Canad. Math. Bull. Vol. **51** (3), 2008 pp. 439–447

On the Maximal Spectrum of Semiprimitive Multiplication Modules

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Abstract. An R-module M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. As defined for a commutative ring R, an R-module M is said to be semiprimitive if the intersection of maximal submodules of M is zero. The maximal spectra of a semiprimitive multiplication module M are studied. The isolated points of $\operatorname{Max}(M)$ are characterized algebraically. The relationships among the maximal spectra of M, $\operatorname{Soc}(M)$ and $\operatorname{Ass}(M)$ are studied. It is shown that $\operatorname{Soc}(M)$ is exactly the set of all elements of M which belongs to every maximal submodule of M except for a finite number. If $\operatorname{Max}(M)$ is infinite, $\operatorname{Max}(M)$ is a one-point compactification of a discrete space if and only if M is Gelfand and for some maximal submodule K, $\operatorname{Soc}(M)$ is the intersection of all prime submodules of M contained in K. When M is a semiprimitive Gelfand module, we prove that every intersection of essential submodules of M is an essential submodule if and only if $\operatorname{Max}(M)$ is an almost discrete space. The set of uniform submodules of M and the set of minimal submodules of M coincide. $\operatorname{Ann}(\operatorname{Soc}(M))M$ is a summand submodule of M if and only if $\operatorname{Max}(M)$ is the union of two disjoint open subspaces A and N, where A is almost discrete and N is dense in itself. In particular, $\operatorname{Ann}(\operatorname{Soc}(M)) = \operatorname{Ann}(M)$ if and only if $\operatorname{Max}(M)$ is almost discrete.

1 Introduction

Several authors have studied topological properties of the maximal spectrum (with Zariski topology) of commutative rings [3, 5, 9]. Specifically, when the Jacobson radical and the nilradical of a ring R coincide, the compactness $\operatorname{Max}(R)$ is equivalent to the normality of $\operatorname{Spec}(R)$. In this position, R is said to be a Gelfand ring. De Marco and Orsatti also gave a algebraic characterization for a semiprimitive Gelfand ring R; in fact, they showed that R is Gelfand if and only if each prime ideal is contained in a unique maximal ideal [3]. The class of regular rings, local rings, zero-dimension rings, rings of continuous function are all examples of Gelfand rings. On the other hand, the socle of a semiprimitive ring which has algebraic properties, is characterized by the isolated points of $\operatorname{Max}(R)$ [9]. Therefore the socle of R can be a good vehicle for studying the relationships among topological properties of $\operatorname{Max}(R)$ and algebraic properties of ring R. One of the purposes of this paper is the generalization of some of the above concepts and to study relationships among topological properties of $\operatorname{Max}(M)$ and the socle of M, when M is a multiplication module.

In this paper all rings are commutative with identity and all modules are unitary. An R-module M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. Multiplication modules and ideals have been investigated by [1, 4, 7, 8, 11, 12] and others. A proper submodule P of M is called prime if

Received by the editors March 3, 2006; revised September 6, 2006.
This work has been supported in part by a grant from IPM (No. 86130030).
AMS subject classification: 13C13.
Keywords: multiplication module, semiprimitive module, Gelfand module, Zariski topology.

$rx \in P$, for $r \in R$ and $x \in M$, implies $r \in (P : M)$ or $x \in P$. In this case, $\mathfrak{p} = (P : M)$ is a prime ideal and we say P is a $\mathfrak{p}$-prime submodule of M. We use $\operatorname{Spec}(M)$ for the spectrum of prime submodules of M. For any submodule N of an R-module M, we define $\mathrm{V}(N)$ to be the set of all prime submodules of M containing N, and $\operatorname{rad} N = \bigcap \mathrm{V}(N)$. Of course, $\mathrm{V}(M)$ is just the empty set and $\mathrm{V}(0)$ is $\operatorname{Spec}(M)$. Note that for any family of submodules N_λ $(\lambda \in \Lambda)$ of M, $\bigcap_{\lambda\in\Lambda} \mathrm{V}(N_\lambda) = \mathrm{V}(\sum_{\lambda\in\Lambda} N_\lambda)$. Thus if $\zeta(M)$ denotes the collection of all subsets $\mathrm{V}(N)$ of $\operatorname{Spec}(M)$, then $\zeta(M)$ contains the empty set and $\operatorname{Spec}(M)$, and $\zeta(M)$ is closed under arbitrary intersection. We shall say that M is a module with a Zariski topology, or a top module for short, if $\zeta(M)$ is closed under finite unions, *i.e.,* for any submodules N and N' of M, there exists a submodule N'' of M such that $\mathrm{V}(N) \cup \mathrm{V}(N') = \mathrm{V}(N'')$, for in this case $\zeta(M)$ satisfies the axioms for the closed subsets of a topological space. It is well known that every multiplication module is a top module, and the converse holds if the module is finitely generated [8].

Throughout this paper, M is a non-zero finitely generated multiplication R-module. We write $\operatorname{Max}(M)$ and $\operatorname{Min}(M)$ for the spectrum of maximal submodules and minimal prime submodules of M, respectively. For any subset X of M, we define

$$\mathrm{V}_M(X) = \mathrm{V}(X) \cap \operatorname{Max}(M) \quad \text{and} \quad \mathrm{V}'(X) = \mathrm{V}(X) \cap \operatorname{Min}(M),$$

$$\mathrm{D}_M(X) = \operatorname{Max}(M) \setminus \mathrm{V}_M(X) \quad \text{and} \quad \mathrm{D}'(X) = \operatorname{Min}(M) \setminus \mathrm{V}'(X).$$

Therefore we consider $\operatorname{Max}(M)$ and $\operatorname{Min}(M)$ as subspaces of $\operatorname{Spec}(M)$. The operators cl and int denote the closure and the interior in $\operatorname{Max}(M)$.

Let x be an element of R-module M. The set $\{r \in R : rx = 0\}$ is an ideal of R, which we write $\operatorname{Ann}(x)$. This ideal is called the *annihilator* of x. A prime ideal $\mathfrak{p}$ of R is called an *associated prime ideal* of M if $\mathfrak{p}$ is the annihilator $\operatorname{Ann}(x)$ of some $x \in M$. The set of associated primes of M is written $\operatorname{Ass}(M)$.

An R-module M is said to be *semiprimitive (reduced)* if the intersection of all maximal (prime) submodules of M is equal to zero. Reduced multiplication modules are studied in [10]. By Lemma 2.1 and [4, Theorem 2.12], it is easy to see that M is semiprimitive (reduced) if and only if $\operatorname{Ann}(M)$ is an intersection of maximal (prime) ideals of R, and if and only if $R/\operatorname{Ann}(M)$ is a semiprimitive (reduced) ring. For example, every faithful multiplication module over a semiprimitive (reduced) ring is a semiprimitive (reduced) module. In particular, every semiprimitive (reduced) ring is a semiprimitive (reduced) module.

A non-zero submodule in a module M is said to be *essential* if it intersects every non-zero submodule non-trivially. The intersection of all essential submodules, or the sum of all minimal submodules, is called the *socle*, and is denoted by $\operatorname{Soc}(M)$. An element $e \in R$ is called an *M-idempotent* in R if $e^2 \equiv e \pmod{\operatorname{Ann}(M)}$.

A space X is said to be *almost discrete* if the set of isolated points of X is dense in X. For example, the one-point compactification and Stone–Cech compactification of a discrete space are almost discrete spaces. We also say that X is *dense in itself* if it has no isolated point [2]. We show that $\operatorname{Ann}(\operatorname{Soc}(M))M$ is a summand submodule of M if and only if $\operatorname{Max}(M)$ is the union of two disjoint open subspaces A and N, where A is almost discrete and N is dense in itself. In particular, $\operatorname{Ann}(\operatorname{Soc}(M)) = \operatorname{Ann}(M)$ if and only if $\operatorname{Max}(M)$ is almost discrete.

2 Isolated Maximal Submodules

In this section we obtain some results about the isolated points of submodule spaces. We denote by $\mathrm{Spec}_0(M)$, $\mathrm{Max}_0(M)$, and $\mathrm{Min}_0(M)$ the sets of isolated points of the spaces $\mathrm{Spec}(M)$, $\mathrm{Max}(M)$, and $\mathrm{Min}(M)$, respectively.

First we need the following lemmas.

Lemma 2.1 *Let P be a proper submodule of M. The following statements are equivalent:*

(i) *P is prime.*
(ii) *$(P : M)$ is a prime ideal of R.*
(iii) *$P = \mathfrak{p}M$ for some prime ideal $\mathfrak{p}$ of R with* $\mathrm{Ann}(M) \subseteq \mathfrak{p}$.

Proof See [4, Corollary 2.11]. ■

Lemma 2.2 *Let I be an ideal of R and let N be a submodule of M. Then*

$$\mathrm{V}(N) \cup \mathrm{V}(IM) = \mathrm{V}(IN) = \mathrm{V}(N \cap IM).$$

Proof See [8, Lemma 3.1]. ■

Lemma 2.3 *Let M be reduced, let N a submodule of M, and $I =$* $\mathrm{Ann}(N)$.

(i) $N \cap IM = 0$.
(ii) $\mathrm{Ann}(N + IM) = \mathrm{Ann}(M)$.

Proof (i) By Lemma 2.2, $\mathrm{V}(N \cap IM) = \mathrm{V}(IN) = \mathrm{V}(0) = \mathrm{Spec}(M)$. Therefore $N \cap IM = 0$.

(ii) Suppose that $r \in \mathrm{Ann}(N + IM)$. Since $rN = 0$, then $r \in I$. Therefore $r^2 \in rI \subseteq \mathrm{Ann}(M)$, and this implies that $r \in \mathrm{Ann}(M)$, since M is reduced. ■

Lemma 2.4 *Let M be reduced and let N be a summand submodule of M. Then there exists an M-idempotent $e \in R$ such that $N = eM$.*

Proof Suppose $M = N \oplus N'$. So there are ideals I and I' such that $N = IM$ and $N' = I'M$. Hence $M = (I + I')M$ implies that $(e + e' - 1)M = 0$, for some $e \in I$ and $e' \in I'$. Then $(e^2 - e)M = ee'M \in N \cap N' = 0$, *i.e.*, $e^2 \equiv e \pmod{\mathrm{Ann}(M)}$. Now for any $x \in N$ we have $x - ex = e'x \in N \cap N' = 0$. This implies that $N = eM$. ■

Lemma 2.5 *Let M be reduced. Then A is a clopen (closed and open) subset of* $\mathrm{Spec}(M)$ *if and only if there exists an M-idempotent $e \in R$ such that $A =$* $\mathrm{V}(eM)$.

Proof Suppose that A is a clopen subset of $\mathrm{Spec}(M)$ and $N = \bigcap A$ and $N' = \bigcap A^c$. Then $A = \mathrm{cl}\, A = \mathrm{V}(\cap A) = \mathrm{V}(N)$ and $A^c = \mathrm{V}(N')$ and $\mathrm{V}(N) \cap \mathrm{V}(N') = \varnothing$. Hence $M = N \oplus N'$, and by Lemma 2.4, there exists an M-idempotent $e \in R$ such that $N = eM$. The converse is trivial. ■

Theorem 2.6 *Let M be semiprimitive and let K be a maximal submodule of M. Then $K = eM$, for some M-idempotent $e \in R$ if and only if $K \in$* $\mathrm{Max}_0(M)$. *Furthermore, in this case, if $K = eM \neq 0$, then $N = (1 - e)M$ is a non-zero minimal submodule of M.*

https://doi.org/10.4153/CMB-2008-044-8 Published online by Cambridge University Press

Proof Suppose that $K = eM$, where $e \in R$ is an M-idempotent. Therefore $e^2 - e \in \operatorname{Ann}(M)$ implies that $\{K\} = \mathrm{D}_M((1-e)M)$. Conversely, suppose $\{K\}$ is an open set in $\operatorname{Max}(M)$. By Lemma 2.5, there exists an M-idempotent $e \in R$ such that $\{K\} = \mathrm{V}_M(eM)$. Now by Lemma 2.2, we have

$$\mathrm{V}_M((1-e)K) = \mathrm{V}_M(K) \cup \mathrm{V}_M((1-e)M) = \mathrm{V}_M(eM) \cup \mathrm{V}_M((1-e)M) = \operatorname{Max}(M).$$

This shows that $K = eM$. For the second part, suppose $x \in N$ is a non-zero arbitrary element. Then $Rx + eM = M$. Thus $R(1-e)x = N$, and this implies that $N = Rx$, *i.e.*, N is a minimal submodule of M. ■

Corollary 2.7 *Let M be semiprimitive and let N be a submodule of M. Then N is a non-zero minimal submodule of M if and only if N is contained in every maximal submodule of M except one,* i.e., $|\mathrm{D}_M(N)| = 1$.

Corollary 2.8 *Let M be semiprimitive. Then* $\operatorname{Soc}(M)$ *is finitely generated if and only if the number of isolated maximal submodules of M is finite. In particular, if M is noetherian,* $\operatorname{Max}_0(M)$ *is finite.*

Proposition 2.9 *Let M be semiprimitive. The following statements are equivalent.*

(i) *Every intersection of essential submodules of M is an essential submodule.*
(ii) $\operatorname{Max}_0(M)$ *is dense in* $\operatorname{Max}(M)$.

Proof (i) $\Rightarrow$ (ii). By hypothesis, $\operatorname{Soc}(M)$ is essential, so Lemma 2.3 implies that $\operatorname{Ann}(\operatorname{Soc}(M))M = 0$. Suppose $x \in \bigcap \operatorname{Max}_0(M)$. Then $Rx = IM$ for some ideal I of R. By Lemma 2.2 and Corollary 2.7, for any minimal submodule N of M,

$$\mathrm{V}_M(IN) = \mathrm{V}_M(N) \cup \mathrm{V}_M(IM) = \mathrm{V}_M(N) \cup \mathrm{V}_M(x) = \operatorname{Max}(M).$$

Therefore, $IN = 0$, and this implies that $I \subseteq \operatorname{Ann}(\operatorname{Soc}(M))$. Consequently, $Rx \subseteq \operatorname{Ann}(\operatorname{Soc}(M))M$, *i.e.*, $x = 0$.

(ii) $\Rightarrow$ (i). By Corollary 2.7, $\operatorname{Soc}(M) = \bigoplus_{e \in E} eM$, where E is a set of M-idempotents in R. Thus we have

$$\operatorname{Ann}(\operatorname{Soc}(M)) = \bigcap_{e \in E} \operatorname{Ann}(eM) = \bigcap_{e \in E} [R(1-e) + \operatorname{Ann}(M)].$$

So by [4, Corollary 1.7], $\operatorname{Ann}(\operatorname{Soc}(M))M = \bigcap_{e \in E}(1-e)M = \bigcap \operatorname{Max}_0(M) = 0$.

To contrast, suppose that $\operatorname{Soc}(M)$ is not essential. Then there exists a non-zero submodule $N = IM$ of M such that $N \cap \operatorname{Soc}(M) = 0$. Therefore by Lemma 2.2,

$$\operatorname{Max}(M) = \mathrm{V}_M(N \cap \operatorname{Soc}(M)) = \mathrm{V}_M(I \operatorname{Soc}(M)).$$

This means that $I \subseteq \operatorname{Ann}(\operatorname{Soc}(M)) \subseteq \operatorname{Ann}(M)$, hence $N = 0$, a contradiction. Thus $\operatorname{Soc}(M)$ is essential. ■

Theorem 2.10 *Let M be reduced.*

(i) $\mathrm{Min}_0(M) = \{\mathfrak{p}M : \mathfrak{p} \in \mathrm{Ass}(M)\}$.

(ii) $P \in \mathrm{Spec}_0(M)$ *if and only if* $P \in \mathrm{Min}_0(M)$ *and P is not semiprime.*

In particular, if M is semiprimitive,

(iii) $\mathrm{Spec}_0(M) = \mathrm{Max}_0(M)$.

Proof (i) Suppose $P \in \mathrm{Min}_0(M)$. Then there exists $x \in \bigcap \mathrm{D}'(P) \setminus P$. Hence $\mathrm{Ann}(x) = (P : M)$, and this implies that $P = \mathrm{Ann}(x)M$. Conversely, suppose $\mathfrak{p} \in \mathrm{Ass}(M)$. Then $\mathfrak{p} = \mathrm{Ann}(x)$, for some $x \in M$. Therefore there exists $P \in \mathrm{Min}(M)$ such that $x \notin P$. But $\mathfrak{p}x = 0$ implies that $\mathfrak{p} \subseteq (P : M)$. Hence by Lemma 2.1, $P = \mathfrak{p}M$. We note that $\mathrm{D}'(x) = \{P\}$, *i.e.*, $P \in \mathrm{Min}_0(M)$.

(ii) Suppose $P \in \mathrm{Min}_0(M)$ and $P \neq \bigcap \mathrm{V}(P) = \mathrm{rad}\, P$. Hence there are $x \in \bigcap D'(P) \setminus P$ and $y \in \mathrm{rad}\, P \setminus P$. Set $I = (x : M)$ and $J = (y : M)$. It is easy to see that $\mathrm{D}(IJM) = \{P\}$, *i.e.*, $P \in \mathrm{Spec}_0(R)$. The opposite inclusion is trivial.

(iii) follows from Theorem 2.6. ∎

Definition 2.11 A multiplication R-module M is said to be Gelfand if $\mathrm{Max}(M)$ is a Hausdorff space.

It is well known that a semiprimitive multiplication module M is Gelfand if and only if every prime submodule of M is contained in a unique maximal submodule, and if and only if $\mathrm{Spec}(M)$ is normal [12].

The following lemma is given in [10].

Lemma 2.12 *For any subset X of M,*

(i) $\mathrm{Ann}(X)M = \bigcap \mathrm{D}(X)$;

(ii) $\mathrm{int}\, \mathrm{V}(X) = \mathrm{D}(\mathrm{Ann}(X)M)$.

Proof (i) Suppose that $P \in \mathrm{D}(X)$. Then $\mathrm{Ann}(X) \subseteq (P : M)$. This implies that $\mathrm{Ann}(X)M \subseteq P$, *i.e.*, $\mathrm{Ann}(X)M \subseteq \bigcap \mathrm{D}(X)$. Conversely, If $y \in \bigcap \mathrm{D}(X)$, then $Ry = IM$, for some ideal I of R, and Lemma 2.2 implies that

$$\mathrm{Spec}(M) = \mathrm{V}(Ry) \cup \mathrm{V}(X) = \mathrm{V}(IM) \cup \mathrm{V}(\langle X \rangle) = \mathrm{V}(I\langle X \rangle).$$

Hence $I \subseteq \mathrm{Ann}(X)$, *i.e.*, $y \in \mathrm{Ann}(X)M$.

(ii) This follows from (i)

$$\mathrm{int}\, \mathrm{V}(X) = \mathrm{Spec}(M) - \mathrm{cl}\ \mathrm{D}(X) = \mathrm{D}(\cap \mathrm{D}(X)) = \mathrm{D}(\mathrm{Ann}(X)M).$$ ∎

Definition 2.13 Let P be a $\mathfrak{p}$-prime submodule of M. We define

$$O_P = \{x \in M : \mathrm{Ann}(x) \nsubseteq \mathfrak{p}\}.$$

Remark 2.14. It is easy to see that $O_P \subseteq P$. By Lemma 2.12, $\mathrm{D}(\mathrm{Ann}(x)M) = \mathrm{int}\ V(x)$, then we have $O_P = \{x \in M : P \in \mathrm{int}\ V(x)\} = \bigcap\{P' \in \mathrm{Spec}(M) : P' \subseteq P\}$.

https://doi.org/10.4153/CMB-2008-044-8 Published online by Cambridge University Press

Theorem 2.15 *Let M be semiprimitive and Gelfand.*

$$\mathrm{Spec}_0(M) = \mathrm{Max}_0(M) = \mathrm{Min}_0(M) = \{\mathfrak{p}M : \mathfrak{p} \in \mathrm{Ass}(M)\}.$$

Proof By Theorem 2.10, it is sufficient to prove $\mathrm{Min}_0(M) \subseteq \mathrm{Max}_0(M)$. Let $P \in \mathrm{Min}_0(M)$. By hypothesis, $P \subseteq K$, for a unique maximal submodule $K \in \mathrm{Max}(M)$. Therefore $\bigcap_{K' \in \mathrm{D}_M(K)} O_{K'} \not\subseteq P$. This means that there exists $0 \neq x \in \bigcap \mathrm{D}_M(K)$. Observe that $x \notin K$, and this implies that K is an isolated point of $\mathrm{Max}(M)$. ■

Theorem 2.16 *Let M be semiprimitive and Gelfand. Then*

$$\mathrm{Ass}(M) = \{\mathfrak{p} \in \mathrm{Max}(R) : \mathfrak{p} = Re + \mathrm{Ann}(M), \textit{ where } e \textit{ is an } M\textit{-idempotent in } R\}.$$

In particular, every prime submodule of M is either an essential submodule or an isolated maximal submodule.

Proof Let $\mathfrak{p} \in \mathrm{Ass}(M)$. Then by Theorem 2.15, $\mathfrak{p}M \in \mathrm{Max}_0(M)$. Hence Theorem 2.6 implies that $\mathfrak{p}M = eM$, for some M-idempotent $e \in R$. Inasmuch as $\mathrm{Ann}(M) \subseteq \mathfrak{p}$, then $Re + \mathrm{Ann}(M) \subseteq \mathfrak{p}$. Also for any $r \in \mathfrak{p}$, $r(1-e)M = 0$. Hence

$$r = re + r(1-e) \in Re + \mathrm{Ann}(M),$$

i.e., $\mathfrak{p} = Re + \mathrm{Ann}(M)$. Conversely, suppose $\mathfrak{p} \in \mathrm{Max}(R)$ and $\mathfrak{p} = Re + \mathrm{Ann}(M)$ for some M-idempotent $e \in R$. Since $(1-e)M \neq 0$, then there exists $x \in M$ such that $(1-e)x \neq 0$. Evidently, $\mathfrak{p} = \mathrm{Ann}((1-e)x) \in \mathrm{Ass}(M)$.

For the second part, suppose P is a non-essential prime submodule. There exists a minimal prime submodule P' contained in P. Since P' is non-essential, $P' \cap N = 0$ for some non-zero submodule N of M. Therefore $V'(N) = \mathrm{Min}(M) \setminus \{P'\}$, *i.e.*, $P' \in \mathrm{Min}_0(M)$. Now Theorem 2.15 implies that $P = P' \in \mathrm{Max}_0(M)$. ■

The following result shows that in a semiprimitive Gelfand module, the set of uniform submodules and the set of minimal submodules coincide.

Proposition 2.17 *Let M be semiprimitive and Gelfand and let N be a submodule of M. Then N is a uniform submodule if and only if N is a minimal submodule.*

Proof Suppose N is a uniform submodule of M. By Corollary 2.7, it is sufficient to show that $|\mathrm{D}_M(N)| = 1$. In contrast, let K', K'' be two distinct elements in $\mathrm{D}_M(N)$. Since $\mathrm{Max}(M)$ is Hausdorff, there are $x', x'' \in M$ such that

$$K' \in \mathrm{D}_M(x') \subseteq \mathrm{D}_M(Rx' \cap N), \quad K'' \in \mathrm{D}_M(x'') \subseteq \mathrm{D}_M(Rx'' \cap N),$$

and $\mathrm{D}_M(x') \cap \mathrm{D}_M(x'') = \varnothing$. Thus $Rx' \cap N \neq 0$ and $Rx'' \cap N \neq 0$. Now we have

$$\mathrm{V}_M((Rx' \cap N) \cap (Rx'' \cap N)) \supseteq \mathrm{V}_M(Rx' \cap Rx'') = \mathrm{V}_M(x') \cup \mathrm{V}_M(x'') = \mathrm{Max}(M).$$

This shows that $(Rx' \cap N) \cap (Rx'' \cap N) = 0$. But N is uniform, a contradiction. The converse is trivial. ■

https://doi.org/10.4153/CMB-2008-044-8 Published online by Cambridge University Press

3 The Socle of M

In this section we obtain some results about the relationships among the algebraic properties of $\mathrm{Soc}(M)$ and the topological properties of $\mathrm{Max}(M)$.

Theorem 3.1 *Let M be semiprimitive. Then the socle $\mathrm{Soc}(M)$ is exactly the set of all elements which belong to every maximal submodule of M except for a finite number. In fact, $\mathrm{Soc}(M) = \{x \in M : \mathrm{D}_M(x)$ is finite$\}$.*

Proof Suppose $x \in \mathrm{Soc}(M)$. Then $x = x_1 + x_2 + \cdots + x_n$, where each x_i belongs to some minimal submodule in M. Thus by Corollary 2.7, $x_1 + x_2 + \cdots + x_n$ belongs to every maximal submodule except for a finite number. This implies that $\mathrm{D}_M(x)$ is finite. Conversely, let $\mathrm{D}_M(x)$ be a finite set. Then $\mathrm{D}_M(x) = \{K_1, K_2, \ldots, K_n\}$. Inasmuch as $\mathrm{Max}(M)$ is a T_1-space, for each $1 \leq i \leq n$, K_i is an isolated point of $\mathrm{Max}(M)$. Now by Theorem 2.6, for each K_i, there exists a minimal submodule N_i such that $M = K_i \oplus N_i$ and $N_i = e_iM$, where e_i is an M-idempotent element of R. Set $y = x - (e_1x + e_2x + \cdots + e_nx)$. Inasmuch as for any $i \neq j$, $e_ie_j \in \mathrm{Ann}(M)$, then $e_iy = 0$, for any $1 \leq i \leq n$. Thus we have

$$\mathrm{Max}(M) = \mathrm{V}_M(x) \cup \mathrm{D}_M(x) = \mathrm{V}_M(x) \cup \{K_1, K_2, \ldots, K_n\} \subseteq \mathrm{V}_M(y).$$

This means that $x = e_1x + e_2x + \cdots + e_nx \in N_1 + N_2 \cdots + N_n \subseteq \mathrm{Soc}(M)$. ∎

Lemma 3.2 *Let M be semiprimitive and Gelfand. If A and B are disjoint closed subsets of $\mathrm{Max}(M)$, then there exists $a \in R$ such that*

$$A \subseteq \operatorname{int} \mathrm{V}_M(aM), \quad B \subseteq \operatorname{int} \mathrm{V}_M((a-1)M).$$

Proof By our hypothesis, the space $\mathrm{Max}(M)$ is Hausdorff and compact. Therefore by [5, Theorem 1.15], there are closed sets E and F in $\mathrm{Max}(M)$ such that

$$A \subseteq \operatorname{int} E \subseteq E, \quad B \subseteq \operatorname{int} F \subseteq F, \quad E \cap F = \varnothing.$$

Hence there are the submodules N and N' such that $E = \mathrm{V}_M(N)$ and $F = V_M(N')$. There are the ideals I and I' such that $N = IM$ and $N' = I'M$. Inasmuch as $M = N + N'$, then $M = (I + I')M$, and this implies that $(a + a' - 1)M = 0$, for some $a \in I$ and $a' \in I'$. Thus we have

$$A \subseteq \operatorname{int} \mathrm{V}_M(N) \subseteq \operatorname{int} \mathrm{V}_M(aM) \quad \text{and} \quad B \subseteq \operatorname{int} \mathrm{V}_M(N') \subseteq \operatorname{int} \mathrm{V}_M((a-1)M).$$

∎

For any subset A of $\mathrm{Spec}(M)$, we define $O_A = \bigcap_{P \in A} O_P$.

Theorem 3.3 *Let M be semiprimitive and Gelfand and let A be a closed subset of $\mathrm{Max}(M)$. Then $O_A \subseteq \mathrm{Soc}(M)$ if and only if every open subset of $\mathrm{Max}(M)$ containing A has a finite complement.*

Proof Suppose $O_A \subseteq \operatorname{Soc}(M)$ and G is an open set of $\operatorname{Max}(M)$ containing A. If $K \in \operatorname{Max}(M) \setminus G$, then by Lemma 3.2, there is $a \in R$ such that $A \subseteq \operatorname{int} \mathrm{V}_M(aM)$ and $K \in \operatorname{int} \mathrm{V}_M((a-1)M)$. Thus $aM \subseteq O_A \subseteq \operatorname{Soc}(M)$. Inasmuch as aM is finitely generated, Theorem 3.1 implies that $\mathrm{D}_M(aM)$ is finite. Now if K is not an isolated maximal submodule, then the open set $\mathrm{D}_M(aM)$ which contains K must be infinite, a contradiction. Therefore $\operatorname{Max}(M) \setminus G$ is a clopen subset of $\operatorname{Max}(M)$, so by Lemma 2.5, there exists an M-idempotent $e \in R$ such that $G = \mathrm{V}_M(eM)$. Hence $eM \subseteq O_A \subseteq \operatorname{Soc}(M)$, and Theorem 3.1 implies that $\operatorname{Max}(M) \setminus G = \mathrm{D}_M(eM)$ is finite. Conversely, let every open subset of $\operatorname{Max}(M)$ containing A have a finite complement and $x \in O_A$. Then $A \subseteq \operatorname{int} \mathrm{V}_M(x)$, so $\operatorname{Max}(M) \setminus \operatorname{int} \mathrm{V}_M(x)$ is finite by our hypothesis and hence $\mathrm{D}_M(x)$ is also finite. Consequently, Theorem 3.1 implies that $x \in \operatorname{Soc}(M)$, *i.e.*, $O_A \subseteq \operatorname{Soc}(M)$. ■

Theorem 3.4 *Let M be semiprimitive and let* $\operatorname{Max}(M)$ *be infinite. Then* $\operatorname{Max}(M)$ *is the one-point compactification of a discrete space if and only if M is Gelfand and for some maximal submodule K,* $\operatorname{Soc}(M)$ *is the intersection of all prime submodules contained in K, (or equivalently,* $\operatorname{Soc}(M) = O_K$*).*

Proof Suppose M is Gelfand and for some maximal submodule K, $\operatorname{Soc}(M) = O_K$. Therefore $\operatorname{Max}(M)$ is a Hausdorff space and K cannot be an isolated point of $\operatorname{Max}(M)$, for otherwise by Theorem 2.6, there is an M-idempotent $e \in R$ such that $K = eM$. Hence $K \in \operatorname{int} \mathrm{V}_M(eM)$, so $eM \subseteq O_K \subseteq \operatorname{Soc}(M)$ and this implies that $\operatorname{Max}(M) \setminus \{K\} = \mathrm{D}_M(eM)$ is finite, a contradiction. Now we will show that K is the only non-isolated point of $\operatorname{Max}(M)$. Suppose that $K' \neq K$ is another non-isolated point of $\operatorname{Max}(M)$. By Lemma 3.2, there is $a \in R$ such that $K \in \operatorname{int} \mathrm{V}_M(aM)$ and $K' \in \operatorname{int} \mathrm{V}_M((a-1)M)$. Thus $aM \subseteq O_K \subseteq \operatorname{Soc}(M)$. Inasmuch as $\operatorname{Max}(M)$ is Hausdorff and $\mathrm{D}_M(aM)$ is a neighborhood of the non-isolated point K', then $\mathrm{D}_M(aM)$ is an infinite set which implies that $aM \not\subseteq \operatorname{Soc}(M)$, a contradiction. Now let G be an open set which contains K. By Theorem 3.3, $\operatorname{Max}(M) \setminus G$ is compact (finite); this means that $\operatorname{Max}(M)$ is the one-point compactification of the space $\operatorname{Max}_0(M)$.

Conversely, let $\operatorname{Max}(M) = Y \cup \{K\}$ be the one-point compactification of a discrete space Y. Obviously, $\operatorname{Max}(M)$ is a Hausdorff space, *i.e.*, M is Gelfand. Hence it is sufficient to show that $\operatorname{Soc}(M) = O_K$. If $x \in O_K$, then $\operatorname{int} \mathrm{V}_M(x)$ is an open set containing K, so $\operatorname{Max}(M) \setminus \operatorname{int} \mathrm{V}_M(x) \subseteq Y$ is compact. Hence $\mathrm{D}_M(x)$ is finite, *i.e.*, $x \in \operatorname{Soc}(M)$. If $x \in \operatorname{Soc}(M)$, then $\mathrm{D}_M(x)$ is finite and hence $K \notin \mathrm{D}_M(x)$, for K is a non-isolated point of $\operatorname{Max}(M)$. Therefore $K \in \mathrm{V}_M(x) = \operatorname{int} \mathrm{V}_M(x)$ implies that $x \in O_K$. ■

Theorem 3.5 *Let M be semiprimitive. Then* $\operatorname{Ann}(\operatorname{Soc}(M)) = Re + \operatorname{Ann}(M)$*, for some M-idempotent $e \in R$ if and only if* $\operatorname{Max}(M)$ *is the union of two disjoint open subspaces A and N, where A is almost discrete and N is dense in itself. In particular,* $\operatorname{Ann}(\operatorname{Soc}(M)) = \operatorname{Ann}(M)$ *if and only if* $\operatorname{Max}(M)$ *is almost discrete.*

Proof First suppose $\operatorname{Ann}(\operatorname{Soc}(M)) = Re + \operatorname{Ann}(M)$, where e is an M-idempotent element of R. We note that by Corollary 2.7, $K \in \operatorname{Max}_0(M)$ if and only if there exists a minimal submodule N of M such that $\mathrm{D}_M(N) = \{K\}$. Thus we have

$$\operatorname{cl} \operatorname{Max}_0(M) = \operatorname{cl} \mathrm{D}_M(\operatorname{Soc}(M)) = \mathrm{V}_M(\operatorname{Ann}(\operatorname{Soc}(M))M) = \mathrm{V}_M(eM).$$

Hence Lemma 2.5 shows that $\operatorname{cl}\operatorname{Max}_0(M)$ is a clopen subset of $\operatorname{Max}(M)$. Now we put $A = \operatorname{cl}\operatorname{Max}_0(M)$ and $N = \operatorname{Max}(M) \setminus \operatorname{cl}\operatorname{Max}_0(M)$ and we are through.

Conversely, let $\operatorname{Max}(M) = A \cup N$, where A and N are two disjoint open subspaces. Then A is almost discrete and N is dense in itself. Inasmuch as A is a clopen subset of $\operatorname{Max}(M)$, then there exists an M-idempotent $e \in R$ such that $A = \mathrm{V}_M(eM)$. We show that $\operatorname{Ann}(\operatorname{Soc}(M)) = Re + \operatorname{Ann}(M)$. Clearly, $e \in \operatorname{Ann}(\operatorname{Soc}(M))$, for if $x \in \operatorname{Soc}(M)$, then $\mathrm{D}_M(x)$ is a finite open set and hence its members are isolated points, *i.e.*, $\mathrm{D}_M(x) \subseteq A = \mathrm{V}_M(eM)$. This implies that $ex = 0$. Therefore $Re \subseteq \operatorname{Ann}(\operatorname{Soc}(M))$. Now if $a \in \operatorname{Ann}(\operatorname{Soc}(M))$, then by Corollary 2.7, $\mathrm{V}_M(eM) = A \subseteq \mathrm{V}_M(aM)$. Thus $a(1-e) \in \operatorname{Ann}(M)$ and this implies that $a \in Re + \operatorname{Ann}(M)$, *i.e.*, $\operatorname{Ann}(\operatorname{Soc}(M)) \subseteq Re + \operatorname{Ann}(M)$. ■

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