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SPECIAL $(p;q)$ RADICALS

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1. Introduction. In [**3**], the study of $(p;q)$ radicals was initiated. In this paper, the integral polynomials $p(x)$ and $q(x)$ which determine the Jacobson radical are characterized and the Jacobson radical is shown to be the only semiprime $(p;q)$ radical for which all fields are semisimple. Also, it is observed that the prime, nil, and Brown-McCoy radicals are not $(p;q)$ radicals. To show that the semiprime $(p;q)$ radicals are special and that they can be determined by subclasses of the class of primitive rings, a classification theorem for $(p;q)$-regular primitive rings is given. Finally, it is shown that the collection of semiprime $(p;q)$ radicals and the collection of semiprime $(p;1)$ radicals coincide.

2. Preliminaries. Let $p(x)$ and $q(x)$ be integral polynomials and R be an associative ring. An element $r \in R$ is *$(p;q)$-regular* if $r \in p(r)Rq(r)$, and R is a *$(p;q)$-regular ring* if every element of R is $(p;q)$-regular. In [**3**], it is shown that every associative ring R contains a largest $(p;q)$-regular ideal $(p(x)Rq(x))$ and that the function which assigns the ideal $(p(x)Rq(x))$ to the ring R is a radical function in the sense of Amitsur and Kurosh. Moreover, A. H. Ortiz has shown (see [**3**]) that a $(p;q)$ radical is semiprime (contains the prime radical) if and only if the constant terms of $p(x)$ and $q(x)$ are ± 1; the Jacobson radical $J(R)$ is the semiprime $(p;q)$ radical given by $((x+1)R)$. It is easy to see that all semiprime $(p;q)$ radicals are hereditary [**3**] and, hence, supernilpotent. We shall study the collection of semiprime $(p;q)$ radicals in this paper.

The word *primitive* unmodified means right primitive; *R-module* means *right R-module.*

3. The Jacobson radical. To begin the characterization of the Jacobson radical in the collection of all semiprime $(p;q)$ radicals, we prove that $J(R)$ is the smallest such radical.

LEMMA 1. *If* $p(0) = 1$, *then* $((x+1)R) = ((x+1)p(x)R)$ *for all rings* R.

Proof. That $((x+1)p(x)R) \subseteq ((x+1)R)$ is obvious. Now, if $r \in ((x+1)R)$ and $p(x) = f(x) + 1$, where $f(x)$ has x as a factor, then $f(r) \in ((x+1)R)$ and, hence, $f(r) \in (f(r)+1)R$. By [**3**, Lemma 4], we see that $R = p(r)R$, or $(r+1)R = (r+1)p(r)R$. Therefore, $((x+1)R) \subseteq ((x+1)p(x)R)$.

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LEMMA 2. *If* $p(0) = 1$, *then* $((x + 1)R) \subseteq (p(x)R)$ *for all rings* R.

Proof. By [**3**, Theorem 3] and Lemma 1, we know that

$$((x + 1)R) \cap (p(x)R) = ((x + 1)p(x)R) = ((x + 1)R)$$

for all rings R. Hence, $((x + 1)R) \subseteq (p(x)R)$.

Similarly, we can show that $((x + 1)R) \subseteq (Rq(x))$ for all rings R whenever $q(0) = 1$.

THEOREM 3. *If* $(p;q)$ *is a semiprime radical, then* $J(R) \subseteq (p(x)Rq(x))$ *for all rings* R.

Proof. $J(R) = ((x + 1)R) \subseteq (p(x)R) \cap (Rq(x)) = (p(x)Rq(x))$.

COROLLARY 4. *Any radical containing the prime radical and properly contained in the Jacobson radical is not a* $(p;q)$ *radical.*

We first characterize the Jacobson radical in the collection of semiprime $(p;1)$ radicals; the characterization of $J(R)$ in the collection of semiprime $(p;q)$ radicals is a consequence of this result and Corollary 16.

THEOREM 5. *If* $p(0) = \pm 1$, *then the following are equivalent*:
(1) $(p(x)R) = J(R)$ *for all rings* R.
(2) $(p(x)F) = (0)$ *for all fields* F.
(3) $p(x) = (ax + 1)p'(x)$, *where* $a \neq 0$ *and, for each prime divisor* m *of* a, *there is an integer* n *such that* m *divides* $p(n)$.

Proof. That (1) implies (2) is obvious.

To prove that (2) implies (3), we note that since $(p(x)\mathbf{Q}) = (0)$, where $\mathbf{Q}$ is the field of rational numbers, $p(x)$ has a linear factor over $\mathbf{Q}$ (hence, over the integers); therefore, $p(x) = (ax + 1)p'(x)$ for some nonzero integer a and integral polynomial $p'(x)$. Now, if m is any prime dividing a, then since $(p(x)R) = (0)$ for $R = GF(m)$, there is an integer n such that m divides $p(n)$.

To prove that (3) implies (1), we first note that if $a = \pm 1$, the result is obvious from Lemma 1. If $a \neq \pm 1$, assume that (3) holds and that

$$J(S) \subset (p(x)S) \text{ (strict containment)}$$

for some ring S. Then $(p(x)T) = T$ and $J(T) = (0)$ for the ring $T = (p(x)S)/J(S) \neq (0)$. Since $J(T) = (0)$, the ring T is a subdirect sum of primitive rings T_β. Because the projection map $T \to T_\beta$ is an epimorphism, we know that $(p(x)T_\beta) = T_\beta$. Since T_β is a primitive ring, it is a dense ring of linear transformations on a vector space V over a division ring D. In the polynomial $p(x) = (ax + 1)p'(x)$, if a is nonzero in D, then for each nonzero v in V there exists t in T such that $vt = v(-a)^{-1} \neq 0$. Since $(p(x)T_\beta) = T_\beta$, we see that $t = p(t)t'$ for some t' in T_β, so $vt = vp(t)t' = v(at + 1)p'(t)t' = (avt + v)p'(t)t' = 0$, which is a contradiction. Now, suppose that m is zero in D for some prime divisor m of a. Then for the number n of the hypothesis there exists an s in T_β such that $vs = vn \neq 0$; moreover, $vs^k = vn^k$, for $k = 1, 2, \ldots$, since s is a linear transformation. As before, we may find some s' in T_β for which $vs = vp(s)s' = vp(n)s' = 0$, which is a contradiction. Hence, $(p(x)R) = J(R)$ for all rings R.

4. The main theorem.

THEOREM 6. *If R is a $(p;1)$-regular primitive ring, where*

$$p(x) = 1 + a_1x + \ldots + a_mx^m,$$

then either (1) R is a dense ring of linear transformations on a vector space V over a division ring D of characteristic c, where c divides a_i, $i = 1, 2, \ldots, m$, or (2) R is a complete ring of $n \times n$ matrices over a division ring, where $n < m$ and for each r in R, the matrix $p(r)$ is nonsingular. Conversely, if either (1′) R is a ring of linear transformations on a vector space over a division ring of characteristic c, where c divides a_i, $i = 1, 2, \ldots, m$, or (2′) R is a complete ring of $n \times n$ matrices over a division ring and $p(r)$ is nonsingular for each r in R, then R is $(p;1)$-regular.

Proof. Suppose that R is a $(p;1)$-regular primitive ring. Then R is a dense ring of linear transformations on a vector space V over a division ring D. If the characteristic of D does not divide a_i, $i = 1, 2, \ldots, m$, then there is a largest positive integer $k + 1$ such that $a_{k+1} \neq 0$ in D. If V has $k + 1$ linearly independent elements $v_0, v_1, \ldots, v_k$, then by the density of R there exists an r in R such that

$$v_0r = v_1, v_1r = v_2, \ldots, v_{k-1}r = v_k, v_kr = -(1/a_{k+1})(v_0 + a_1v_1 + \ldots + a_kv_k).$$

Since R is $(p;1)$-regular, $r \in p(r)R$ and $0 \neq v_1 = v_0r = v_0p(r)s = 0 \cdot s = 0$. Therefore, there are at most k linearly independent elements of V and R is a complete ring of $n \times n$ matrices over a division ring, where $n \leqq k$. Since $r \in p(r)R$ for all r in R, [**3**, Lemma 4] implies that $R = p(r)R$ and, hence, $1 = p(r)s$ for some s in R. Conversely, if (1′) holds, then $r = p(r)r$ for all r in R, and if (2′) holds, $1 = p(r)s$, which implies that $r = p(r)sr$.

If M is a faithful irreducible R-module for the primitive ring R and if I is a nonzero ideal of R, then M is also a faithful irreducible I-module. R and I are dense rings of linear transformations on M regarded, respectively, as a vector space over the commuting ring $C_R(M)$ and as a vector space over $C_I(M)$. Evidently, $C_R(M) \subseteq C_I(M)$, so the characteristics of $C_R(M)$ and $C_I(M)$ are the same. Thus, we can prove the following.

LEMMA 7. *If R is a primitive ring and $p(0) = \pm 1$, then R is $(p;1)$-regular or $(p;1)$-semisimple.*

Proof. Suppose that $I = (p(x)R)$ and that I is not zero. Since R is primitive, R is a dense ring of linear transformations on a vector space V over a division ring D. Also, I is a dense ring of linear transformations on V over a division ring D' whose characteristic is the same as the characteristic of D. Since I is $(p;1)$-regular, Theorem 6 shows that either (1) the common characteristic of D and D' divides the nonconstant coefficients of $p(x)$, or (2) I is a complete ring of $n \times n$ matrices over D'. If (1) holds, then by Theorem 6 (1′), the ring R is $(p;1)$-regular. If (2) holds, then I has a unity and, hence, $R = I \oplus J$ for some ideal J of R. However, since R is prime and $IJ = (0)$, it follows that $J = (0)$. Therefore, $I = R$.

THEOREM 8. *If* $p(0) = 1$, *then* $(p(x)R) = \bigcap_\alpha \{P_\alpha : P_\alpha$ *is primitive and* R/P_α *is* $(p;1)$-*semisimple*$\}$.

Proof. Since

$$[(p(x)R) + P_\alpha]/P_\alpha \cong (p(x)R)/[(p(x)R) \cap P_\alpha],$$

the ring $[(p(x)R) + P_\alpha]/P_\alpha$ is both $(p;1)$-semisimple and $(p;1)$-regular; hence, $(p(x)R) \subseteq P_\alpha$. Therefore, $(p(x)R) \subseteq \bigcap_\alpha P_\alpha$. Now, suppose that $r \notin (p(x)R)$ and that $p(x) = 1 + a_1x + \ldots + a_mx^m$. Then $\langle r \rangle$, the two-sided ideal generated by r, is not $(p;1)$-regular. Hence, there is an s in $\langle r \rangle$ such that $s \notin p(s)R$. By Zorn's Lemma, there exists a modular right ideal M_s, maximal with respect to containing $p(s)R$ and not containing s, where the modular element is $a_1s + a_2s^2 + \ldots + a_ms^m$. Moreover, M_s is a maximal modular right ideal in R; for, if I is a right ideal of R and I properly contains M_s, then the statements $s \in I$, $p(s)R \subseteq I$, and $p(0) = 1$ imply that $R = I$. Now, M_s contains the primitive ideal $P_s = (M_s{:}R)$, where

$$(M_s{:}R) = \{r\colon r \in R \text{ and } Rr \subseteq M_s\}.$$

Since R/P_s is a primitive ring, by Lemma 7 it is either $(p;1)$-regular or $(p;1)$-semisimple. If R/P_s is $(p;1)$-regular, then $s + P_s \in p(s + P_s)(R/P_s)$, or $s - p(s)t \in P_s$, for some t in R. This last relation and the fact that $p(s)R \subseteq M_s$ imply that $s \in M_s$, which is a contradiction. Therefore, R/P_s is $(p;1)$-semisimple; moreover, r is not contained in P_s since $s \notin P_s$. Hence, $r \notin \bigcap_\alpha \{P_\alpha\colon P_\alpha$ is primitive and R/P_α is $(p;1)$-semisimple$\}$.

5. Special $(p;q)$ radicals. A class $\mathfrak{M}$ of rings is *special* if $\mathfrak{M}$ satisfies the three conditions (Andrunakievič [**1**]):

$S1$: Every ring in $\mathfrak{M}$ is a prime ring.

$S2$: Every nonzero ideal of a ring in $\mathfrak{M}$ is a ring in $\mathfrak{M}$.

$S3$: If I is a ring of $\mathfrak{M}$ and I is an ideal of a ring R, then R/I^* is a ring of $\mathfrak{M}$, where I^* is the two-sided annihilator of I.

Our use of the following formulation in the proof of Theorem 10 was suggested by E. H. Connell.

LEMMA 9. *A class* $\mathfrak{M}$ *of rings is special if and only if* $\mathfrak{M}$ *satisfies* $S1$, $S2$ *and* $S3'$: *If* $I \in \mathfrak{M}$ *and* I *is a large ideal of a ring* R, *then* $R \in \mathfrak{M}$. (I *is a large ideal of* R *if* $I \cap J \neq (0)$ *for all nonzero ideals* J *of* R).

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THEOREM 10. *If $\mathfrak{M}$ is a special class of rings and ρ is a hereditary radical, then the class $\mathfrak{N}$ of rings of $\mathfrak{M}$ that are ρ-semisimple is a special class.*

Proof. It is immediate that $S1$ and $S2$ hold. To prove $S3'$, let I be a ring in $\mathfrak{N}$ and I be a large ideal of R. Since $I \in \mathfrak{N} \subseteq \mathfrak{M}$ and $\mathfrak{M}$ is a special class, we have $R \in \mathfrak{M}$. Since I is a large ideal of R and $I \cap \rho R = \rho I = (0)$, we conclude that $\rho R = (0)$. Thus, $R \in \mathfrak{N}$.

COROLLARY 11. *If $p(0) = 1$, then the class $\mathfrak{N}$ of all $(p;1)$-semisimple primitive rings is a special class.*

Proof. In Theorem 10, let $\mathfrak{M}$ be the class of primitive rings.

If $\mathfrak{M}$ is a class of rings, we denote by $\mathfrak{S}_{\mathfrak{M}}$ the upper radical determined by $\mathfrak{M}$ [2, p. 6]; if $\mathfrak{M}$ is a special class, then the radical $\mathfrak{S}_{\mathfrak{M}}$ is called a *special radical.* Combining Theorem 8 and Corollary 11 we obtain a principal result.

THEOREM 12. *If $p(0) = \pm 1$, then $(p;1)$ is a special radical.*

Proof. Let $\mathfrak{M} = \{R{:}R$ is a $(p;1)$-semisimple primitive ring$\}$. Then by [**2**, p. 139, Lemma 80], we conclude that $\mathfrak{S}_{\mathfrak{M}}\ (R) = \bigcap_\alpha\{I_\alpha{:}\ R/I_\alpha \in \mathfrak{M}\} = \bigcap_\alpha\{I_\alpha{:}\ I_\alpha$ is a primitive ideal and R/I_α is $(p;1)$-semisimple$\} = (p(x)R)$ for all rings R.

COROLLARY 13. *If $p(0)q(0) = \pm 1$, then $(p;q)$ is a special radical; in fact, a special class for which $(p;q)$ is the upper radical is the union of the class of $(p;1)$-semisimple right-primitive rings and the class of $(1;q)$-semisimple left-primitive rings.*

Proof. Since $(p;1)$ is a special radical and, as interchanging right and left primitivity in previous reasoning shows, so is $(1;q)$, by [**4**, Lemma 6] and [**3**, Theorem 3], we have $[(p;1) \wedge (1;q)](R) = (p(x)R) \cap (Rq(x)) = (p(x)Rq(x)) = (p;q)(R)$. Therefore, by [**4**, Proposition 11] we have $(p;q) = \mathfrak{S}_{\mathfrak{M} \cup \mathfrak{N}}$, where $\mathfrak{M} = \{R{:}\ R$ is a $(p;1)$-semisimple right-primitive ring$\}$ and $\mathfrak{N} = \{R{:}\ R$ is a $(1;q)$-semisimple left-primitive ring$\}$.

COROLLARY 14. *If $(p;q) = (p';q')$ on all right- and left-primitive rings, then $(p;q) = (p';q')$ on all rings.*

Proof. If $(p';q') \nleq (p;q)$, then we may assume that there is a ring $R \neq (0)$ such that R is $(p';q')$-radical and $(p;q)$-semisimple. By [**2**, Lemma 80], R is a subdirect sum of rings R_α in $\mathfrak{M} \cup \mathfrak{N}$, where $\mathfrak{M}$ and $\mathfrak{N}$ are the classes defined in the proof immediately above. Since R projects onto R_α, each R_α is $(p';q')$-radical, but being a member of $\mathfrak{M} \cup \mathfrak{N}$, it is $(p;q)$-semisimple and either right or left primitive.

The next theorem is based on a conjecture of David M. Morris. It is important here because it provides a simple method for completing the desired characterization of $J(R)$. It also eliminates the need to use more than one kind of primitivity in discussing $(p;q)$-radical rings.

THEOREM 15. *If* $p(0) = \pm 1$, *then* $(p(x)R) = (Rp(x))$ *for all rings* R.

Proof. Let R be a right-primitive ring. If R is $(p;1)$-regular, then Theorem 6 applies. If (1) of Theorem 6 holds, then clearly $r \in Rp(r)$ for all r in R. If (2) holds and $r \in R$, then $r = p(r)s$ for some s in $(p(x)R)$. But $r = p(r)^{-1} \cdot p(r)r = p(r)^{-1}rp(r) = sp(r)$; hence, R is $(1;p)$-regular. If, on the other hand, the right-primitive ring R is $(1;p)$-regular, it is a subdirect sum of left-primitive rings R_α, each of which is $(1;p)$-regular. Moreover, if R is represented faithfully as a ring of linear transformations on a vector space (a right R-module) over a division ring D, and if each R_α is represented correspondingly (for a left R_α-module) over a division ring D_α, then the characteristics of D and D_α are the same. Consequently, we can obtain a coordinatewise computation, based on the analogue of Theorem 6 for $(1;p)$-regular left-primitive rings R_α, like the computation just displayed; for, alternative (1) of the analogue is valid for all R_α or (2) is valid for all R_α. Therefore, we can conclude that R is $(p;1)$-regular.

That the $(p;1)$-regular and $(1;p)$-regular left-primitive rings are the same, follows in like manner. Applying Corollary 14, we see that $(p;1) = (1;p)$.

COROLLARY 16. *If* $p(0) = \pm 1$ *and* $q(0) = \pm 1$, *then* $(p(x)Rq(x)) = (p(x)q(x)R)$.

Proof. By [**3**, Theorem 3], we find that

$$(p(x)Rq(x)) = (p(x)R) \cap (Rq(x)) = (p(x)R) \cap (q(x)R) = (p(x)q(x)R).$$

Thus, since we can reduce the semiprime $(p;q)$ radicals to semiprime $(p;1)$ radicals, we finally have the result sought from the outset.

THEOREM 17. *If* $p(0) = \pm 1$ *and* $q(0) = \pm 1$, *then the following are equivalent.*

(1) $(p(x)Rq(x)) = J(R)$ *for all rings* R.

(2) $(p(x)Fq(x)) = (0)$ *for all fields* F.

(3) $p(x) = (ax + 1)p'(x)$ *or* $q(x) = (ax + 1)q'(x)$, *where* $a \neq 0$ *and, for each prime divisor* m *of* a, *there is an integer* n *such that* m *divides* $p(n)q(n)$.

Proof. This is immediate from Theorem 5 and Corollary 16.

COROLLARY 18. (D. M Morris). *If the leading and constant coefficients of* $p(x)$ *and* $q(x)$ *are* ± 1, *then* $(p(x)Rq(x)) = J(R)$ *if and only if* $(x \pm 1)$ *divides* $p(x)$ *or* $(x \pm 1)$ *divides* $q(x)$.

COROLLARY 19. *Any radical which properly contains the Jacobson radical and which is zero on all fields is not a* $(p;q)$ *radical*; *hence, the Brown-McCoy radical is not a* $(p;q)$ *radical.*

It is also interesting to notice that if $(p;q)$ is a semiprime radical and $(p(x)Rq(x))$ is contained between $J(R)$ and the Brown-McCoy radical $G(R)$ of R, for all rings R, then $(p(x)Rq(x)) = J(R)$ since $J(F) = G(F) = 0$ for all fields F. Hence, $J(R)$ is the only semiprime $(p;q)$ radical that coincides with the nil radical on rings with *D.C.C.* on left ideals [**2**, p. 43, Theorem 13].

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