies of L_p. That is, $f \in \tilde{L}_p$ if and only if $f = (f_0, f_1, f_2, \dots)$ such that each $f_n \in L_p$ and

$$\|f\|^p \equiv \|f_0\|_p^p + \|f_1\|_p^p + \dots$$

is finite. It is well known and easy to see that $(\tilde{L}_p, \| \ \|)$ is a Banach space isometrically isomorphic to $L_p(\tilde{X}, \tilde{\mathcal{F}}, \tilde{\mu})$ where $\tilde{X}$ is the union of disjoint copies X_n, $(n = 0, 1, \dots)$, of X, each $\tilde{\mathcal{F}} \cap X_n$ is a copy $\mathcal{F}_n$ of $\mathcal{F}$, and $\tilde{\mu}|\mathcal{F}_n \equiv \mu_n$ is a copy of μ, such that f_n has support in $(X_n, \mathcal{F}_n, \mu_n)$.

Definition. The *unilateral backward T-shift* is the operator $\tilde{T}$ on $\tilde{L}_p$ such that $\tilde{T}(f_0, f_1, \dots) = (Tf_1, Tf_2, \dots)$.

There being no risk of confusion, we shall write $\| \ \|$ for the norm in either L_p or $\tilde{L}_p$. It easy to see that $\|T^n\| = \|\tilde{T}^n\|$, $(n \geqq 0)$. Moreover, we have $\|M(T)\| \leqq \|M(\tilde{T})\|$, as strong (p, p) bounds of sublinear operators. This fact is expressed in the following theorem.

THEOREM 3. *If $\tilde{T}$ has a* DEE *with constant C, then T also has a* DEE *with constant C.*

Proof. Consider any $f \in L_p$. For any $n \geqq 1$, let $\tilde{f}_n$ be the $\tilde{L}_p$ function whose first n coordinates are equal to f and whose remaining coordinates are 0. We observe that all but the first $n - i$ coordinates of $\tilde{T}^i\tilde{f}_n$ are 0. Hence we have

$$M(\tilde{T})\tilde{f}_n = (M_n f, M_{n-1} f, \dots, M_1 f, 0, 0, \dots),$$

where $M_i \equiv M_i(T)$, $i \geqq 1$. By the DEE for $\tilde{T}$, we have

$$\|M(\tilde{T})\tilde{f}_n\| \leqq C\|\tilde{f}_n\|.$$

Raising both sides to the p-th powers and then dividing by n, this yields

$$\frac{1}{n} \sum_{m=1}^{n} \|M_m f\|^p \leqq C^p \|f\|^p.$$

Since $\|M_m f\| \uparrow \|M(T)f\|$, the left-hand side coverges monotonely, as $n \to \infty$, to $\|M(T)f\|^p$. Hence T has a DEE with constant C.

If T is a bounded positive operator on L_p, $1 < p < \infty$, then so is $\tilde{T}$ on $\tilde{L}_p$. For any nonnegative function f on $(\tilde{X}, \tilde{\mathcal{F}}, \tilde{\mu})$, $\tilde{T}f$ can be defined as the monotone limit of $\tilde{T}f_n$ for any sequence $f_n \in \tilde{L}_p$ such that $0 \leqq f_n \uparrow f$ a.e.

This definition is independent of the choice of the sequence. The same extension can be made if $\tilde{T}$ is replaced by $\tilde{T}^*$, and, furthermore,

$$\int f \cdot \tilde{T}^* g d\tilde{\mu} = \int g \cdot \tilde{T} f d\tilde{\mu},$$

for any pair of nonnegative functions f, g. (What is said of $\tilde{T}$ is also true for T.) We have the following theorem.

THEOREM 4. *If T is a bounded positive operator on L_p, $1 < p < \infty$, and there exists a finite, positive* a.e. *function f on $(\tilde{X}, \tilde{\mathscr{F}}, \tilde{\mu})$ such that* (i) $\tilde{T}^* f^{p-1} \leqq f^{p-1}$ *and* (ii) $\tilde{T}^n f \leqq Hf$, $(n \geqq 0)$, *for a positive constant H, then T has a* DEE *with constant $Hp/(p - 1)$.*

Proof. Define the operator S on nonnegative measurable functions g on $(\tilde{X}, \tilde{\mathscr{F}}, \tilde{\mu})$ by $Sg = f^{-1}\tilde{T}(fg)$. It is easy to verify directly that S extends to a bounded positive operator on $B \equiv L_p(\tilde{X}, \tilde{\mathscr{F}}, f^p d\tilde{\mu})$ such that for all non-zero $g \in B$,

$$\|S^n g\|_B / \|g\|_B = \|\tilde{T}^n(fg)\| / \|fg\|, \quad (n \geqq 1),$$

and that the same relation holds if we replace $(S^n, \tilde{T}^n)$ by $(S_n, \tilde{T}_n)$ or by $(M(S), M(\tilde{T}))$. The mapping that takes g to fg is an invertible isometry from B onto $\tilde{L}_p$. Consequently, S and $\tilde{T}$ have DEEs with the same constant, if either has one.

We shall show that relative to the measure space $(\tilde{X}, \tilde{\mathscr{F}}, f^p d\tilde{\mu})$, S extends to a positive L_1 contraction which is L_∞ power bounded by H. Clearly S is positive, and the last assertion follows readily from condition (ii). That S is an L_1 contraction follows from condition (i) by the following computation: with $g \geqq 0$ a.e.,

$$\int (Sg) \cdot f^p d\tilde{\mu} = \int \tilde{T}(fg) \cdot f^{p-1} d\tilde{\mu} = \int fg \cdot \tilde{T}^* f^{p-1} d\tilde{\mu}$$
$$\leqq \int fg \cdot f^{p-1} d\tilde{\mu} = \int g \cdot f^p d\tilde{\mu}.$$

Hence by Theorem 2, S (as an operator on B) has a DEE with constant $Hp/(p - 1)$. From the previous paragraph, so does $\tilde{T}$, and by Theorem 3, so does T too.

Remarks (4). Theorem 4 remains valid if $(\tilde{X}, \tilde{\mathscr{F}}, \tilde{\mu}; \tilde{T})$ is replaced by $(X, \mathscr{F}, \mu; T)$. This is obvious from the proof.

4. Power bounded codisjunctive operators. We now proceed to prove our main result. Let T be a codisjunctive operator on L_p $(1 < p < \infty)$ power bounded by K. Let $q = p/(p - 1)$. From [**11**], there exists a

σ-endomorphism Φ of $(X, \mathcal{F}, \mu)$ and a function h with support $= \Phi X$ such that $T^*g = h \cdot \Phi g$ for all $g \in L_q$. Further,

$$\text{supp } T^*g = \Phi \text{ (supp } g).$$

From Theorem 4.3 in [**11**], T^{*n} is induced by Φ^n and some function h_n, $(n \geqq 0)$. It is also clear from that theorem that T has a linear modulus $|T|$ such that $|T|^*$ is induced by Φ and $|h|$, and that $|T|$ is power bounded by K. From this, and the fact $M(T)$ is majorized by $M(\,|T|\,)$, we can assume, without loss of generality, that T is positive in addition. Hence $h_n \geqq 0$ a.e. $(n \geqq 0)$. Now $\|T^n\| \leqq K$ implies $\|T^{*n}\| \leqq K$, $(n \geqq 0)$, in the dual operator norm. By Theorem 4.2 in [**11**], for each $n \geqq 0$, there exists a bounded nonnegative function, here denoted by $D(T^{*n})$, such that

$$(2) \quad \|T^{*n}\|^q = \|D(T^{*n})\|_\infty,$$

and that for any nonnegative function g,

$$(3) \quad \int h_n{}^q \cdot \Phi^n g d\mu = \int g \cdot D(T^{*n}) d\mu.$$

LEMMA 1. *For any nonnegative function g on X, and any $n \geqq 0$,*

$$(4) \quad T^n(T^{*n}g)^{q-1} = D(T^{*n})g^{q-1} \leqq K^q g^{q-1}.$$

Proof. For any nonnegative function f,

$$\begin{aligned}\int f \cdot T^n(T^{*n}g)^{q-1} d\mu &= \int T^{*n}f \cdot (T^{*n}g)^{q-1} d\mu \\ &= \int h_n \Phi^n f \cdot (h_n \Phi^n g)^{q-1} d\mu = \int h_n \Phi^n f \cdot h_n{}^{q-1} \Phi^n g^{q-1} d\mu \\ &= \int h_n{}^q \cdot \Phi^n (f g^{q-1}) d\mu = \int f g^{q-1} D(T^{*n}) d\mu,\end{aligned}$$

by (3). From this, the equality part of (4) follows. The inequality part follows from (2).

We shall now establish (i) and (ii) in Theorem 4. Take any strictly positive function g_0 on X, and any sequence of nonnegative, finite a.e. functions g_n, $(n \geqq 1)$, each with support $= X \backslash \Phi X$. For each $n \geqq 0$, let

$$g_{n,i} = T^{*n-i} g_i, \quad (i = 0, \ldots, n).$$

Each $g_{n,i}$ is finite a.e., since T^* is disjunctive. Let

$$g = (g_{n,0} + \ldots + g_{n,n})_{n \geqq 0} \quad \text{and} \quad f = g^{q-1}.$$

Now

https://doi.org/10.4153/CJM-1983-033-1 Published online by Cambridge University Press

$$\operatorname{supp} g_{n,0} = \Phi^n X, \quad \text{and}$$

$$\operatorname{supp} g_{n,i} = \Phi^{n-i}(X \backslash \Phi X) = \Phi^{n-i} X \backslash \Phi^{n-i+1} X, \quad (i = 1, \dots, n).$$

Hence the supports of $g_{n,i}$, $(i = 0, \dots, n)$, are mutually disjoint, and the support of the n-th coordinate of g is

$$\Phi^n X \cup (\Phi^{n-1} X \backslash \Phi^n X) \cup \dots \cup (X \backslash \Phi X) = X.$$

Hence $g > 0$ and so $f > 0$ a.e. From the definition of $\tilde{T}$, we easily get

$$\tilde{T}^*(k_n)_{n \geqq 0} = (T^* k_{n-1})_{n \geqq 0},$$

taking k_{-1} to be 0. Now $f^{p-1} = g$, since $(p - 1)(q - 1) = 1$. Hence

$$\text{(i)} \quad \tilde{T}^* f^{p-1} = \left(T^* \sum_{i=0}^{n-1} g_{n-1,i}\right)_{n \geqq 0} = \left(\sum_{i=0}^{n-1} g_{n,i}\right)_{n \geqq 0} \leqq f^{p-1}.$$

Here the void sum (which occurs when $n = 0$) is interpreted as 0.

To get (ii), first we observe that for every $m \geqq 0$,

$$\tilde{T}^m (f_n)_{n \geqq 0} = (T^m f_{m+n})_{n \geqq 0}.$$

Since $g_{n,i}$, $(i = 0, \dots, n)$, have disjoint supports,

$$f = \left(\sum_{i=0}^{n} g_{n,i}^{q-1}\right)_{n \geqq 0}.$$

So

$$\tilde{T}^m f = \left(\sum_{i=0}^{n+m} T^m g_{n+m,i}^{q-1}\right)_{n \geqq 0}.$$

Now if $m + n \geqq i > n$, then for any nonnegative function k, we have

$$\int k T^m g_{n+m,i}^{q-1} d\mu = \int (T^{*m} k)(g_{n+m,i}^{q-1}) d\mu = 0,$$

since $\operatorname{supp} T^{*m} k \subset \Phi^m X$ while

$$\operatorname{supp} g_{n+m,i} \subset X \backslash \Phi^{m-(i-n-1)} X \subset X \backslash \Phi^m X.$$

(Note that $\Phi^j X$ decreases with increase of j.) Hence

$$(5) \quad T^m g_{n+m,i}^{q-1} = 0, \quad (n < i \leqq m + n).$$

On the other hand, we have

https://doi.org/10.4153/CJM-1983-033-1 Published online by Cambridge University Press

(6) $T^m g_{n+m,i}^{q-1} = T^m (T^{*m} g_{n,i})^{q-1} \leqq K^q g_{n,i}^{q-1}, \quad (0 \leqq i \leqq n),$

by Lemma 1. From (5) and (6), we conclude that

$$\text{(ii)} \quad \tilde{T}^m f \leqq \left(K^q \sum_{i=0}^{n} g_{n,i}^{q-1} \right)_{n \geqq 0} = K^q f.$$

Applying Theorem 4, we arrive at our main result:

THEOREM 5. *Let T be a codisjunctive operator on L_p, $1 < p < \infty$, such that* sup $\|T^n\| \equiv K < \infty$, *then T has a* DEE *with constant $K^q q$, where $q = p/(p - 1)$, and the pointwise ergodic property, i.e. $T_n f$ converges a.e. ($f \in L_p$).*

Proof. It remains only to prove the convergence part. This follows from the DEE and the mean ergodic theorem, as shown in [**10**], on observing that $T^n f/n \to 0$ a.e. for all $f \in L_p$. The latter fact follows from

$$\int \sum_{n=1}^{\infty} \left| \frac{1}{n} T^n f \right|^p d\mu = \sum_{n=1}^{\infty} \left\| \frac{1}{n} T^n f \right\|_p^p \leqq K^p \left(\sum_{n=1}^{\infty} n^{-p} \right) \|f\|_p^p < \infty,$$

as observed by M. A. Akcoglu (in the case $K = 1$, see [**10**]).

5. Dualization and variation of the method. The method that we have used actually shows that if the constant in Theorem 2(b) is $c_p(H)$, then the constant in Theorem 5 can be taken as $c_p(K^q)$, where $q = p/(p - 1)$. This result can be dualized, along with the method that leads to it. Consider the (unilateral) forward T-shift on $\tilde{L}_p$ derived from an operator T on L_p, $1 \leqq p < \infty$, denoted again by $\tilde{T}$, which is defined by

$$\tilde{T}(f_0, f_1, \ldots) = (0, Tf_0, Tf_1, \ldots).$$

Since the first i coordinates of $\tilde{T}^i \tilde{f}_n$ are 0, we have

$$M(\tilde{T})\tilde{f}_n \geqq (M_1 f, \ldots, M_n f, 0, 0, \ldots),$$

with the first n pairs of corresponding coordinates equal. From this, Theorem 3 (see its proof) is true for our new $\tilde{T}$. By a similar method as before, we have the following theorem.

THEOREM 6. *Fix $1 < p < \infty$.*

(a) *If all positive L_∞ contractions that are simultaneously L_1 power bounded by H have a* DEE *on L_p with constant $c_p(H)$, then all disjunctive operators that are L_p power bounded by K have a* DEE *with constant $c_p(K^p)$ and the pointwise ergodic property.*

(b) *In particular, all disjunctive L_p contractions have a* DEE *with constant* $p/(p-1)$ *and the pointwise ergodic property.*

Proof. Part (a). Let T be a disjunctive operator on L_p that is power bounded by K. As before, we can assume T to be positive, without loss of generality. According to the dualized method to that embodied in the proof of Theorem 5, to prove part (a), we need only establish (i′) $\tilde{T}f \leqq f$, and (ii′) $\tilde{T}^{*n}f^{p-1} \leqq H \cdot f^{p-1}$, $(n \geqq 0)$, for $H = K^p$, for a strictly positive, finite a.e. measurable function f on $(\tilde{X}, \tilde{\mathscr{F}}, \tilde{\mu})$. We can take

$$f = (T^n f_0 + \ldots + Tf_{n-1} + f_n)_{n \geqq 0},$$

for any sequence of nonnegative, finite a.e. functions f_n, $(n \geqq 0)$, with supp $f_0 = X$ and supp $f_n = X \backslash \Phi X$, $(n \geqq 1)$, where Φ is the associated σ-endomorphism of T. We omit the details.

Part (b) follows from part (a) and the DEE for L_1 simultaneously L_∞ contractions (i.e., case $H = 1$ of Theorem 2(b)).

Remarks (5). Concerning part (b), case $H = 1$ of Theorem 2(b) was proved originally by N. Dunford and J. T. Schwartz [**6**, Section 4.7], but with a constant that is not sharp. The sharp constant, $p/(p-1)$, was later observed by A. M. Garsia and B. Jamison (see [**4**] or [**9**, Corollary 2.2.1]). A different proof of part (b), by geometric dilation to isometries, is given in [**11**, Theorem 5.1]. Using truncated analogues of the forward T-shift, Chacon and McGrath [**3**] proved the DEE for a positive L_p contraction T, $(1 < p < \infty)$, with the property that $\|T^n f_n\| = \|f_n\|$, $(n \geqq 1)$, for a sequence of positive a.e. L_p functions f_n. This includes the isometry case, but not part (b) above. We note in this connection that the technique used in this paper stems partly from a study of the method used in [**3**, **4**].

The construction involved in the proofs of Theorems 5 and 6, being general, is somewhat complicated. There is a variation of the method that need only deal with two special cases with simpler constructions and that may put it in a better perspective. First, we have the following lemma.

LEMMA 2. *For each σ-endomorphism Φ on $(X, \mathscr{F}, \mu)$, there exists a decomposition of X into disjoint subsets Y and Z, such that $\Phi Y = Y$, $\Phi Z \subset Z$, and Z is a disjoint union of Z_i, $i \geqq 0$, with $\Phi Z_i = Z_{i+1}$, $i \geqq 0$. (Y and Z_i, $i \geqq$ some i_0, may be null.)*

Proof. Let $Z_0 = X \backslash \Phi X$, $Z_i = \Phi^i Z_0$, $(i \geqq 1)$, $Z = \bigcup_{i=0}^{\infty} Z_i$ and $Y = X \backslash Z$. The assertions follow readily.

https://doi.org/10.4153/CJM-1983-033-1 Published online by Cambridge University Press

From this lemma, if either T or T^* is disjunctive, with associated σ-endomorphism Φ, then l_p^2-direct sum decomposition

$$L_p(X) = L_p(Y) \oplus L_p(Z)$$

reduces T, and the DEE for T need only be proved separately on $L_p(Y)$ and on $L_p(Z)$. It also follows easily from Lemma 2 that, in either case, $T^n f \to 0$ a.e. for all $f \in L_p(Z)$, if T is power bounded.

For the T in Theorem 5 and case $X = Y$, i.e., $\Phi X = X, f = g^{q-1}$, where $g = (g_0, T^* g_0, T^{*2} g_0, \dots)$ and g_0 is a positive a.e. function on X, will satisfy the conditions in Theorem 4 with $H = K^q$.

For the case $X = Z$, we modify the construction as follows. Take a finite, nonnegative a.e. function g' with support $= Z_0$. Then

$$\text{supp } T^{*n} g' = Z_n, \quad (n \geqq 0), \quad \text{and}$$

$$g \equiv g' + T^* g' + T^{*2} g' + \dots$$

is positive, finite a.e. on X, and so is $f \equiv g^{q-1}$. We can then show that conditions (i) (ii) in Theorem 4 are true for T in lieu of $\tilde{T}$ with $H = K^q$ and hence T has a DEE with constant $K^q q$. (See Remarks (4).)

The T in Theorem 6 can be treated similarly.

Finally, these modified constructions can be adapted to reduce the DEE problem for power bounded positive L_p operators. For $1 \leqq p \leqq \infty$ and $K \geqq 1$, denote by $\mathscr{P}(p;K)$ the class of positive L_p operators power bounded by K.

THEOREM 7. *Let $1 < p < \infty$. The class $\mathscr{P}(p;K)$ will admit of a* DEE *on L_p with a constant $c_p(K)$ if so does either*

(I) $\mathscr{P}(p;K) \cap \mathscr{P}(1;1)$ *or*

(II) $\mathscr{P}(p;K) \cap \mathscr{P}(\infty;1)$.

Proof. Let $T \in \mathscr{P}(p;K)$. Without loss of generality, we assume that T operates on L_p of a σ-finite measure space.

Case (I). Take $0 < \theta < 1$ and $0 < g' \in L_q$, where $q = p/(p - 1)$. Define

$$g = \sum_0^\infty (\theta T)^{*n} g'.$$

Then $0 < g \in L_q$, and $f \equiv g^{q-1} \in L_p$ satisfies

$$(\theta T)^* f^{p-1} \leqq f^{p-1} \equiv g,$$

i.e., condition (i) in Theorem 4 for θT in place of $\widetilde{T}$. S constructed in the proof thereof is then in the subclass (I). The first part of the same proof gives the DEE for θT *on* L_p with constant $c_p(K)$. The same DEE holds for T since $M(\theta T) \uparrow M(T)$ as $\theta \uparrow 1$.

Case (II). Similarly, take $0 < f' \in L_p$ and define

$$f = \sum_{0}^{\infty} (\theta T)^n f',$$

which belongs to L_p and satisfies $(\theta T)f \leqq f$. The corresponding S is in the subclass (II) and by the same token as in case (I), T has a DEE with constant $c_p(K)$.

References

1. M. A. Akcoglu, *A pointwise ergodic theorem in L_p-spaces*, Can. J. Math. *27* (1975), 1075-1082.

2. A. P. Calderon, *Ergodic theory and translation-invariant operators*, Proc. Nat. Acad. Sci. U.S.A. *59* (1968), 349-353.

3. R. V. Chacon and S. A. McGrath, *Estimates of positive contractions*, Pacific J. Math. *30* (1969), 609-620.

4. R. V. Chacon and J. Olsen, *Dominated estimates of positive contractions*, Proc. Amer. Math. Soc. *20* (1969), 266-271.

5. J. L. Doob, *Stochastic processes* (Wiley, New York, 1952).

6. N. Dunford and J. T. Schwartz, *Convergence almost everywhere of operator averages*, J. Math. Mech. *5* (1966), 399-401.

7. M. Fukamiya, *On dominated ergodic theorems in L_p ($p \geqq 1$)*, Tôhoku Math. J. *46* (1939), 150-153.

8. A. M. Garsia, *A simple proof of E. Hopf's maximal ergodic theorem*, J. Math. Mech. *14* (1965), 381-382.

9. ——— *Topics in almost everywhre ccnvergence*, Lectures in advanced mathematics, vol. 4 (Markham, Chicago, 1970).

10. A. Ionescu Tulcea, *Ergodic properties of isometries in L_p-spaces*, $1 < p < \infty$, Bull. Amer. Math. Soc. *70* (1964), 366-371.

11. C. H. Kan, *Ergodic properties of Lamperti operators*, Can. J. Math. *30* (1978), 1206-1214.

National University of Singapore,
Kent Ridge, Singapore