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SYSTOLIC FILLINGS OF SURFACES

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Abstract

A *filling* of a closed hyperbolic surface is a set of simple closed geodesics whose complement is a disjoint union of hyperbolic polygons. The *systolic length* is the length of a shortest essential closed geodesic on the surface. A geodesic is called systolic, if the systolic length is realised by its length. For every $g \geq 2$, we construct closed hyperbolic surfaces of genus g whose systolic geodesics fill the surfaces with complements consisting of only two components. Finally, we remark that one can deform the surfaces obtained to increase the systole.

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1. Introduction

Fillings of surfaces have become increasingly important in the study of the mapping class groups, Teichmüller spaces and moduli spaces of surfaces which have their origin in the work of Thurston [5]. Let $\mathcal{M}_g$ denote the moduli space of oriented closed hyperbolic surfaces of genus g. It is a very well known and difficult problem to construct a spine of $\mathcal{M}_g$. Thurston has proposed the set χ_g (so-called Thurston set) of all closed surfaces F_g of genus g, whose systolic geodesics fill the surface, as a candidate spine of $\mathcal{M}_g$ and has provided a sketch of a proof [5]. But the proof is difficult to complete. Moreover, many things about the set χ_g are unknown, for example connectivity, dimension and contractibility.

Fillings of surfaces have been studied extensively by Alexander, Parlier and Pettet [1, 2], Aougab and Huang [3] and others. In [2], the authors bound the cardinality of a filling set of systolic geodesics from below by $\pi\sqrt{g(g-1)}/\log(4g-2)$ [2, Theorem 3]. Anderson, Parlier and Pettet have constructed a sequence of surfaces S_{g_k} in the Thurston set χ_{g_k} with large Bers constant, where g_k is large enough [1, Theorem 1.1]. Furthermore, they have studied the shape of χ_g, comparing it with the set $\mathcal{Y}_g$ of trivalent surfaces, by giving a lower bound on the Hausdorff distance between χ_g and $\mathcal{Y}_g$ (see [1, Section 4]).

More recently, Fanoni and Parlier have studied fillings of punctured surfaces [4]. They have constructed hyperbolic surfaces of signature $(0, n)$ for $n \geq 4$, with a filling set of systolic geodesics of cardinality n [4, Proposition 5.3]. Furthermore, they have shown that the cardinality of a filling set of systoles of a surface $S_{g,n}$ is at least $\pi(4g-4+n)/4l$, where l is the systolic length [4, Theorem 4].

In this article, we construct closed hyperbolic surfaces with systolic fillings. More precisely, for genus $g=2$, we construct a hyperbolic surface S_2 (the so-called Bolza surface), where the set of all systolic geodesics has cardinality 12 and provides a triangulation of the surface. In [2], Anderson, Parlier and Pettet have already constructed hyperbolic surfaces with $2g+2$ systolic geodesics filling the surface and, furthermore, there are subsets with cardinality $2g$ of these $2g+2$ systolic geodesics filling the surface. What is new here is that, for each $g \geq 3$, we construct a hyperbolic surface S_g of genus g, whose set of systolic geodesics has exactly $2g$ curves and fills the surface (see Theorem 4.1). Furthermore, for $g \geq 3$, these are the surfaces with the minimum number of systolic geodesics among such surfaces in χ_g that are known so far. Our construction is combinatorial and uses *decorated fat graphs*. Finally, we remark that one can deform these surfaces $S_g, g \geq 3$, continuously in the Thurston set to increase the systolic lengths.

2. Preliminaries

In this section, we recall some notions on fat graphs and systoles (the shortest length essential geodesics) of surfaces and discuss the connection between them. We conclude the section with a proof of Proposition 2.1 on hyperbolic polygons, which will be used in the subsequent sections. The idea behind the proof of this proposition is similar to the proof of [4, Proposition 5.3].

A *fat graph* is a graph equipped with a cyclic order on the set of edges emanating from each vertex. If the degree of each vertex of a fat graph is even and at least four, then we call it a *decorated fat graph*. A cycle in a decorated fat graph is called a *standard cycle* if every two consecutive edges in the cycle are opposite each other with respect to the cyclic order on the set of edges emanating from their common vertex. For more details on fat graphs, we refer the reader to [7, Section 2].

A surface S will always be a closed Riemann surface with constant curvature -1; such a surface is called a *hyperbolic surface*. A *filling* of S is a set of simple closed geodesics whose complement is a disjoint union of polygonal regions. For a nonnegative integer k, the kth complexity $\mathcal{T}_k(\Omega)$ of a system of curves (in particular, a filling system) Ω on a surface S is defined as the number of elements in

$$\Big\{\gamma \in C(S) \setminus \Omega \,\Big|\, \sum_{\delta \in \Omega} i(\gamma, \delta)| = k\Big\},$$

where $C(S)$ denotes the set of all simple closed geodesics on S and $i(\gamma, \delta)$ denotes the geometric intersection number between γ and δ on S.

The *systolic length* $\mathrm{sys}(S)$ of a hyperbolic surface S is the length of a shortest essential geodesic on the surface. A simple closed geodesic on S realising $\mathrm{sys}(S)$ is called a *systolic geodesic* or, simply, a systole. The set of all systolic geodesics of S is denoted by $\mathrm{SLG}(S)$. We are interested in the surfaces S, where $\mathrm{SLG}(S)$ is a filling.

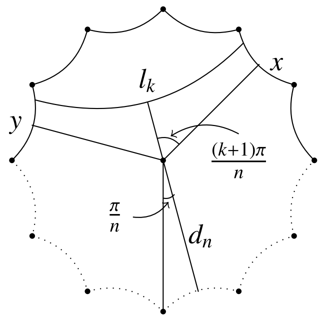

FIGURE 1. The polygon $\mathcal{P}_n$.

Given a filling Ω of a surface S, it naturally corresponds to a decorated fat graph $\Gamma(\Omega)$, where the vertices are the intersection points of the curves in Ω, the edges are the subarcs of the curves in Ω between the vertices, and the fat graph structure is provided by the orientation of the surface. The standard cycles of $\Gamma(\Omega)$ correspond to the curves in Ω as the curves in a filling are pairwise in minimal position and, in particular, intersect transversally.

PROPOSITION 2.1. *Let $\mathcal{P}_n$ be a right-angled regular hyperbolic n-sided polygon, where $n \geq 5$. If x and y are two nonconsecutive sides of $\mathcal{P}_n$, then*

$$d_{\mathbb{H}}(x, y) \geq t_n, \tag{2.1}$$

where t_n is the length of a side of $\mathcal{P}_n$ and $d_{\mathbb{H}}$ is the distance function on the hyperbolic plane $\mathbb{H}$. Furthermore, the inequality in (2.1) is strict if and only if the minimum number of sides between x and y in $\mathcal{P}_n$ is greater than one.

PROOF. First, we compute the length t_n. Any two consecutive vertices and the centre of $\mathcal{P}_n$ form a hyperbolic triangle with the interior angles $\pi/4, \pi/4$ and $2\pi/n$. The side opposite the vertex with interior angle $2\pi/n$ is the side of $\mathcal{P}_n$ in the triangle. Thus, using the hyperbolic cosine rule II (see [6, Section 7.12]) on this triangle,

$$t_n = \cosh^{-1}\left(\frac{\cos^2(\pi/4) + \cos(2\pi/n)}{\sin^2(\pi/4)}\right) = \cosh^{-1}\left(1 + 2\cos\left(\frac{2\pi}{n}\right)\right).$$

Let d_n be the length of the perpendicular from the centre to a side of $\mathcal{P}_n$ (see Figure 1). From [6, Theorem 7.11.3],

$$d_n = \cosh^{-1}\left(\frac{1}{\sqrt{2}\sin(2\pi/n)}\right).$$

Now consider two nonconsecutive sides x and y of $\mathcal{P}_n$. Let k be the minimum number of edges in $\mathcal{P}_n$ between x and y. If $k = 1$, then the distance $d_{\mathbb{H}}(x, y)$ is realised by the length of the side between them which is the common perpendicular of x and y and,

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therefore, equality holds in (2.1). Next we assume that $k \geq 2$. Let l_k be the common perpendicular of x and y. The perpendiculars from the centre of $\mathcal{P}_n$ to x, y, together with the arcs of x, y and l_k, form a pentagon in which all the angles are right angles except the interior angle at the centre which is equal to $2(k+1)\pi/n$. The perpendicular from the centre to l_k divides the pentagon into two congruent sharp corners (Lambert quadrilateral) with the only non right angle equal to $(k+1)\pi/n$ (see Figure 1). Now, using [6, Theorem 7.17.1, formula (ii)] on the sharp corner,

$$\cosh(l_k/2) = \cosh d_n \sin((k+1)\pi/n),$$

which implies that $l_k > l_{k'}$, if $k > k'$. In particular, $l_k > l_1 = t_n$ for all $k > 1$, which completes the proof. □

3. Genus two

In this section, we construct a hyperbolic surface of genus two (which is the so-called Bolza surface) in χ_2. We prove the following theorem.

Theorem 3.1. *There exists a closed hyperbolic surface S_2 of genus two such that:*

(1) $\mathrm{SLG}(S_2)$ *provides a triangulation of S_2 and, in particular, $S_2 \in \chi_2$;*
(2) $|\mathrm{SLG}(S_2)| = 12$*;*
(3) $\mathrm{sys}(S_2) = 2\cosh^{-1}(1+\sqrt{2})$*; and*
(4) $\mathcal{T}_i(\mathrm{SLG}(S_2)) = 0$ *for* $0 \leq i \leq 5$.

Proof. Let us consider a decorated fat graph Γ_2 with four standard cycles, as given in Figure 2. A simple Euler characteristic argument implies that the genus of the fat graph is two.

The graph Γ_2 has two boundary components, which are given by

$$\partial_1 = a_1 d_2 \bar{c}_2 \bar{b}_2 \bar{a}_2 \bar{d}_1 c_1 b_1 \quad \text{and} \quad \partial_2 = \bar{a}_1 b_2 \bar{c}_1 \bar{d}_2 a_2 \bar{b}_1 c_2 d_1.$$

Now, consider two labelled right-angled regular hyperbolic polygons $P_i = P_i(\partial_i)$ for $i = 1, 2$, as shown in Figure 3, which correspond to the boundaries of Γ_2. Note that the boundary words of the polygons provide a side pairing.

The polygons P_1 and P_2, with the labellings as indicated in Figure 3, project onto a closed hyperbolic surface of genus two when we glue the edges labelled by the same letter with the same subscript by hyperbolic isometries. We denote the resulting surface by S_2.

The sides $\{a_1, a_2\}, \{b_1, b_2\}, \{c_1, c_2\}$ and $\{d_1, d_2\}$ project onto simple closed geodesics on S_2. We denote these geodesics by a, b, c and d, respectively, and define

$$\Omega_2 = \{a, b, c, d\}.$$

Let $\tilde{\gamma}^i_j$ be the geodesic segments on the polygon P_i joining two diagonally opposite vertices labelled by v_j, where $i = 1, 2$ and $j = 1, \ldots, 4$ (see Figure 3). The geodesic

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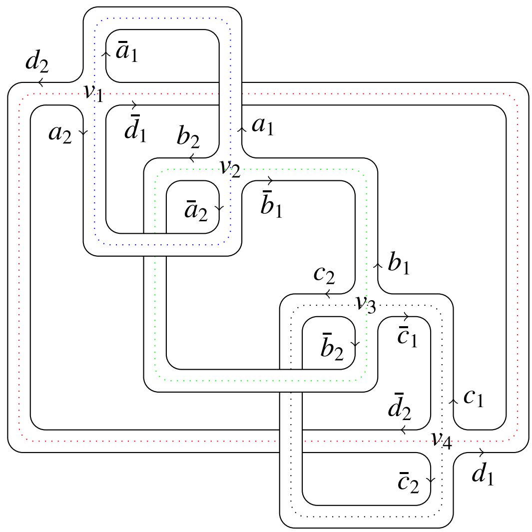

FIGURE 2. The graph Γ_2.

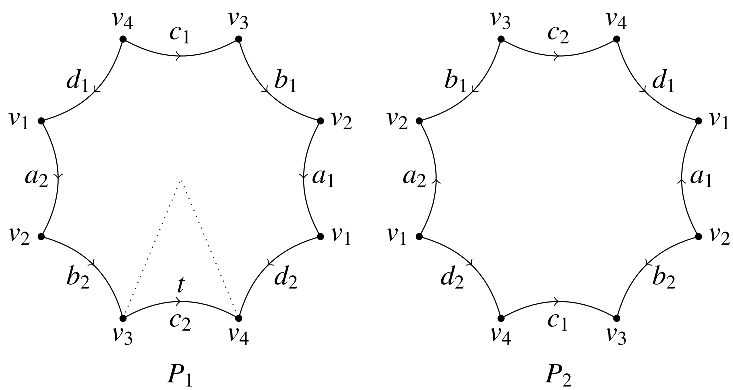

FIGURE 3. Labelled polygons P_1 and P_2.

segments $\tilde{\gamma}^i_j$ project onto simple closed geodesics on S_2; we denote these geodesics by γ^i_j. Now we define

$$X_2 := \Omega_2 \cup \{\gamma^i_j \mid i = 1, 2 \text{ and } j = 1, \ldots, 4\}.$$

Next, we prove two lemmas.

LEMMA 3.2. *If t is the hyperbolic length of each side of P_i and d is the distance between two diagonally opposite vertices, then*

$$\cosh t = \cosh(d/2) = 1 + \sqrt{2}.$$

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PROOF. Consider the hyperbolic triangle T with vertices at any two consecutive vertices and the centre of the polygon P_i. Then T is an equilateral triangle with each interior angle $\pi/4$. Using the hyperbolic cosine rule II (see [6, Section 7.12]),

$$\cosh t = \cosh(d/2) = \frac{\cos^2(\pi/4) + \cos(\pi/4)}{\sin^2(\pi/4)} = 1 + \sqrt{2}. \qquad \square$$

LEMMA 3.3. Length$(\alpha) = 2\cosh^{-1}(1 + \sqrt{2})$ *for all* $\alpha \in X_2$.

PROOF. Suppose $\alpha \in X_2$. Then its length is either d, if $\alpha = \gamma^i_j$, or $2t$, if $\alpha \in \Omega_2$. In both of these cases, it follows from Lemma 3.2 that length$(\alpha) = 2\cosh^{-1}(1 + \sqrt{2})$. $\square$

The set X_2 is a filling of S_2 with $|X_2| = 12$ and it provides a triangulation of S_2. The proof of (1), (2) and (3) of Theorem 3.1 will be completed once we prove the following claim.

CLAIM 3.4. $\mathrm{SLG}(S_2) = X_2$.

PROOF OF CLAIM 3.4. The subset $\Omega_2 \subset X_2$ is a filling of S_2 with complement P_1 and P_2. It is straightforward to see in Figure 3 that each pair of edges labelled by the same letter with the same subscript is in a different polygon, which implies that, if γ is a simple closed geodesic, then it intersects the union of the curves in Ω_2 at least twice, that is, $\mathcal{T}_k(\Omega_2) = 0$, if $k \leq 1$.

Let $\gamma \in C(S_2) \setminus X_2$. Then γ cannot cross only two consecutive sides of P_i, otherwise it would be null homotopic. Hence, it crosses two nonconsecutive sides x, y. Therefore, by Proposition 2.1, we have length$(\gamma) \geq 2d_{\mathbb{H}}(x, y)$, which implies that length$(\gamma) > 2t_1 = 2\cosh^{-1}(1 + \sqrt{2})$. Therefore Lemma 3.3 yields the claim. $\square$

Now we focus on the proof of (4) of Theorem 3.1. Let γ be a simple closed geodesic on S_2 which is not in $\mathrm{SLG}(S_2)$. Then γ intersects two nonconsecutive sides of P_i, which implies that $\sum_{\alpha\in\Omega_2} i(\gamma, \alpha) \geq 2$ and $\sum_{\alpha\in\mathrm{SLG}(S_2)\setminus\Omega_2} i(\gamma, \alpha) \geq 4$. Therefore we have $\sum_{\alpha\in\mathrm{SLG}(S_2)} i(\gamma, \alpha) \geq 6$ and $\mathcal{T}_i(\mathrm{SLG}(S_2)) = 0$, for $i = 0, 1, \ldots, 5$. $\square$

4. Higher genus

In this section, we consider the closed surfaces of genus $g \geq 3$. We prove the following theorem.

THEOREM 4.1. *Let* $g \geq 3$ *be any integer. There exists a closed hyperbolic surface* S_g *of genus* g*, such that:*

(1) $\mathrm{SLG}(S_g)$ *fills* S_g*, and, in particular,* $S_g \in \chi_g$*;*
(2) $|\mathrm{SLG}(S_g)| = 2g$*;*
(3) *the complement of* $\mathrm{SLG}(S_g)$ *in* S_g *is the disjoint union of two right-angled hyperbolic regular polygons;*

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(4) $\mathrm{sys}(S_g) = 2t_{4g}$, *where* t_{4g} *is given in Proposition 2.1; and*
(5) $\mathcal{T}_i(\mathrm{SLG}(S_g)) = 0$, *if* $i < 2$.

The proof of Theorem 4.1, depends on the following essential proposition.

Proposition 4.2. *There exists a filling* Ω_g *of the closed topological surface* Σ_g *of genus* g *such that:*

(1) $|\Omega_g| = 2g$;
(2) $T_k(\Omega_g) = 0$, *if* $k \le 1$; *and*
(3) *the number of connected components in* $\Sigma_g \setminus \Omega_g$ *is two.*

Proof. Consider the decorated 4-regular graph $\Gamma_g = (E, \sim, \sigma_1, \sigma_0)$ given by:

(1) $E = \{a'_i, a''_i, \bar{a}'_i, \bar{a}''_i \mid i = 1, 2, \dots, 2g\}$;
(2) $V = E/\sim = \{v_i \mid i = 0, 1, \dots, 2g-1\}$, where

$$v_i = \begin{cases} \{\bar{a}'_1, a''_{2g}, a''_1, \bar{a}'_{2g}\} & \text{for } i = 0, \\ \{a'_i, a''_{i+1}, \bar{a}''_i, \bar{a}'_{i+1}\} & \text{for } i = 1, \dots, 2g-2, \\ \{a'_{2g-1}, \bar{a}''_{2g}, \bar{a}''_{2g-1}, a'_{2g}\} & \text{for } i = 2g-1; \end{cases}$$

(3) $\sigma_1(a'_i) = \bar{a}_i$ and $\sigma_1(\bar{a}'_i) = a'_i$ and σ_1 is similarly defined on $\{a''_i, \bar{a}''_i\}, i = 1, \dots 2g$;
(4) $\sigma_0 = \prod_{i=0}^{2g-1} \sigma_{v_i}$, where $\sigma_{v_0} = (\bar{a}'_1, a''_{2g}, a''_1, \bar{a}'_{2g})$, $\sigma_{v_{2g-1}} = (a'_{2g-1}, \bar{a}''_{2g}, \bar{a}''_{2g-1}, a'_{2g})$ and $\sigma_{v_i} = (a'_i, a''_{i+1}, \bar{a}''_i, \bar{a}'_{i+1})$, for $2 \le i \le 2g-2$.

Note that, for a formal definition of a fat graph and examples with such descriptions of fat graphs, we refer the reader to [7, Section 2].

The fat graph Γ_g has $2g$ standard cycles and the set of standard cycles is given by

$$\mathrm{SC}(\Gamma_g) = \{a_i = [(a'_i, a''_i)],\ i = 1, \dots, 2g\}.$$

The fat graph has two boundary components δ_1, δ_2 given by

$$\delta_1 = \underbrace{a''_{2g}}_{1}, \underbrace{\bar{a}''_{2g-1}, \dots, \bar{a}''_3, \bar{a}''_2, \bar{a}''_1}_{2g-1}, \underbrace{\bar{a}'_{2g}}_{1}, \underbrace{a'_{2g-1}, \dots, a'_3, a'_2, a'_1}_{2g-1} \quad \text{and} \tag{4.1}$$

$$\delta_2 = \underbrace{\bar{a}'_1, a''_2, \bar{a}'_3, a''_4, \dots, a''_{2g-2}, \bar{a}'_{2g-1}}_{2g-1}, \underbrace{\bar{a}''_{2g}}_{1}, \underbrace{a''_1, \bar{a}'_2, a''_3, \bar{a}'_4, \dots, \bar{a}'_{2g-2}, a''_{2g-1}}_{2g-1}, \underbrace{a'_{2g}}_{1}. \tag{4.2}$$

A simple Euler characteristic argument implies that the genus of Γ_g is g. Let Σ_g be the oriented closed topological surface obtained by attaching a topological disc to each boundary component of the fat graph. The set of standard cycles of Γ_g provides the required filling set Ω_g of Σ_g, where the boundary components δ_1 and δ_2 correspond to the components in the complement of the filling. □

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PROOF OF THEOREM 4.1. Let us consider two regular right-angled hyperbolic $4g$-sided polygons P_g and Q_g equipped with a side pairing given by the boundary words

$$\omega(\delta_1) = \underbrace{a''_{2g}}_{1}\ \underbrace{\bar{a}''_{2g-1}\ldots\bar{a}''_3\bar{a}''_2\bar{a}''_1}_{2g-1}\ \underbrace{\bar{a}'_{2g}}_{1}\ \underbrace{a'_{2g-1}\ldots a'_3a'_2a'_1}_{2g-1} \quad \text{and}$$

$$\omega(\delta_2) = \underbrace{\bar{a}'_1a''_2\bar{a}'_3a''_4\ldots a''_{2g-2}\bar{a}'_{2g-1}}_{2g-1}\ \underbrace{\bar{a}''_{2g}}_{1}\ \underbrace{a''_1\bar{a}'_2a''_3\bar{a}'_4\ldots\bar{a}'_{2g-2}a''_{2g-1}}_{2g-1}\ \underbrace{a'_{2g}}_{1}$$

of the polygons P_g and Q_g, respectively, which are the same as the boundaries given in equations (4.1) and (4.2) in the proof of Proposition 4.2. We obtain the closed hyperbolic surface S_g of genus g by gluing the side pairing of the polygons P_g and Q_g using hyperbolic isometries. The sides of P_g, Q_g labelled by $a'_i, a''_i, \bar{a}'_i, \bar{a}''_i$ project to simple closed geodesics a_i, $i = 1, 2, \ldots, 2g$, on S_g. The length of a_i is twice the length of a side of P_g, which is equal to $2t_{4g}$ by Proposition 2.1. We define $\Omega_g = \{a_i \mid i = 1, 2, \ldots, 2g\}$. Now we claim that $\mathrm{SLG}(S_g) = \Omega_g$. The rest of the proof follows from the next lemma. □

LEMMA 4.3. *Let γ be an essential simple closed geodesic on S_g with the property that $\gamma \notin \{a_i \mid i = 1, 2, \ldots, 2g\}$. Then*

$$\mathrm{length}(\gamma) > 2t_{4g}.$$

PROOF. It is easy to see in the boundary words $\omega(\delta_1)$ and $\omega(\delta_2)$ that each pair of edges with identical labelling is in a different polygon, which implies that $T_k(\Omega_g) = 0$ for $k \leq 1$. Therefore γ intersects the union of curves in Ω_g at least twice.

If γ is the projection of a geodesic arc joining two opposite vertices in the polygons, then

$$\begin{aligned}\mathrm{length}(\gamma) &= 2\cosh^{-1}\Big(\frac{1+\cos(\pi/2g)}{\sin(\pi/2g)}\Big)\\ &> 2\cosh^{-1}\Big(2+2\cos\frac{\pi}{2g}\Big) \quad (\text{since } g \geq 3)\\ &> 2\cosh^{-1}\Big(1+2\cos\frac{\pi}{2g}\Big) = 2t_g.\end{aligned}$$

If γ intersects only two consecutive sides of the polygons, then it will be homotopically trivial. In the remaining cases, γ intersects two nonconsecutive sides x, y, say. Let $n(x, y)$ be the minimum number of sides of the polygon between x and y. We choose x and y so that $n(x, y)$ is maximum. For such a choice, we have $n(x, y) > 1$; otherwise γ will be one of the curves in Ω_g. Therefore, by Proposition 2.1,

$$\mathrm{length}(\gamma) \geq 2d_{\mathbb{H}}(x, y) > 2t_g. \qquad □$$

REMARK 4.4. Let $P_g(\epsilon)$ and $Q_g(\epsilon)$ be two hyperbolic $4g$-gons, $g \geq 3$, with alternative angles $\pi/2 + \epsilon$ and $\pi/2 - \epsilon$ and side length $t_g(\epsilon)$, where

$$t_g(\epsilon) = \cosh^{-1}\Big(\frac{\cos\epsilon + 2\cos(\pi/2g)}{\cos\epsilon}\Big).$$

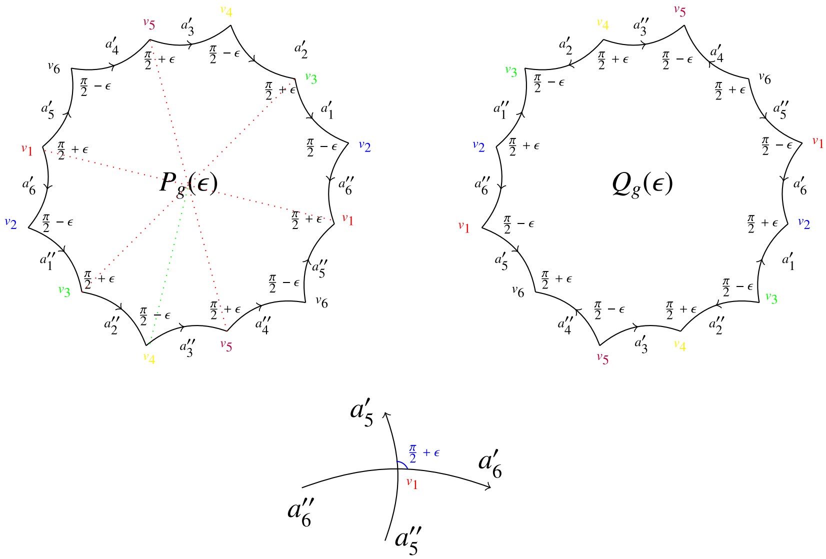

FIGURE 4. The polygons $P_g(\epsilon)$, $Q_g(\epsilon)$ and a local picture at the vertex v_1 on $S_g(\epsilon)$.

Such polygons can be obtained by attaching together $4g$ copies of hyperbolic triangles with interior angles $\pi/4+\epsilon/2, \pi/4-\epsilon/2$ and $\pi/2g$ in an appropriate way. Note that it is straightforward to see that $t_g(\epsilon)$ is a monotonically increasing function in ϵ. We consider the side pairing given by the boundary words

$$\omega(\delta_1) = \underbrace{a_{2g}''}_{1}\ \underbrace{\bar{a}_{2g-1}''\ldots\bar{a}_3''\bar{a}_2''\bar{a}_1''}_{2g-1}\ \underbrace{\bar{a}_{2g}'}_{1}\ \underbrace{a_{2g-1}'\ldots a_3'a_2'a_1'}_{2g-1} \quad \text{and}$$

$$\omega(\delta_2) = \underbrace{\bar{a}_1'a_2''\bar{a}_3'a_4''\ldots a_{2g-2}''\bar{a}_{2g-1}'}_{2g-1}\ \underbrace{\bar{a}_{2g}''}_{1}\ \underbrace{a_1''\bar{a}_2'a_3''\bar{a}_4'\ldots\bar{a}_{2g-2}'a_{2g-1}''}_{2g-1}\ \underbrace{a_{2g}'}_{1},$$

as in the proof of Theorem 4.1 (see Figure 4 for the case when $g=3$).

We denote the surface provided by the configuration above by $S_g(\epsilon)$, and we denote the set of simple closed geodesics which are the projections of the boundary sides of these polygons by $\Omega_g(\epsilon)=\{a_i(\epsilon)\mid i=1,2,\ldots,2g\}$. By arguments similar to those in the proof of Theorem 4.1, $\mathrm{SLG}(S_g(\epsilon))=\Omega_g(\epsilon)$ for $0\le\epsilon<\pi(g-2)/2g$, and $\mathrm{sys}(S_g(\epsilon)) = 2t_g(\epsilon)$. Thus we have a continuous family of surfaces $\{S_g(\epsilon)\mid 0\le\epsilon<\pi(g-2)/2g\}$ in the Thurston set χ_g with the property

$$\mathrm{sys}(S_g(t_1)) < \mathrm{sys}(S_g(t_2)) \quad \text{when } 0\le t_1<t_2<\frac{\pi(g-2)}{2g},$$

and hence a deformation in the Thurston set which increases the systolic length.

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