CORRESPONDENCE

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Sirs,

Mr L. R. Packer's article 'The distribution of the sum of n rectangular variates' (*J.S.S.* x, 52–61), made me refer to some notes which I had prepared after seeing G. J. Lidstone's paper 'Accuracy of arithmetical approximation' (*T.F.A.* XVII, 27). At the foot of p. 27 Lidstone refers to the frequency curve corresponding to the sum of s items. The determination of this frequency curve is Mr Packer's problem. In a letter to Mr Lidstone of May 1943 I wrote that this frequency was a curve which I had called $J_r(a_0, a_1, \ldots, a_r; x)$ in an entirely different connexion in *J.I.A.* LXV, 296–298, the general points $a_0, a_1, \ldots, a_r$ on which

$$J_r(a_0, a_1, \ldots, a_r; x)$$

was based being replaced by the equidistant points

$$-\tfrac{1}{2}s,\ -\tfrac{1}{2}s+1,\ \ldots,\ \tfrac{1}{2}s-1,\ \tfrac{1}{2}s.$$

If $a_0, a_1, \ldots, a_r$ are $r+1$ points taken in order $J_r(a_0, a_1, \ldots, a_r; x)$ is defined formally as being equal to zero outside the range (a_0, a_r) and to the polynomial $J_r(a_s, a_{s+1}; x)$ for

$$a_s \leqslant x \leqslant a_{s+1},\ s=0,\ 1,\ 2,\ \ldots,\ r-1,$$

where the set of polynomials $J_r(a_s, a_{s+1}; x)$ is defined by

$$J_r(a_{s-1}, a_s; x) - J_r(a_s, a_{s+1}; x) = r\frac{(a_s-x)^{r-1}}{(a_s-a_0)(a_s-a_1)\ldots(a_s-a_{s-1})(a_s-a_{s+1})\ldots(a_s-a_r)},$$

with starting and ending values

$$J_r(a_{-1}, a_0; x) = J_r(a_r, a_{r+1}; x) = 0.$$

This somewhat complicated definition means that

$$J_r(a_0, a_1, \ldots, a_r; x)$$

is made up of separate polynomials of degree $r-1$ in each interval (a_s, a_{s+1}). Consecutive sections have osculatory contact of the

$(r-2)$th order with each other at the junction points, and the first and last sections have osculatory contact of the $(r-2)$th order with the base-line at points a_0 and a_r respectively.

$J_r(a_0, a_1, \ldots, a_r; x)$ was constructed for the purpose of possessing the property that if $f(x)$ is any function possessing differential coefficients up to the rth in the interval then the Newtonian Divided Difference Function

$$f_r(a_0, a_1, \ldots, a_r) = (1/r!)\int_{a_0}^{a_r} J_r(a_0, a_1, \ldots, a_r; x) f^{(r)}(x)\, dx. \quad (1)$$

This property and others are proved in *J.I.A.* LXV, 297–298. It may also be shown that

$$J_{r+1}(0, 1, 2, \ldots, r+1; x) = \int_0^1 J_r(0, 1, 2, \ldots, r; x-\lambda)\, d\lambda, \quad (2)$$

which provides the necessary link with Mr Packer's $f(x\ r)$ (see Packer, p. 54, l. 4) since from definition $J_1(0, 1; x)$ is clearly $f(x, 1)$.

Since $J_r(a_0, a_1, \ldots, a_r; x)$ may in some circumstances be regarded as a frequency curve (becoming in fact normal when equidistant base points are indefinitely increased in number) it may be of interest to show how to calculate the first few moments.

Take $f(x) = x^{r+1}/(r+1)$ in relationship (1).

$$\text{First moment} = \int_{a_0}^{a_r} x J_r(a_0, a_1, \ldots, a_r; x)\, dx$$
$$= f_r(a_0, a_1, \ldots, a_r) = (a_0 + a_1 + \ldots + a_r)/(r+1).$$

Take $f(x) = x^{r+2}/(r+1)(r+2)$

$$\text{Second moment} = \int_{a_0}^{a_r} x^2 J_r(a_0, a_1, \ldots, a_r; x)\, dx$$
$$= 2f_r(a_0, a_1, \ldots, a_r) = 2(\Sigma a_0^2 + \Sigma a_0 a_1)/(r+1)(r+2);$$
$$\sigma^2 = 2(\Sigma a_0^2 + \Sigma a_0 a_1)/(r+1)(r+2) - (\Sigma a_0^2 + 2\Sigma a_0 a_1)/(r+1)^2$$
$$= [\Sigma a_0^2 - (\Sigma a_0)^2/(r+1)]/(r+1)(r+2).$$

When $a_i = i$, this becomes $\sigma^2 = r/12$.

Yours faithfully,

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