SYMMETRIC PERIODIC ORBITS IN THE ANISOTROPIC KEPLER PROBLEM

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ABSTRACT. The anisotropic Kepler problem has a group of symmetries with three generators; they are symmetries respect to zero velocity curve and the two axes of motion's plane. For a fixed negative energy level it has four homothetic orbits. We describe the symmetric periodic orbits near these homothetic orbits. Full details and proofs will appear elsewhere (Casasayas-Llibre).

1. INTRODUCTION AND EQUATIONS OF MOTION.

The anisotropic Kepler problem was introduced by Gutzwiller (1973) to model certain quantum mechanical systems. But for us it has a mathematical interest because it is an easy model in order to study usual tools in the analysis of the n-body problem as non-integrability, collision manifold,... (Devaney, 1981).

This problem deals with the motion of a body which is attracted by a gravitational potential and has an anisotropic mass. It is described by the Hamiltonian system

$$\begin{aligned} \dot{q} &= M^{-1}p, \\ \dot{p} &= -\nabla V(q), \end{aligned} \tag{1}$$

where

$$q = (q_1,q_2) \in R^2-\{(0,0)\} \text{ and } p = (p_1,p_2) \in R^2$$

are the position and momentum coordinates of the body,

$$M^{-1} = \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} ,$$

is the masses matrix and μ, $1 \leq \mu \leq +\infty$, is the mass parameter and

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$V(q) = -|q|^{-1}$ is the potential energy. The total energy function is given by the Hamiltonian

$$H(q,p) = \tfrac{1}{2}\, p^t M^{-1} p + V(q).$$

System (1) is actually a one parameter family of Hamiltonian systems with two degrees of freedom depending analytically on the parameter μ. When $\mu=1$, we have the Kepler problem and $\mu > 1$ introduces the anisotropic matrix M which means that q_2 is the "heavy axis".

Equations (1) have a singularity of collision when q=0 which can be studied using the "blow up" technique of McGehee (Devaney, 1981).

Thereafter we will assume that the energy level H=h is fixed and takes a prescribed negative value. Otherwise, if $h \geq 0$ there are no periodic orbits.

2. SYMMETRIES OF THE PROBLEM

The anisotropic Kepler problem has the following symmetries:

$$S_1(q_1,q_2,p_1,p_2,t) = (q_1,-q_2,-p_1,p_2,-t),$$

$$S_2(q_1,q_2,p_1,p_2,t) = (-q_1,q_2,p_1,-p_2,-t),$$

$$S_3(q_1,q_2,p_1,p_2,t) = (q_1,q_2,-p_1,-p_2,-t),$$

which can be interpreted in the following way.

Let $\gamma(t) = (q_1(t),q_2(t),p_1(t),p_2(t))$ be a solution of (1), then

$$S_1(\gamma(t)) = (q_1(-t),-q_2(-t),-p_1(-t),p_2(-t))$$

is another solution, see Figure 1. In a similar way $S_2(\gamma(t))$ and $S_3(\gamma(t))$ are solutions of (1), see Figures 2 and 3.

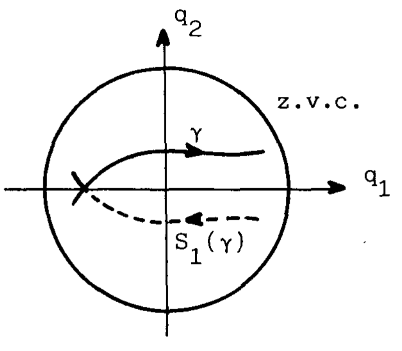

Figure 1.

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Orbits which cross orthogonally the q_1-axis (resp. q_2-axis) are the symmetric orbits respect to S_1 (resp. S_2), that is $S_1(\gamma) = \gamma$ (resp. $S_2(\gamma)=\gamma$). Orbits which have some point on the zero velocity curve (z.v.c.) are the symmetric orbits respect to S_3.

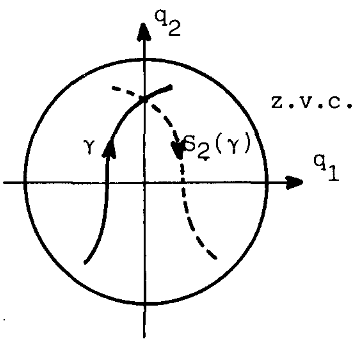


Figure 2.

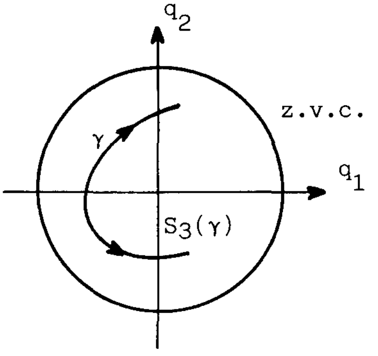


Figure 3.

If an orbit crosses two times orthogonally the q_1-axis (resp. q_2-axis) then it is a symmetric periodic orbit for S_1 (resp. S_2). Similarly, if an orbit has two points on the zero velocity curve then it is a symmetric periodic orbit for S_3, for more details see Devaney (1976). This fact is essential for our study of the symmetric periodic orbits.

3. SYMMETRIC PERIODIC ORBITS FOR THE KEPLER PROBLEM ($\mu=1$).

For the Kepler problem is not difficult to prove the following.

(1) There is a bijection between the symmetric periodic orbits respect to S_i, for each i=1,2, and two copies of the segment $(0,-1/h)$. One copy corresponds to the direct ellipses and the other one to the retrograde ellipses, see Figures 4 and 5.

(2) There is a bijection between the symmetric periodic orbits respect to S_1 and S_2 and the two points $\pm(2h)^{-1}$. They correspond to the circular orbits, see Figure 6.

(3) There is a bijection between the symmetric orbits (but not periodic) respect to S_3 and the circle. They are orbits of elliptic collision. If we regularize the equations then the orbits become periodic, see figure 7.

So when $\mu=1$ we have a complete description for the symmetric periodic orbits (note that the system is integrable).

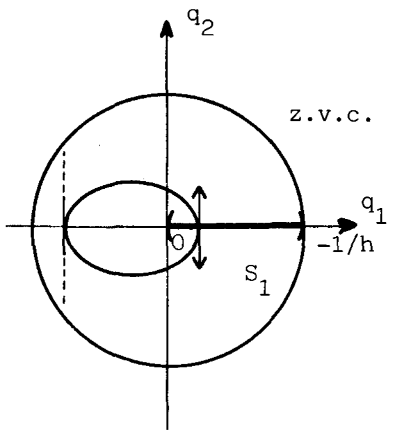
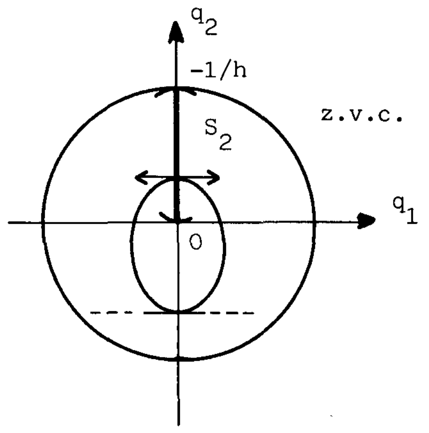


Figure 4.



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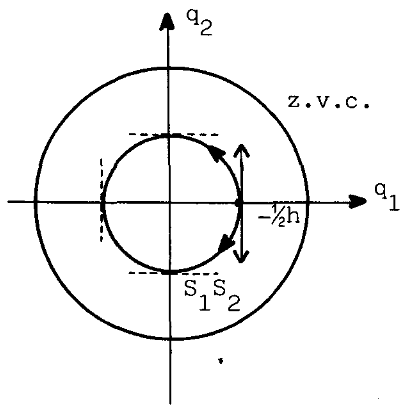

Figure 6.

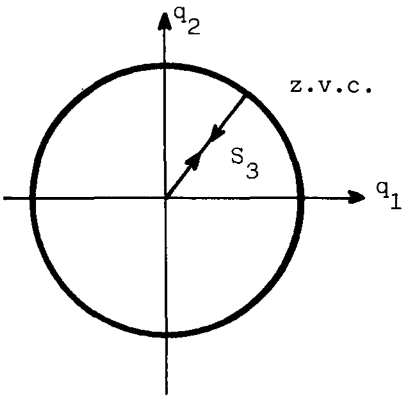

Figure 7.

4. SYMMETRIC PERIODIC ORBITS FOR $\mu > 9/8$.

A solution $(q(t),p(t))$ of the anisotropic Kepler problem is called homothetic if we can obtain $(q(t_1),p(t_1))$ from $(q(t_2),p(t_2))$ through a dilation, for every t_1, t_2 where the solution is defined. An orbit $(q(t),p(t))$ of system (1) is called a collision (resp. ejection) orbit if there exists t_o such that $q(t) \to 0$ as $t \uparrow t_o$ (resp. $t \downarrow t_o$).

It is known that the anisotropic Kepler problem has only four homothetic orbits π_i for $i = 1,2,3,4$, which are also of ejection-collision type, see Figure 8 (Devaney, 1981). We have studied the neighborhood of these orbits in order to prove the following theorem (see Casasayas-Llibre).

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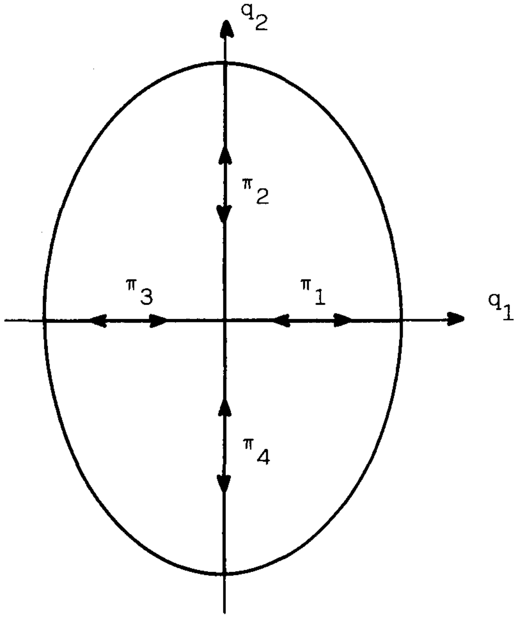

Figure 8.

THEOREM. There exists a positive integer number n_o such that for every $n,m \geq n_o$ the following statements hold.

(i) There are four symmetric periodic ejection-collision orbits respect to S_3 (resp. S_2) such that the number of crossings with the q_2-axis is 2n (resp. 2n+1), see Figures 9 and 10 (resp. Figures 11 and 12). There are similar figures for the region $q_2 \leq 0$.

(ii) There are two symmetric periodic orbits respect to S_2 (resp. S_2 and S_3) such that the qualitative behaviour is given in Figure 13 (resp. Figure 14). There are similar figures for the region $q_2 \geq 0$ obtained by the change $\theta = \theta + \pi$.

(iii) There is one symmetric periodic orbit respect to S_2 (resp. S_3) such that the qualitative behaviour is shown in Figure 15 (resp. Figure 16). When m=n the orbit is also symmetric respect to S_1.

(iv) There are four (resp. two) symmetric periodic orbits respect to S_3 (resp. S_2 and S_3) such that the qualitative behaviour is given in Figures 17 and 18 (resp. Figure 19). There are similar figures for the region $q_2 \leq 0$ obtained by the change $\theta = \theta + \pi$.

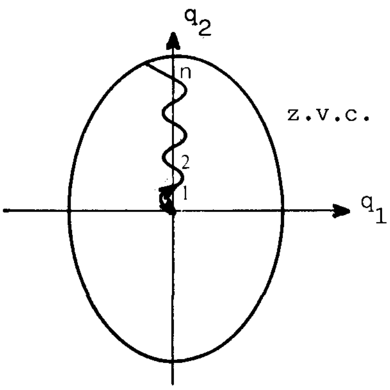

Figure 9.

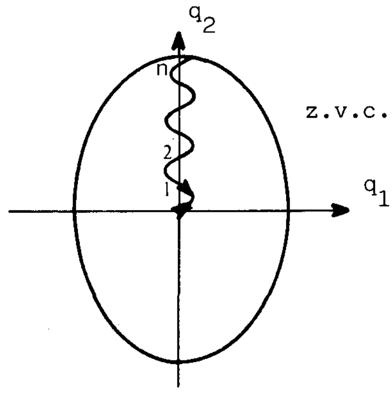

Figure 10.

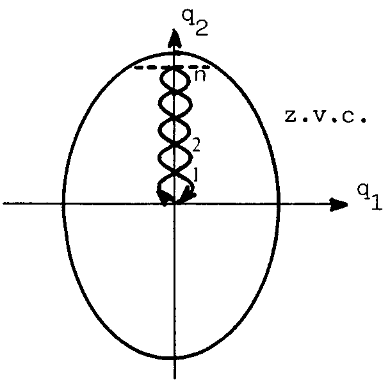

Figure 11.

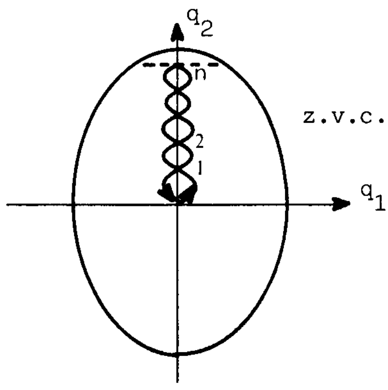

Figure 12.

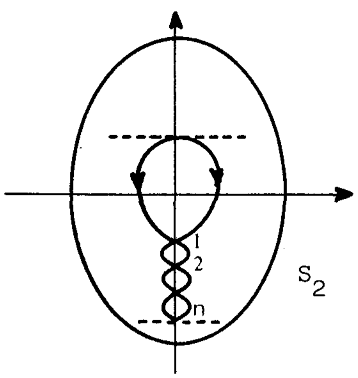

Figure 13.

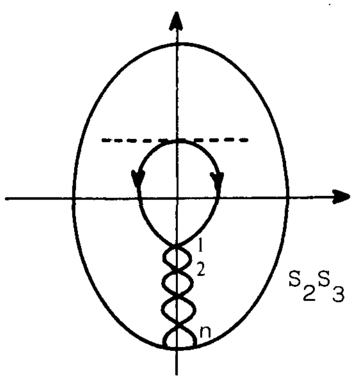

Figure 14.

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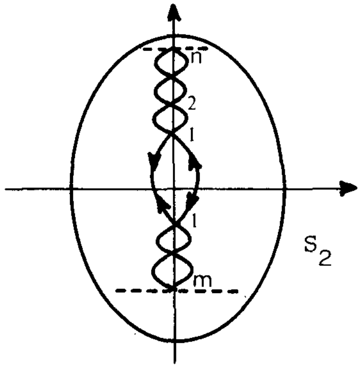

Figure 15.

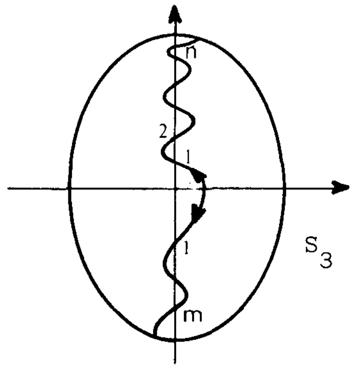

Figure 16.

S_3

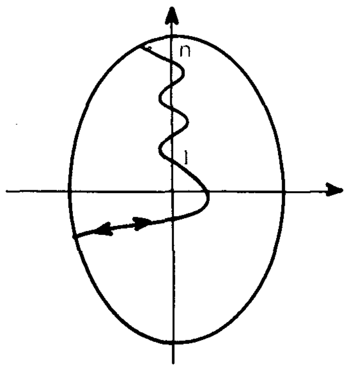

Figure 17.

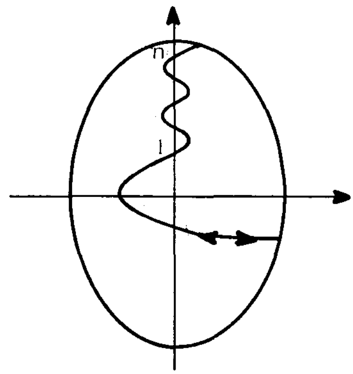

Figure 18.

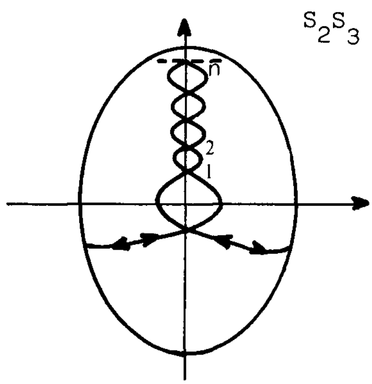

Figure 19.

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