ON THE EXISTENCE OF SOLUTION OF THE GREENSPAN-CARRIER EQUATION

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Introduction

We are concerned with the flow of a viscous incompressible electrically conducting fluid of constant properties past a semi-infinite rigid plate. The governing boundary layer equations were derived by Greenspan and Carrier [2] in 1959. Numerical solutions of these equations subject to different boundary conditions have been considered by Stewartson and Wilson [5], Wilson [8], and recently by Bramley [1]. In this paper, we consider boundary conditions corresponding to flow past a thin plate, and show that the system admits at least one solution when a parameter is suitably restricted. The existence proof is a modification of methods used by Weyl [7], Serrin and McLeod [4], and Tam [6].

The Boundary Value Problem

The boundary value problem to be considered is

(1′) $$f''' + ff'' - \gamma g g'' = 0, \qquad \gamma > 0;$$

(2′) $$g'' + \varepsilon f g' - \varepsilon f' g = 0, \qquad \varepsilon > 0;$$

(3′) $$f(0) = f'(0) = g(0) = 0,$$

(4′) $$f'(\infty) = g'(\infty) = 2.$$

We observe that the parameter γ can be absorbed by writing $h = \sqrt{\gamma}\, g$. Instead of the above system, we consider the boundary value problem (BVP):

(1) $$f''' + ff'' - hh'' = 0,$$

(2) $$h'' + \varepsilon f h' - \varepsilon f' h = 0,$$

(3) $$f(0) = f'(0) = h(0) = 0,$$

(4) $$f'(\infty) = 2, \; h'(\infty) = 2\sqrt{\gamma}.$$

We replace (4) by the additional initial conditions

$$f''(0) = \beta, \quad h'(0) = \alpha, \tag{5}$$

and consider the initial value problem (1), (2), (3) and (5). The existence of a solution in a neighbourhood of $x = 0$ is guaranteed by the local existence theorem. Further, if (1) and (2) are written as a first order system, it is readily seen that the system is Lipschitzian as long as f' and h' remain uniformly bounded. In that case, the solution of the initial value problem can be continued for all x. Our aim is to show that there exists some (α, β) for which the solution of (1), (2), (3) and (5) can be continued and has the correct limiting behaviour as required by (4).

Preliminary Observations

Throughout this paper, the quantity x_i is used as a generic symbol. Unless specified explicitly, the x_i's used in different lemmas are unrelated. Further, in considering the limit of a function $f(x)$ as x tends to infinity, we shall simply write $\lim f$.

From the physics of the problem, we have the additional information that the desired solution must be such that f' and h' are positive. We shall therefore limit our attention to the case of α, $\beta > 0$ only. For subsequent reference, the initial values of f and h, and derivatives up to the fifth order as given in (3), (5) and derived from (1), (2) are listed as follows:

$$f(0) = f'(0) = 0, f''(0) = \beta, f'''(0) = f''''(0) = 0, f^{v}(0) = -\beta^2$$

$$h(0) = 0, \ h'(0) = \alpha, \ h''(0) = h'''(0) = 0, \ h''''(0) = \varepsilon\alpha\beta, \ h^{v}(0) = 0.$$

We first observe that solutions of the initial value problem (1), (2), (3) and (5) have the following property:

LEMMA 1. *For $\alpha\beta > 0$, suppose f and h are solutions of* (1), (2), (3) *and* (5), *then $f'' > 0$, $h'' > 0$ for $x > 0$.*

PROOF. We obtain on differentiating (2)

$$h''' + \varepsilon f h'' - \varepsilon f'' h = 0. \tag{6}$$

Suppose f'' and h'' do not remain positive, then either f'' and h'' vanish together, or one vanishes before the other. If f'' and h'' vanish simultaneously at x_0, say, then it follows from the differential equations and the analyticity of the solutions that $f^{(n)}(x) = h^{(n)}(x) = 0$ for $n \geqq 2$ in $[x_0, \infty)$. Indeed, by extending the solution to the left of x_0, we see that $f^{(n)}(x) = h^{(n)}(x) = 0$ for $n \geqq 2$ in $(0, \infty)$, which contradicts $f''(0) = \beta > 0$. Hence f'' and h'' do not vanish simultaneously.

If f'' vanishes before h'', then there exists an x_0 such that $f''(x_0) = 0$; and $f''(x) > 0$, $h''(x) > 0$ for $0 < x < x_0$. From (1), we have $f'''(x_0) = h(x_0)h''(x_0) > 0$,

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which clearly contradicts the fact that $f'' > 0$ in $(0,x_0)$. Hence f'' does not vanish before h''. In the remaining case, a contradiction is obtained in the same way by using (6).

It follows from Lemma 1 that, for $\alpha, \beta > 0$, f, h, f', h', f'' and h'' are all positive for $x > 0$. Hence, we have from (2)

$$f'h - fh' > 0, \qquad x > 0,$$

$$\left(\frac{f}{h}\right)' > 0, \qquad x > 0, \tag{7}$$

which implies that the function f/h is monotonic increasing.

LEMMA 2. *For $\varepsilon \leqq 1$, if there exists an x_1 at which $f'(x_1) > h'(x_1)$, then $f'(x) > h'(x)$ for $x > x_1$.*

PROOF. Since $h'(0) = \alpha > f'(0) = 0$, the hypothesis $f'(x_1) > h'(x_1)$ implies there is an $x_0 < x_1$ at which $f'(x_0) = h'(x_0)$. Suppose the lemma is false; then there exists an $x_2 > x_1$ such that $f'(x_2) = h'(x_2)$. Further, there exists an $\bar{x}$, $x_0 < \bar{x} < x_2$, at which $f''(\bar{x}) - h''(\bar{x}) = 0$ and $f'''(\bar{x}) - h'''(\bar{x}) \leqq 0$.

Subtracting (6) from (1), we have at $\bar{x}$:

$$(f''' - h''') = (1 - \varepsilon)(h - f)f''. \tag{8}$$

Hence, for $\varepsilon < 1$, it follows that $h(\bar{x}) - f(\bar{x}) \leqq 0$. Since $f' > h'$ for $\bar{x} \leqq x < x_2$ we have $f(x_2) > h(x_2)$. It then follows from (2) that $h''(x_2) < 0$, which clearly contradicts Lemma 1. Hence we have $f' > h'$ for $x > x_1$.

For $\varepsilon = 1$, we obtain again by subtracting (6) from (1)

$$(f''' - h''') + (f + h)(f'' - h'') = 0,$$

from which follows

$$(f'' - h'') = \beta \exp\left[-\int_0^x (f + h)dt \right] > 0.$$

Hence there is no $\bar{x}$ at which $f'' - h'' = 0$. Again we conclude $f' > h'$ for $x > x_1$.

No analogous statement can be made for $\varepsilon > 1$. Indeed, we observe that if $(f' - h')$ vanishes at x_0 and x_2, then from (8) we have

$$h(\bar{x}) \geqq f(\bar{x}),$$

and hence from (1), $f'''(\bar{x}) \geqq 0$. We shall make use of this observation in the next section.

We now want to show that, for $\gamma > 1$, no solution to the boundary value problem exists. This non-existence result has been shown by Reuter and Stewartson [3]. It is included here for the sake of completeness, and because it motivates the approach used in the existence proof.

LEMMA 3. *If $\gamma > 1$, the BVP has no solution for $\varepsilon > 0$.*

PROOF. Suppose to the contrary that a solution exists; then we have lim $\dfrac{f}{h} = \dfrac{1}{\sqrt{\gamma}} < 1$, and $\lim h'' = \lim f'' = 0$. Now since $h > f$ in a neighbourhood of $x = 0$, we have either (i) $h = f$ at some x_0, or (ii) $h > f$ for all x. If (i) occurs, then since f/h is an increasing function, we have $f/n > 1$ for $x > x_0$, contradicting the condition that $\lim f/h = 1/\sqrt{\gamma} < 1$. If (ii) occurs, then by subtracting (1) from (6), and using $h > f$, we readily obtain

$$(h''' - f''') + (1 + \varepsilon)(hh'' - ff'') > 0.$$

Hence, it follows from (1)

$$(h''' - f''') + (1 + \varepsilon)f''' > 0,$$

and so

$$h''' + \varepsilon f''' > 0,$$

for all $x > 0$. Integrating, we have

$$h'' + \varepsilon f'' > \varepsilon\beta > 0,$$

which clearly contradicts $\lim h'' = \lim f'' = 0$. Hence, the BVP has no solution for $\varepsilon > 0$, $\gamma > 1$.

The proof of Lemma 3 suggests that for a solution of the BVP to exist, it is necessary to have $f > h$ at some x. Further, the observation made following the proof of Lemma 2 shows that for $\varepsilon > 1$, it is possible to have $f''' \geqq 0$, which is obviously an undesirable feature, bearing in mind the fact that f'' is related to the vorticity of the fluid, and we expect the vorticity to decay, that is $f''' < 0$. We are thus led to consider only situations in which $f''' < 0$, a condition that is guaranteed when α and β satisfy a certain relation to be defined in the next section.

The Existence Proof

The proof is greatly simplified by the following transformation. Suppose $[f(x), h(x)]$ is a solution pair of (1), (2) and (3). Then for any positive k, $[kf(kx), kh(kx)]$ is also a solution pair. This transformation is well known in the existence proof for the Blasius equation [7]. If the prime is now used to denote differentiation with respect to kx, then equations (1) and (2) and the initial condition (3) remain invariant. The first derivatives df/dx and dh/dx transform to $k^2\, df(kx)/d(kx)$ and $k^2\, dh(kx)/d(kx)$. The freedom gained by introducing k can now be used to fix one of the two parameters $h'(0) = \alpha$; $f''(0) = \beta$. The problem is thus reduced to showing that for a fixed α, the initial value problem (1), (2), (3) and (5) has a solution such that f' and h' tend to finite constants, and that there is some β^* for which $f'(\infty)/h'(\infty) = 1/\sqrt{\gamma}$, when γ is restricted to some interval.

Suppose for $\beta=\beta^*, f'(\infty)=c$, then by choosing $k^2=2/c$ we have the required conditions $df(\infty)/dx=2$ and $dh(\infty)/dx=2\sqrt{\gamma}$. In the following, we suppose this transformation has been made, and we take $h'(0)=1$. We then proceed to show that there is an open set S of values of β for which $f'(\infty)/h'(\infty)$ is a continuous function of β and it maps S into the half line $(\delta_0(\varepsilon),\infty)$ where $\delta_0(\varepsilon)>1$. This implies that there exists a positive number $\sqrt{\gamma_0(\varepsilon)}=1/\delta_0(\varepsilon)$ such that for $0<\gamma<\gamma_0$, the BVP has at least one solution. The existence proof is established by proving a series of lemmas, the contents of which also exhibit some salient features of the solution curves.

LEMMA 4. *There is a non-empty open set S of values of β such that for $\beta\in S$, there is a point x_β at which $f(x_\beta)>h(x_\beta)$, and $f'''<0$ for $x\in(0,x_\beta]$.*

PROOF. We have $f''(0)=\beta, f'''(0)=f''''(0)=0$ and $f^{v}(0)=-\beta^2$, so that f''' is negative in a right neighbourhood of $x=0$.

As long as $f'''<0$, we have $ff''>hh''$, $f''<\beta$, and $f<\beta x^2/2$. Since $h''>0$ $h'(0)=1$ implies $h>x$, it follows that $h''<\beta^2x/2$, and hence we have

$$h<x+\frac{\beta^2x^3}{12}.$$

Now, from $f'''+ff''>0$, we readily obtain

$$f''>\beta\exp\left[-\frac{\beta x^3}{6}\right],$$

and from (6), we have

$$\begin{aligned} h'' &= \varepsilon\exp\left[-\varepsilon\int_0^x f\,dt\right]\int_0^x f''h\exp\left[\varepsilon\int_0^t f\,du\right]dt \\ &< \varepsilon\beta\int_0^x h\,dt \\ &< \frac{\varepsilon\beta x^2}{2}\left(1+\frac{\beta^2x^2}{24}\right). \end{aligned}$$

Hence, we have

$$f''-h''>\beta\exp\left[-\frac{\beta x^3}{6}\right]-\frac{\varepsilon\beta x^2}{2}\left(1+\frac{\beta^2x^2}{24}\right)\equiv P(x;\beta,\varepsilon). \tag{9}$$

Clearly, $P(x;\beta,\varepsilon)$ is positive in a right neighbourhood of $x=0$, in which $f'''<0$. We write $u=\beta x^3/6$, and consider

$$P(u;\beta,\varepsilon)=\beta e^{-u}-\frac{\varepsilon\beta}{2}\left(\frac{6}{\beta}\right)^{\frac{2}{3}}u^{\frac{2}{3}}\left[1+\frac{\beta^2}{24}\left(\frac{6}{\beta}\right)^{\frac{2}{3}}u^{\frac{2}{3}}\right].$$

Let $J = (0, u^*)$ be the maximal open interval in which $P(u; \beta, \varepsilon) > 0$. Then by using $e^{-u} > (1-u)$, it is clear that u^* is greater than the zero of

$$\beta(1-u) - \frac{\varepsilon\beta}{2}\left(\frac{6}{\beta}\right)^{\frac{2}{3}} u^{\frac{2}{3}}\left[1 + \frac{\beta^2}{24}\left(\frac{6}{\beta}\right)^{\frac{2}{3}} u^{\frac{2}{3}}\right],$$

which is clearly less than $u = 1$. Since $u^{\frac{2}{3}} > u$ for $0 < u < 1$, u^* is in turn greater than the zero of

$$\beta(1-u^{\frac{2}{3}}) - \frac{\varepsilon\beta}{2}\left(\frac{6}{\beta}\right)^{\frac{2}{3}} u^{\frac{2}{3}}\left[1 + \frac{\beta^2}{24}\left(\frac{6}{\beta}\right)^{\frac{2}{3}} u^{\frac{2}{3}}\right],$$

which is

$$\tilde{u} = \left[\frac{(B^2 + 4A\beta)^{\frac{1}{2}} - B}{2A}\right]^{\frac{3}{2}} = N^{\frac{3}{2}};$$

where

$$A = \frac{\varepsilon\beta^3}{48}\left(\frac{6}{\beta}\right)^{4/3}; \quad B = \frac{\varepsilon\beta}{2}\left(\frac{6}{\beta}\right)^{\frac{2}{3}} + \beta.$$

In terms of x, we have $P(x; \beta\varepsilon) > 0$ for $x \in J = (0, x^*)$, where

$$x^* > \tilde{x} \equiv \left(\frac{6}{\beta}\right)^{\frac{1}{3}} N^{\frac{1}{2}}.$$

Since

$$B^2 + 4A\beta > \frac{\varepsilon}{4}(6)^{4/3}\beta^{8/3}\left(\varepsilon + \frac{1}{3}\right),$$

it is readily verified that

$$N > \frac{12}{6^{\frac{2}{3}}}\frac{1}{\beta^{\frac{1}{3}}}\left[\left(1 + \frac{1}{3\varepsilon}\right)^{\frac{1}{2}} - 1\right] = N_0. \tag{10}$$

For $x \in J$, we obtain by integrating (9)

$$f' - h' > \beta x \exp\left[-\frac{\beta x^3}{6}\right] - \frac{\varepsilon\beta x^3}{6} - \frac{\varepsilon\beta^3 x^5}{240} - 1,$$

$$f - h > \frac{\beta x^2}{2}\exp\left[-\frac{\beta x^3}{6}\right] - \frac{\varepsilon\beta x_4}{24} - \frac{\varepsilon\beta^3 x^6}{1440} - x.$$

Using $P(x; \beta, \varepsilon) > 0$ for $x \in J$, we have

$$f - h > \frac{5}{24}\varepsilon\beta x^4 + \frac{\varepsilon\beta^3 x^6}{720} - x.$$

At $x = \tilde{x}$, we have

$$f(\tilde{x}) - h(\tilde{x}) > \frac{5\varepsilon\beta}{24}\left(\frac{6}{\beta}\right)^{4/3} N^2 + \frac{\varepsilon\beta^3}{720}\left(\frac{6}{\beta}\right)^2 N^3 - \left(\frac{6}{\beta}\right)^{\frac{1}{3}} N^{\frac{1}{2}}$$

$$\equiv R(\beta, \varepsilon).$$

Since $N_0 = 0(1/\beta^{\frac{1}{3}})$, it is clear that $R(\beta, \varepsilon) > 0$ for β sufficiently large. In fact, if we choose β so that

$$\text{(11)} \qquad \frac{\varepsilon\beta^3}{720}\left(\frac{6}{\beta}\right)^2 N_0^{5/2} - \left(\frac{6}{\beta}\right)^{\frac{1}{3}} \geqq 0,$$

$f(\tilde{x}) - h(\tilde{x})$ will be positive. The inequality (11) implies

$$\beta \geqq \left[\frac{30(24)}{12^{5/2}\,\varepsilon}\right]^2 \left[\left(1 + \frac{1}{3\varepsilon}\right)^{\frac{1}{2}} - 1\right]^{-5}.$$

Using $(1 + 1/3\varepsilon)^{\frac{1}{2}} < 1 + (1/3\varepsilon)^{\frac{1}{2}}$, we can simplify the above to

$$\beta \geqq \frac{75\sqrt{3}}{4}\varepsilon^{\frac{1}{2}} = \beta_0,$$

so that β is in S if $\beta \geqq \beta_0$. Hence S is non-empty as claimed.

Since the solution of the initial value problem depends continuously on $h'(0)$ and $f''(0)$, the set S is open.

LEMMA 5. *$f'' > h''$ for all x if $\varepsilon = 1$ and $f'' > h''$ for all x if $\varepsilon > 1$ and $\beta \in S$.*

PROOF. The first part of the lemma is simply a restatement of what has been proved in the last part of Lemma 2. For $\varepsilon > 1$ and $\beta \in S$, since $f''(0) = \beta > 0$ and $h''(0) = 0$, it is clear that $f'' - h'' > 0$ in a right neighbourhood of $x = 0$. Suppose x_1 is the first zero of $f'' - h''$. Then it is clear that $f'''(x_1) - h'''(x_1) \leqq 0$. From (8) we have $h(x_1) \geqq f(x_1)$. But $\beta \in S$ implies that there exists an x_2 such that $f(x_2) = h(x_2)$ and that $f''' < 0$ for $0 < x \leqq x_2$. If $x_2 \geqq x_1$, we have $f'''(x_1) < 0$. It then follows from (1) that $f(x_1) > h(x_1)$, contradicting the supposition that $h(x_1) \geqq f(x_1)$. If $x_2 < x_1$, then since f/h is increasing, we have $f(x_1) > h(x_1)$. Again, the same contradiction results. Hence $f'' - h''$ must remain positive for all x.

LEMMA 6. *If $\varepsilon < 1$, and $\beta \in S$, then $f'' - h''$ vanishes at only one point and then remains negative.*

PROOF. Suppose to the contrary that $f'' > h''$ for all x. Then from (1) we have

$$\text{(12)} \qquad f''' + ff'' < hf''.$$

Since $\beta \in S$, let x_0 be the first zero of $f(x) - h(x)$. Integrating (12), and using the fact that $f''(x_0) < \beta$ and $f/h > 1$ for $x > x_0$, we have

$$f''(x) < \beta \exp\left[-\int_{x_0}^{x} (f - h)dt\right].$$

In the same manner, we have

$$h''(x) > h''(x_0) \exp\left[-\varepsilon\int_{x_0}^{x} (f - h)dt\right].$$

Hence $f'' > h''$ implies

$$\beta > h''(x_0) \exp\left[(1-\varepsilon)\int_{x_0}^{x}(f-h)dt\right],$$

which clearly is impossible since $h''(x_0) > 0$ and $f - h > 0$ for $x > x_0$. Hence $f'' - h''$ must vanish at some point. Let x_1 be the first zero of $f'' - h''$. Clearly, we have $f'''(x_1) - h'''(x_1) \leqq 0$. By subtracting (6) from (1), we have

$$(f''' - h''') + (f+h)(f'' = h'') = \left(\frac{1}{\varepsilon} - 1\right)h''. \tag{13}$$

If $f'''(x_1) = h'''(x_1)$, the above equation implies $f'''(x_1) = h'''(x_1) = 0$. It then follows from (8) that $f(x_1) = h(x_1)$; and for $\beta \in S$, $f'''(x_1) < 0$, which contadicts $f'''(x_1) = 0$.. Hence, we must have $f'''(x_1) < h'''(x_1)$. Now (13) implies $h'''(x_1) < 0$ and (8) implies $f(x_1) > h(x_1)$. If $f'' - h''$ vanishes at a second point, say x_2, then $f'''(x_2) - h'''(x_2) \geqq 0$, and it follows from (8) that $h(x_2) \geqq f(x_2)$. This is impossible since f/h is an increasing function. Hence $h'' > f''$ for $x > x_1$.

We now use Lemmas 5 and 6 to obtain some further results regarding the properties of the solutions of the initial value problem when $\beta \in S$.

LEMMA 7. *For $\beta \in S$, f' and h' tend to finite limits; and $f''' < 0$ for all x.*

PROOF. We have shown in Lemma 5 that for $\varepsilon \geqq 1$, $f'' > h''$ for all x if $\beta \in S$. Hence, we have $f''' + (f-h)f'' < 0$, and

$$f'' < \beta \exp\left[-\int_0^x (f-h)dt\right].$$

Since $f - h > 0$ ultimately, f'' ultimately tends to zero exponentially and hence f'' is integrable. That $f'' > h''$ implies h'' is also integrable.

For $\varepsilon < 1$, and $\beta \in S$, we have seen in Lemma 6 that there is an x_1 at which $f'' = h''$, and $f(x) > h(x)$, $h''(x) > f''(x)$ for $x > x_1$. Hence we have

$$h''' + \varepsilon f h'' < \varepsilon h'' h, \qquad x > x_1;$$

and

$$h'' < h''(x_1) \exp\left[-\varepsilon\int_{x_1}^{x}(f-h)dt\right],$$

which implies that h'', and hence f'', are integrable. Hence both f' and h' tend to finite constants as x tends to infinity.

To show that $f''' < 0$, suppose the contrary is true. Then there exists an x_2 at which $f'''(x_2) = 0$ and $f''''(x_2) \geqq 0$. By definition, $\beta \in S$ implies that $f'''(x) < 0$ for $x \in (0, x_\beta]$. Since $f'''(x_2) = 0$, we must have $x_2 > x_\beta$. That $f(x_\beta) > h(x_\beta)$ and f/h is increasing then imply $f(x_2) > h(x_2)$.

Now $f'''(x_2) = 0$ and $h''(x_2) > 0$ imply that at x_2, we have

$$\frac{h''}{f''} = \frac{f}{h} \text{ and } \frac{f'}{h'} > \frac{f}{h}.$$

We have from (1)

$$f'''' + ff''' + f'f'' = hh''' + h'h''. \tag{14}$$

Substituting $h''' = \varepsilon(f''h - fh'')$ in (14), we readily obtain

$$\frac{f''''(x_2)}{f''(x_2)} < \varepsilon[h^2(x_2) - f^2(x_2)],$$

so that $f''''(x_2) < 0$, contradicting the condition that $f''''(x_2) \geqq 0$. Hence we have $f''' < 0$ for all x.

It then remains to show that $f'(\infty)/h'(\infty)$ is a continuous function of β, and that there is a $\gamma_0 < 1$ such that for a given $\tilde{\gamma} < \gamma_0$, there is at least one $\tilde{\beta}$ for which $f'(\infty)/h'(\infty) = 1/\sqrt{\tilde{\gamma}}$. We proceed by first proving the following lemma.

LEMMA 8. *$f'(\infty)/h'(\infty)$ is a continuous function of β.*

PROOF. We observe that for $\beta \in S$, f and h both tend to linear functions of x, and so

$$\lim_{x \to \infty} \frac{f(x)}{h(x)} = \frac{f'(\infty)}{h'(\infty)}.$$

Hence, it suffices to show $f(\infty)/h(\infty)$ is a continuous function of β.

Now from (2), a simple manipulation gives

$$\frac{f(x)}{h(x)} = \frac{1}{\varepsilon} \int_0^x \frac{h''}{h^2} dt.$$

Regardless of the value of ε, it is clear from Lemma 7 that $h'' < \beta$. For β in any open set (β_0, β_1), we have

$$\frac{1}{\varepsilon} \frac{h''(x)}{h^2(x)} < \frac{\beta_1}{\varepsilon} \frac{1}{x^2} \equiv \mu(x). \tag{15}$$

Now $\mu(x)$ is clearly integrable in any open interval (a, ∞) for $a > 0$. Hence $f(\infty)/h(\infty)$ is a continuous function of β.

We now complete the existence proof by showing that there is at least one β for which $f'(\infty)/h'(\infty) = \sqrt{1/\gamma}$, for a given $\gamma < \gamma_0 < 1$.

We first obtain upper and lower bounds for $f(\infty)/h(\infty)$ as follows:
We have

$$\frac{f(\infty)}{h(\infty)} = \frac{f(1)}{h(1)} + \frac{1}{\varepsilon} \int_1^\infty \frac{h''}{h^2} dt$$

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$$< \frac{\beta}{2} + \frac{\beta}{\varepsilon} \int_1^\infty \frac{dt}{t^2} = \beta\left(\frac{\varepsilon + 2}{2\varepsilon}\right).$$

From $f''' + ff'' > 0$, and $h'' < \varepsilon f'h + \varepsilon fh'$, we readily obtain

$$f > \beta x^2/2 \ \exp\left[-\int_0^x f dt\right] \text{ and } h < x \exp\left[\varepsilon \int_0^x f dt\right].$$

These inequalities hold for all $x > 0$. Hence, we have

$$\frac{f(x)}{h(x)} > \frac{\beta x}{2} \exp\left[-(1+\varepsilon)\int_0^x f dt\right].$$

Further, for $\beta \in S$, we have $f < \beta x^2/2$, so that

$$\frac{f(x)}{h(x)} > \frac{\beta x}{2} \exp\left[-(1+\varepsilon)\frac{\beta x^3}{6}\right], \qquad x > 0.$$

The right side of the above inequality has the maximum value $[\beta^2/4e(1+\varepsilon)]^{\frac{1}{3}}$, and since f/h is an increasing function, we have

$$\frac{f(\infty)}{h(\infty)} > \left[\frac{\beta^2}{4e(1+\varepsilon)}\right]^{\frac{1}{3}}.$$

Hence, $f(\infty)/h(\infty)$ becomes unbounded as β tends to infinity; but for any $\beta \in S$, it is bounded from above by $\beta(\frac{1}{2} + 1/\varepsilon)$. In particular, for $\beta = \beta_0 = 75\sqrt{3\varepsilon}/4$, we have

$$\frac{f(\infty)}{h(\infty)} < \frac{75\sqrt{3\varepsilon}}{4}\left(\frac{1}{2} + \frac{1}{\varepsilon}\right) \equiv \delta_0(\varepsilon).$$

That $f(\infty)/h(\infty)$ depends continuously on β implies that its value ranges over $(\tilde{\delta}, \infty)$ where $\tilde{\delta} < \delta_0(\varepsilon)$. Hence for any γ in $0 < \gamma < \gamma_0(\varepsilon) \equiv [1/\delta_0(\varepsilon)]^2$, there is at least one $\beta \in S$ for which

$$\frac{f(\infty)}{h(\infty)} = \frac{f'(\infty)}{h'(\infty)} = \frac{1}{\sqrt{\gamma}}.$$

Since $\delta_0(\varepsilon) > 1$, it is clear that the value $\gamma_0(\varepsilon)$ obtained is less than unity, and that it is less than the "true" value of γ below which a solution can exist because of the rough approximations used.

Concluding Remarks

We first sum up the results in the main theorem:

THEOREM. *The boundary value problem has at least one solution for* $0 < \gamma < \gamma_0$, *where*

$$\gamma_0^{\frac{1}{2}} = \frac{2\varepsilon}{2+\varepsilon} \frac{4}{75\sqrt{3\varepsilon}} < 1.$$

We next observe that when a solution to the BVP exists, then f and g can be expected to have the same displacement thickness, as defined below. We have seen from Lemma 1 that h', and hence g', has no stationary value. Hence if a solution exists, g' must tend to 2 from below. We define

$$G(x) = \int_0^x (g' - 2)dt,$$

and we expect

$$\lim_{x\to\infty} \int_0^x (g' - 2)dt = G(\infty) = -K,$$

where $0 < K < \infty$ is the displacement thickness for g. Substituting $g = 2x + G$ into (2′), we have

$$G'' + \varepsilon f G' = \varepsilon f' G + 2\varepsilon x f' - 2\varepsilon f.$$

We expect G' to tend to zero sufficiently fast so that $\lim_{x\to\infty} (G'' + \varepsilon f G') = 0$. Hence, if $f \sim 2x - K_1$, where K_1 is the displacement thickness for f, it follows from $\lim_{x\to\infty} (f'G + 2xf' - 2f) = 0$ that $K_1 = K$.

Lastly, we mention that the thick plate problem where the initial condition $h(0) = 0$ is replaced by $h'(0) = 0$ can be handled in the same manner.

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