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JORDAN SUBALGEBRAS OF BANACH ALGEBRAS

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1. Introduction

We recall that a JC-algebra (Størmer (3)) is a norm closed Jordan algebra of self-adjoint operators on a Hilbert space. Recently, Alfsen, Shultz, and Størmer (1) have introduced a class of abstract normed Jordan algebras called JB-algebras, and have proved that every special JB-algebra is isometrically isomorphic to a JC-algebra. We show that this result brings to a satisfactory conclusion the discussion in (2) of certain wedges W in Banach algebras and their related Jordan algebras $W-W$, and leads to two characterisations of the bicontinuously isomorphic images of JC-algebras.

It was proved in (2) that if W is a closed type-0 locally multiplicative wedge in a Banach algebra and the set $\{\|(1+w)^{-1}\|: w \in W\}$ is bounded, then $W-W$ is a closed Jordan algebra and behaves in many ways like a JC-algebra with positive cone W. It can now be seen that $W-W$ is in fact bicontinuously isomorphic to a JC-algebra. We prove also that for a closed type-0 wedge W in a Banach algebra, to be locally multiplicative is equivalent to having $xyx \in W$ whenever $x,y \in W$. As a corollary we obtain two characterisations of those Jordan subalgebras R of a Banach algebra that are bicontinuously isomorphic to JC-algebras. The first involves subadditivity of the spectral radius r, and in the second subadditivity is replaced by a submultiplicative property:

$$x,y \in R, \quad \lambda \in \boldsymbol{R} \cap \mathrm{Sp}(xy) \Rightarrow |\lambda| \leq r(x)r(y).$$

For JC-algebras A it is obvious that the stronger submultiplicative property

$$r(xy) \leq r(x)r(y) \quad (x,y \in A)$$

holds, but it remains an open question whether this holds on their bicontinuously isomorphic images R.

Finally we show that if W is a closed type-0 locally multiplicative cone but the set of inverses $(1+w)^{-1}$ is not necessarily bounded, the completion of $W-W$ with respect to the spectral radius norm is a special JB-algebra. Thus in this more general case $W-W$ remains isomorphic to a dense Jordan subalgebra of a JC-algebra.

2. Notation

B will denote a complex Banach algebra with unit, $\mathrm{Inv}(B)$ will denote the set of invertible elements of B, and for $a \in B$, Sp(a) and $r(a)$ will denote the spectrum and spectral radius of A.

A *wedge* in B is a non-void subset W of B such that

$$x,y \in W, \alpha \in \boldsymbol{R}^+ \Rightarrow x+y, \alpha x \in W.$$

A wedge W in B is of *type*-0 if

$$x \in W \Rightarrow 1+x \in \operatorname{Inv}(B) \text{ and } (1+x)^{-1} \in W,$$

is *locally multiplicative* if

$$x,y \in W, xy = yz \Rightarrow xy \in W,$$

and is a *cone* if $W \cap (-W) = \{0\}$.

H will denote a complex Hilbert space, and $BL(H)$ the Banach algebra of all bounded linear operators on H. A *JC-algebra* is a real linear subspace A of $BL(H)$, closed with respect to the operator norm, consisting of self-adjoint operators, and satisfying

$$a,b \in A \Rightarrow ab+ba \in A.$$

The positive cone A^+ in a JC-algebra A is the set of elements of A that are positive operators in the usual sense (that is operators a with $(ax,x) \geq o$ $(x \in H)$).

Following Alfsen, Shultz and Størmer (**1**), a *JB-algebra* is a real Banach space X which is a Jordan algebra with respect to a product $x \circ y$, which has a unit element, and which satisfies (for all $x,y \in X$)

$$\|x \circ y\| \leq \|x\|\,\|y\|, \quad \|x^2\| = \|x\|^2, \quad \|x^2\| \leq \|x^2+y^2\|.$$

X is a *special* JB-algebra if it is also a subset of an associative algebra and $x \circ y = \frac{1}{2}(xy+yx)$, where xy is the associative product.

3.

Theorem 1. *Let W be a closed type-0 locally multiplicative wedge in B, let $R = W - W = \{x-y : x,y \in W\}$, and suppose that the set $\{\|(1+w)^{-1}\| : w \in W\}$ is bounded. Then r is a norm on R equivalent to the given norm, and R with the norm r is a special JB-algebra.*

Proof. We recall from (**2**, Theorem 3) that R is a closed real linear subspace and Jordan subalgebra of B, that r is subadditive on R, that all elements of R have their spectra contained in the reals, and that

$$W = \{x \in R : \operatorname{Sp}(x) \subset \boldsymbol{R}^+\}. \tag{1}$$

By (**2**, Theorem 9), $W \cap (-W) = \{0\}$, and so, by (**2**, Theorem 4), r is a norm on R satisfying the inequality

$$r(\tfrac{1}{2}(xy+yx)) \leq r(x)r(y) \quad (x,y \in R).$$

By hypothesis, there exists a real constant M such that

$$\|(1+w)^{-1}\| \leq M \quad (w \in W). \tag{2}$$

We prove that

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$$\|x\| \leqslant (2\|1\| + 3M)r(x) \quad (x \in R). \tag{3}$$

Since r is a norm, this is trivial if $r(x) = 0$; and we may therefore suppose, by normalisation, that $r(x) = \frac{1}{2}$. Then $\mathrm{Sp}(1 + x) \subset [\frac{1}{2}, \frac{3}{2}]$. By (1), $1 + x$ is therefore of the form $\frac{1}{2} + u$ with $u \in W$. Since W is of type-0 it follows that $(1 + x)^{-1} \in W$, and we have $\mathrm{Sp}((1 + x)^{-1}) \subset [\frac{2}{3}, 2]$. Again by (1), it follows that $(1 + x)^{-1} = \frac{2}{3} + w$ with $w \in W$, and so $1 + x = \frac{3}{2}(1 + \frac{3}{2}w)^{-1}$. The inequality (2) now gives $\|1 + x\| \leqslant \frac{3}{2}M$, and (3) is proved.

We have now proved that r is a norm on R equivalent to the given norm, and so R with the norm r is a real Banach space. It is obvious that $r(x^2) = (r(x))^2$ for all $x \in R$, and, since the squares of elements of R belong to W, the proof will be complete if we show that

$$r(u) \leqslant r(u + v) \quad (u,v \in W). \tag{4}$$

Let $u,v \in W$. Then $u + v \in W$, and, by (1),

$$r(u + v) - (u + v) \in W.$$

Since W is a wedge, it follows that

$$r(u + v) - u = v + r(u + v) - (u + v) \in W,$$

and so $\mathrm{Sp}(r(u + v) - u) \subset \boldsymbol{R}^+$. Thus (4) is proved, and the proof is complete.

Corollary 2. *Let W, R be as in Theorem 1. Then R is bicontinuously isomorphic to a JC-algebra A, and W corresponds under this isomorphism to the positive cone A^+ of A.*

Proof. By Theorem 1 and (**1**, Lemmas 9.3 and 9.4), R with the norm r is isometrically isomorphic to a JC-algebra A. Since the norm r is equivalent on R to the given norm, the isomorphism is bicontinuous with respect to the given norm. The identification of the image of W with A^+ follows at once from the fact that W is the set of squares of elements of R (**2**, Theorem 9) and the corresponding fact for A^+.

Corollary 3. *Let W be a closed type-0 locally multiplicative wedge. Then the set $\{\|(1 + w)^{-1}\| : w \in W\}$ is bounded if and only if W is a normal cone (that is, there exists a constant $\kappa > 0$ with $\|x + y\| \geqslant \kappa\|x\| (x,y \in W)$).*

Proof. Let $E = \{\|(1 + w)^{-1}\| : w \in W\}$. By (**2**, Proposition 10(i)), E is bounded if W is a normal cone. Conversely, suppose that E is bounded. By Theorem 1 there exists a positive constant κ with

$$r(x) \geqslant \kappa\|x\| \quad (x \in R).$$

By (**2**, Proposition 10(ii)), this shows that W is a normal cone.

Theorem 4. *Let W be a closed type-0 wedge in B. Then the following statements are equivalent:*

(i) *W is locally multiplicative,*
(ii) $x,y \in W \Rightarrow xyx \in W$.

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Proof. That (i) implies (ii) was proved in (**2**, Theorem 5). Suppose conversely that (ii) holds. Given $w \in W$, it is clear that $w^n \in W$ $(n = 1, 2, \ldots)$, the fact that $1 \in W$ giving the case $n = 2$. We prove that

$$\mathrm{Sp}(w) \subset \boldsymbol{R}^+ (w \in W). \tag{5}$$

The invertibility of $1 + w$ gives

$$\mathrm{Sp}(w) \cap (-\boldsymbol{R}^+) \subset \{0\} \quad (\mathrm{w} \in \mathrm{W}). \tag{6}$$

We argue as in the proof of (**2**, Proposition 1). Suppose that $w \in W$ and that $\rho e^{i\theta} \in \mathrm{Sp}(w)$ with $\rho > 0$, $\theta \in \boldsymbol{R}$, $0 < |\theta| < \pi$. Choose the greatest positive integer n with $n|\theta| < \pi$. Then $n|\theta| \geqslant \pi/2$. Take $b = w^n$ and observe that $\rho^n e^{in\theta}$ is of the form $-\gamma + i\delta$ with $\gamma \geqslant 0$ and $\delta \in \boldsymbol{R} \backslash \{0\}$. Then $(\gamma + b)^2 \in W$ and $-\delta^2 \in \mathrm{Sp}((\gamma + b)^2)$, contradicting (6). Thus $\mathrm{Sp}(w) \subset \boldsymbol{R}$ and (5) follows from (6).

We prove next that

$$w \in W, r(w) < 1 \Rightarrow 1 - w \in W. \tag{7}$$

Given $w \in W$ with $r(w) < 1$, we have $(1 - w)^{-1} = 1 + b$ with $b = \Sigma_{k=1}^{\infty} w^k \in W$. Thus $1 - w = (1 + b)^{-1} \in W$.

Now let $x, y \in W$ with $xy = yx$, and suppose first that $x \in \mathrm{Inv}(B)$ and $r(x) < 1$. By (5) we have $r(1 - x) < 1$, and by (7) $1 - x \in W$. Since the binomial series for $(1 - t)^{-1/2}$ has positive coefficients, we therefore have

$$x^{-1/2} = \{1 - (1 - x)\}^{-1/2} \in W,$$

and $x^{-1/2}y = yx^{-1/2}$. By condition (ii), we have in turn $x^{-1/2}yx^{-1/2} \in W$, $x^{1/2}yx^{1/2} = x(x^{-1/2}yx^{-1/2})x \in W$. Therefore

$$xy = x^{1/2}yx^{1/2} \in W,$$

and it is clear that this still holds without the condition $r(x) < 1$. Finally, given arbitrary $x, y \in W$ with $xy = yx$ and $\epsilon > 0$, we have $\epsilon + x \in W \cap \mathrm{Inv}(B)$, and $(\epsilon + x)y = y(\epsilon + x)$. Therefore $(\epsilon + x)y \in W$. Since W is closed, we have $xy \in W$, and the proof is complete.

Corollary 5. *Let R be a Jordan subalgebra of B containing* 1. *Then R is bicontinuously isomorphic to a JC-algebra if and only if it satisfies the following conditions*:

(i) *R is closed,*
(ii) $\mathrm{Sp}(x) \subset \boldsymbol{R}$ $(x \in R)$,
(iii) *r is subadditive on R,*
(iv) $\{\|(1 + x^2)^{-1}\| : x \in R\}$ *is bounded.*

Proof. Suppose first that R satisfies the stated conditions (i)–(iv), and let $W = \{x \in R : \mathrm{Sp}(x) \subset \boldsymbol{R}^+\}$. By (**2**, Theorem 3), W is a closed type-0 locally multiplicative wedge and $R = W - W$. Since elements of $W \cap \mathrm{Inv}(B)$ have square roots in W, the set $\{\|(1 + w)^{-1}\| : w \in W\}$ is bounded. By Corollary 2, R is bicontinuously isomorphic to a JC-algebra.

Conversely, suppose that ϕ is a bicontinuous isomorphism of R onto a JC-algebra

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A, and let $W=\phi^{-1}(A^+)$. Then W is a closed cone and $R=W-W$. If $a,b\in A$ with $ab=ba=1$, and $x=\phi^{-1}(a)$, $y=\phi^{-1}(b)$, then $xy=yx=1$. This is not quite obvious since ϕ is not an isomorphism for the associative structure. However, since ϕ is a Jordan isomorphism, we have $xy+yx=2$ and $xyx=x$. Thus $(xy)^2=xy$ and $(yx)^2=yx$, and we have in turn $4-4xy+xy=(2-xy)^2=2-xy$, $xy=1$, $yx=1$. Since A^+ is of type-0, it now follows that W is of type-0. Since $aba\in A^+$ $(a,b\in A^+)$ and ϕ is a Jordan isomorphism, we have $xyx\in W$ $(x,y\in W)$. By Theorem 4, W is locally multiplicative, and Theorem 3 of (2) now shows that R satisfies (ii) and (iii). That R satisfies (i) and (iv) is clear from the boundedness of ϕ and ϕ^{-1} and from the inequality $\|(1+a^2)^{-1}\|\leqslant 1$ $(a\in A)$.

The next lemma may appear obvious at first sight but involves the difficulty that the spectra of elements are defined in terms of the complex Banach algebras B and $BL(H)$ whereas the Jordan isomorphism is defined only on the Jordan subalgebra R.

Lemma 6. *Let R be a Jordan subalgebra of B containing* 1, *and let ϕ be a bicontinuous isomorphism of R onto a JC-algebra. Then*

$$\mathrm{Sp}(\phi(x))=\mathrm{Sp}(x)\quad(x\in R).$$

Proof. Let $W=\{x\in R:\mathrm{Sp}(x)\subset \boldsymbol{R}^+\}$. By Corollary 5 and (2, Theorem 3), W is a closed type-0 locally multiplicative wedge. Also $\{\|(1+w)^{-1}\|:w\in W\}$ is bounded. Therefore, by Theorem 1, the spectral radius is a norm on R equivalent to the given norm, and so there exists a constant $\kappa>0$ such that

$$r(x)\geqslant\kappa\|x\|\quad(x\in R).$$

We deduce that

$$x,y\in R, xy=yx\ \Rightarrow\ \|x+iy\|\geqslant\frac{\kappa^2}{4}(\|x\|+\|y\|).\tag{8}$$

Given $x,y\in R$ with $xy=yx$, we have

$$(\|x\|+\|y\|)\|x+iy\|\geqslant\|x-iy\|\,\|x+iy\|\geqslant\|x^2+y^2\|\geqslant r(x^2+y^2)\geqslant r(x^2)\geqslant\kappa^2\|x\|^2.$$

Similarly $(\|x\|+\|y\|)\|x+iy\|\geqslant\kappa^2\|y\|^2$, and so

$$(\|x\|+\|y\|)\|x+iy\|\geqslant\frac{\kappa^2}{2}(\|x\|^2+\|y\|^2)\geqslant\frac{\kappa^2}{4}(\|x\|+\|y\|)^2,$$

which proves (8).

We prove next that

$$x\in R\cap\mathrm{Inv}(B)\Rightarrow x^{-1}\in R.\tag{9}$$

Let $x\in R\cap\mathrm{Inv}(B)$, and let C be the least closed complex subalgebra of B containing 1 and x. Since $\mathrm{Sp}(x)\subset\boldsymbol{R}$, the spectrum of x relative to C coincides with its spectrum relative to B. Therefore $x^{-1}\in C$, and there exist real polynomials p_n,q_n in x such that $x^{-1}=\lim_{n\to\infty}(p_n+iq_n)$. It follows from (8) that there exist $p,q\in R\cap C$ such that $\lim_{n\to\infty}p_n=p$, $\lim_{n\to\infty}q_n=q$. Thus $x^{-1}=p+iq$. We have $px,qx\in R\cap C$ and $(px-1)+iqx=0$. Therefore, by (8), $px-1=0$, $x^{-1}=p\in R$; (9) is proved.

Given $x,y\in R$ with $xy=yx=1$, we have, as in the proof of Corollary 5,

$\phi(x)\phi(y) = \phi(y)\phi(x) = 1$. It follows, by (9) that if $x \in R \cap \mathrm{Inv}(B)$, then $\phi(x)$ is invertible in $BL(H)$. Finally, since $\mathrm{Sp}(x) \subset \boldsymbol{R}$ for all $x \in R$, we now have $\mathrm{Sp}(\phi(x)) = \mathrm{Sp}(x)$.

We now consider the characterisation of bicontinuous isomorphic images of JC-algebras in terms of a submultiplicative property of the spectral radius in place of the subadditive property in Corollary 5.

Theorem 7. *Let R be a Jordan subalgebra of B containing* 1. *Then R is bicontinuously isomorphic to a JC-algebra if and only if it satisfies*

(i) *R is closed,*

(ii) $\mathrm{Sp}(x) \subset \boldsymbol{R} \quad (x \in R)$,

(iii) $x,y \in R, \lambda \in \boldsymbol{R} \cap \mathrm{Sp}(xy) \Rightarrow |\lambda| \leq r(x)r(y)$,

(iv) $\{\|(1+x^2)^{-1}\| : x \in R\}$ *is bounded.*

Proof. Suppose first that conditions (i)–(iv) hold, and let $W = \{x \in R : \mathrm{Sp}(x) \subset \boldsymbol{R}^+\}$. Minor modifications of the proof of (2, Theorem 8) show that W is a closed type-0 locally multiplicative wedge and that $R = W - W$. Thus Corollary 2 gives the required bicontinuous isomorphism.

Suppose conversely that ϕ is a bicontinuous isomorphism of R onto a JC-algebra A. We prove first that

$$x,y,z \in R \Rightarrow \mathrm{Sp}(z(xy-yx)z) \subset i\boldsymbol{R}. \tag{10}$$

Given $x,y,z \in R$, let $a = \phi(x)$, $b = \phi(y)$, $c = \phi(z)$. We have

$$(z(xy+yx)z)^2 - (z(xy-yx)z)^2 = 2zxyz^2yxz + 2zyxz^2xyz$$

and, in turn, $yz^2y \in R$, $xyz^2yx \in R$, $zxyz^2yxz \in R$. Therefore $(z(xy-yx)z)^2 \in R$, and, since ϕ is a Jordan isomorphism,

$$\begin{aligned}\phi((z(xy-yx)z)^2) &= \phi((z(xy+yx)z)^2) - 2\phi(zxyz^2yxz) - 2\phi(zyxz^2xyz)\\ &= (c(ab+ba)c)^2 - 2cabc^2bac - 2cbac^2abc\\ &= (c(ab-ba)c)^2.\end{aligned}$$

Since $ic(ab-ba)c$ is self-adjoint, we have $\mathrm{Sp}((c(ab-ba)c)^2) \subset -\boldsymbol{R}^+$. Therefore, by Lemma 6, $\mathrm{Sp}((z(xy-yx)z)^2) \subset -\boldsymbol{R}^+$, and (10) is proved.

Let $W = \phi^{-1}(A^+)$. Then W is a closed type-0 locally multiplicative cone and $W = \{x \in R : \mathrm{Sp}(x) \subset \boldsymbol{R}^+\}$. Let $x,y \in R$ and $\lambda \in \boldsymbol{R}$ with $\lambda > r(\frac{1}{2}(xy+yx))$. Then $2\lambda - (xy+yx) \in W \cap \mathrm{Inv}(B)$ and so $2\lambda - (xy+yx) = z^{-2}$ with $z \in W$. Therefore

$$\begin{aligned}2\lambda - 2xy &= 2\lambda - (xy+yx) - (xy-yx)\\ &= z^{-1}\{1 - z(xy-yx)z\}z^{-1}\end{aligned}$$

By (10), it follows that $\lambda - xy \in \mathrm{Inv}(B)$; and we have proved that $\lambda \notin \mathrm{Sp}(xy)$ whenever $\lambda > r(\frac{1}{2}(xy+yx))$. Replacing x by $-x$, we see that $\lambda \notin \mathrm{Sp}(xy)$ whenever $-\lambda > r(\frac{1}{2}(xy+yx))$, and so

$$\lambda \in \mathbf{R} \cap \mathrm{Sp}(xy) \Rightarrow |\lambda| \leq r(\tfrac{1}{2}(xy + yx)).$$

But, by (**2**, Theorem 4), $r(\frac{1}{2}(xy + yx)) \leq r(x)r(y)$, and so (iii) is proved; and (i), (ii), (iv) have been proved in Corollary 5.

I owe the following example to M. A. Youngson. This not only shows that we can have $W \cap (-W) = \{0\}$ without having boundedness of $\{\|(1 + w)^{-1}\| : w \in W\}$, but also settles a question asked in (**2**, p.247) concerning the existence of square roots.

Example 8. Take B to be the complex Banach algebra $C_1[0, 1]$ of all continuous complex functions on $[0, 1]$ with continuous first derivatives there, with the usual norm $\|x\| = \sup\{|x(s)| : 0 \leq s \leq 1\} + \sup\{|x'(s)| : 0 \leq s \leq 1\}$. Let V be the set of all non-negative real valued functions belonging to B. Plainly V is a closed type-0 locally multiplicative cone. As is no doubt well known, V is not a normal cone; for example consider $u,v \in V$ given by $u(s) = s^n$, $v(s) = 1 - s^n$ $(0 \leq s \leq 1)$. It follows that the set $\{\|(1 + w)^{-1}\| : w \in V\}$ is not bounded, as can also be verified directly without difficulty. Moreover, the element w of V given by $w(s) = s$ $(0 \leq s \leq 1)$ has no square root in V; so that $W \cap (-W) = \{0\}$ is not sufficient to give the existence of square roots of elements of W (see (**2**, p. 247)). In this connection it should also be noted that if elements of W have at most one square root in W, then $W \cap (-W) = \{0\}$. For if $h \in W \cap (-W)$, then h^2 is an element of W with the square roots h and $-h$ in W.

In the light of Example 8 it is of interest, when $W \cap (-W) = \{0\}$, to consider the completion of $W - W$ with respect to the spectral radius norm. By using the full force of the characterisation of special JB-algebras in (**1**), we show that the completion is still a special JB-algebra.

Theorem 9. *Let W be a closed type-*0 *locally multiplicative cone in B, and let $R = W - W$. Then r is a norm on R, and the completion of R with the natural extension of r and of the Jordan product on R, is a special JB-algebra.*

Proof. Since $W \cap (-W) = \{0\}$, Theorems 3 and 4 in (**2**) show that r is a norm on the Jordan algebra R and that

$$r(x \circ y) \leq r(x)r(y) \quad (x,y \in R), \tag{11}$$

where $x \circ y = \frac{1}{2}(xy + yx)$. Let S denote the completion of R with respect to the norm r and let r denote also the natural extension of the norm r to S. Then S with the extended norm r is a real Banach space. By (11), the Jordan product extends by continuity to a mapping: $S \times S \to S$ which we denote also by $x \circ y$. It is routine to check that $x \circ y$ is a distributive and commutative product on S, that $\alpha(x \circ y) = (\alpha x) \circ y = x \circ (\alpha y)$ for $\alpha \in \mathbf{R}$ and that $x^2 \circ (y \circ x) = (x^2 \circ y) \circ x$. Thus S with this product is a Jordan algebra.

Given $x,y \in S$, choose Cauchy sequences $\{x_n\}$, $\{y_n\}$ in R corresponding to x,y respectively. Then, since the proof of (4) does not use the boundedness of the set $\{(1 + w)^{-1} : w \in W\}$, we have

$$r(x^2 + y^2) = \lim_{n\to\infty} r(x_n^2 + y_n^2) \geq \lim_{n\to\infty} r(x_n^2) = r(x^2),$$

and $r(x^2) = \lim_{n\to\infty} r(x_n^2) = \lim_{n\to\infty}(r(x_n))^2 = (r(x))^2$. We have now proved that (S,r) is a JB-algebra. To prove that S is special, let f be a real polynomial in three variables that vanishes on all special Jordan algebras but not on M_3^8, the exceptional formally real simple Jordan algebra of finite dimension. Then $f(x,y,z) = 0$ for all $x,y,z \in R$ and therefore also (by continuity) for all $x,y,z \in S$. Therefore, by (**1**, Theorem 9.4), S is a special JB-algebra.

Corollary 10. *Let W, R be as in Theorem 9. Then the Jordan algebra R is isomorphic to a dense Jordan subalgebra of a JC-algebra.*

Proof. Let S be the completion of R with respect to the norm r. Then S with the extended norm r is a special JB-algebra and is therefore isometrically isomorphic to a JC-algebra.

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