NOTES ON CONGRUENCES ON REGULAR SEMIGROUPS. I

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Abstract

Four properties of congruences on a regular semigroup \( S \) are studied and compared. Let \( \mathcal{R}, \mathcal{L}, \) and \( \mathcal{D} \) denote Green's relations and let \( V = \{(a, b) \in S \times S | a \) and \( b \) are mutually inverse\}. A congruence \( \rho \) on \( S \) is (1) \textit{rectangular} provided \( \rho \cap \mathcal{D} = (\rho \cap \mathcal{L}) \circ (\rho \cap \mathcal{R}) \), (2) \textit{\( \mathcal{V} \)-commuting} provided \( \rho \circ V = V \circ \rho \), (3) \textit{(\( \mathcal{L}, \mathcal{R} \))-commuting} provided \( \mathcal{L} \circ \rho = \rho \circ \mathcal{L} \) and \( \mathcal{R} \circ \rho = \rho \circ \mathcal{R} \), and (4) \textit{idempotent-regular} provided each idempotent \( \rho \)-class is a regular subsemigroup of \( S \).

A rectangular congruence is \((\mathcal{L}, \mathcal{R})\text{-commuting}\) and a \( \mathcal{V} \)-commuting congruence is idempotent-regular. If \( \rho \) is idempotent-regular and \((\mathcal{L}, \mathcal{R})\text{-commuting}\) then \( \rho \) is \( \mathcal{V} \)-commuting. Examples and conditions are given to show what other implications among the four properties hold. In addition to characterizations of the properties, these are studied in the presence of other conditions on \( S \). For example, if \( S \) is a stable regular semigroup, then each congruence under \( \mathcal{D} \) is rectangular.


We introduce several properties of congruences on regular semigroups and study their inter-relationships. Let \( \rho \) be a congruence on a regular semigroup \( S \).

(1) \( \rho \) is \textit{rectangular} if whenever \( apb \) and \( a\mathcal{D}b \), there exists \( x, y \in S \) with \( a\mathcal{R}x \mathcal{L}b \mathcal{R}y \mathcal{L}a \) and \( apxypb \).

(2) \( \rho \) is \textit{\( \mathcal{V} \)-commuting} if \( \rho \circ V = V \circ \rho \) where \( V = \{(a, b) \in S \times S | a \) and \( b \) are mutually inverse\}.

(3) \( \rho \) is \textit{(\( \mathcal{L}, \mathcal{R} \))-commuting} if \( \rho \circ \mathcal{L} = \mathcal{L} \circ \rho \) and \( \rho \circ \mathcal{R} = \mathcal{R} \circ \rho \).

(4) \( \rho \) is \textit{idempotent-regular} if \( I^2 = I \in S/\rho \) implies that the \( \rho \)-class \( I \) is a regular subsemigroup of \( S \).

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Properties (2) and (3) were introduced by the authors [6]. Property (4) was introduced by Nambooripad [9]. These notions have been applied by one of the authors in [7].

Among the results that we establish is the fact that if $S$ is a stable regular semigroup, then each congruence under $\mathcal{D}$ is rectangular. We give various characterizations of properties (1), (2), and (3), some of which involve the "fullness" of multiplication of certain equivalence classes. We characterize the property $\rho \circ V = V \circ \rho$ in terms of properties of the decomposition induced by $\rho$ on the set of idempotents. We discuss the class $(VC)$ of semigroups on which each congruence is $V$-commuting and show that $(VC)$ contains the full transformation semigroups. Also, we present several examples to delineate the results. We are indebted to K. Byleen for an observation about the four-spiral semigroup (Example 5.3) and to J. D. Lawson for the initial part of Example 5.2.

1. Notation and conventions

We assume throughout the paper that $S$ is a regular semigroup. Let $\rho$ be a congruence on $S$. We denote the natural map $S \to S/\rho$ by $\rho$ or $\phi$ and for $C \in S/\rho$ we denote $\phi^{-1}(C)$ by $\bar{C}$ to indicate it as a subset of $S$ rather than as an element of $S/\rho$. Elements of $S/\rho$ will be represented by upper case letters with $E$, $F$, $I$, and $K$ being idempotents. Elements of $S$ will be lower case letters with $e$, $f$, $i$, $j$, $k$, and $h$ being idempotents.

The notation $p \perp q$ indicates that $p$ and $q$ are mutually inverse elements, and $V = \{(p, q) : p \perp q\}$. For a relation $\mathcal{K}$ on $S$, we will use the notation $(a, b) \in \mathcal{K}$ and $a \mathcal{K} b$ interchangeably and $\mathcal{K}_x$ will represent the $\mathcal{K}$-class of $x$. Terminology from Clifford and Preston [3] will be used, and we make free use of Green's translational lemmas and the results of Miller and Clifford on the algebra of $\mathcal{D}$-classes.

2. Rectangular congruences

As above, a congruence $\rho$ on a regular semigroup $S$ is said to be rectangular provided $\rho \cap \mathcal{D} = (\rho \cap L) \circ (\rho \cap R)$. We note that $\rho \cap \mathcal{D} \supset (\rho \cap L) \circ (\rho \cap R)$ always.

**Proposition 2.1.** Any congruence $\rho$ on a semigroup $S$ satisfies the following:

(a) $(\rho \cap R) \circ L = L \circ (\rho \cap R)$;
(b) $(\rho \cap L) \circ R = R \circ (\rho \cap L)$;
(c) $(\rho \cap L) \circ (\rho \cap R) = (\rho \cap R) \circ (\rho \cap L)$.
PROOF. We show \((\rho \cap R) \circ L \subseteq L \circ (\rho \cap R)\). The other containments follow similarly. Let \((a, b) \in (\rho \cap R) \circ L\). Then there is \(x\) so that \(a(\rho \cap R)xLb\). Since \(bLx\) there is a \(t\) so that \(b = tx\). Hence \(ta(\rho \cap R)tx = b\) and \(aLta\). Thus \((a, b) \in L \circ (\rho \cap R)\).

PROPOSITION 2.2. Let \(\rho\) be a congruence on a regular semigroup \(S\) and \(\rho \subseteq D\). If \(\rho\) is rectangular, then \(\rho \circ L = L \circ \rho\) and \(\rho \circ R = R \circ \rho\), that is, \(\rho\) is \((L, R)\)-commuting.

PROOF. Since \(\rho\) is rectangular and \(\rho \subseteq D\), we have \(\rho \subseteq (L \cap \rho) \circ (R \cap \rho) = (R \cap \rho) \circ (L \cap \rho)\). Therefore \(\rho \circ S \subseteq (L \cap \rho) \circ \rho \circ (R \cap \rho) \subseteq (L \cap \rho) \circ L \circ (R \cap \rho) \subseteq L \circ (R \cap \rho) \subseteq L \circ \rho\). The other parts are similar.

EXAMPLE 2.3. It follows from [3, Theorem 10.58] that if \(X\) is a finite set and if \(\rho\) is a congruence on the transformation semigroup \(T_X\) on \(X\), then \(\rho\) is rectangular. If, however, \(X\) is infinite, then \(\rho\) might not be rectangular. For instance, let \(X = N = \{1, 2, 3, 4, \ldots\}\), and take \(\rho = \{(a, b) | dr(a, b) < \infty\}\) where \(D(a, b) = \{x \in N | a(x) \neq b(x)\}\) and \(dr(a, b) = \max\{|a(D(a, b))|, |b(D(a, b))|\}\). For some \((a, b) \in \rho \cap D\) we will show that there is no \(c \in T_N\) so that \(a(R \cap \rho)c(L \cap \rho)b\), that is, there is no \(c \in T_N\) so that \(\ker(a) = \ker(c), c(N) = b(N), dr(a, c) < \infty,\) and \(dr(c, b) < \infty\). Choose \(b: N \to N\) to be the identity and \(a: N \to N\) by \(a(1) = a(2) = 1\) and \(a(n) = n\) for \(n > 3\). Suppose there is \(c \in T_N\) satisfying the above. Since \(b\) is one-to-one and \(dr(b, c) < \infty\), then \(D(b, c)\) will be finite. Hence there is \(M\) so that for \(n \geq M, c(n) = n\). Hence \(c\) maps \(\{1, 2, \ldots, M\}\) onto \(\{1, 2, \ldots, M\}\), but \(c(1) = c(2)\). This is a contradiction. Hence \(\rho\) is not rectangular.

A congruence \(\rho\) on a regular semigroup \(S\) is said to be \(K\)-covering provided for each \(a \in S, \phi(H_a) = H_{\phi(a)}\).

THEOREM 2.4. If a congruence \(\rho\) on a regular semigroup \(S\) is \(K\)-covering, then \(\rho\) is rectangular. If \(\rho \subseteq D\), the converse holds.

PROOF. Suppose \(\rho\) is \(K\)-covering. We show \((\rho \cap D) \subseteq (L \cap \rho) \circ (R \cap \rho)\). Let \((a, b) \in \rho \cap D\) and \(t \in R_b \cap L_a\). Since \(tLapbRt\) we have \(\phi(t)K\phi(a)\), and therefore \(\phi(H_t) = H_{\phi(t)} = H_{\phi(a)} = \phi(H_a)\). So, there exists \(x \in H_t\) with \(\phi(x) = \phi(a)\). Now \((a, x) \in L \cap \rho\) and \((x, b) \in R \cap \rho\), so \((a, b) \in (L \cap \rho) \circ (R \cap \rho)\).

Suppose \(\rho\) is rectangular and \(\rho \subseteq D\). We show that for \(a \in S, H_{\phi(a)} \subseteq \phi(H_a)\); the other containment is clear. Let \(X \in H_{\phi(a)}\) and \(\phi(a) = A\). By Theorem 2.2, \(\rho\) is \((L, R)\)-commuting. Since \(XLA\), there exists \((by [4])x \in X\) and \(p \in A\) with \(xLp\). Then \(aLpx\) implies there is \(q\) so that \(aLpqx\). Similarly, since \(XRA\), there is
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y \in \bar{X} and r \in \bar{A} with yRr. Then aprAy implies there is s so that aRspy. Hence qLas with q, s \in \bar{X}. Since \rho is rectangular, there is w \in \bar{X} with qLwRs. Thus w \in H_a and X = \phi(w) \in \phi(H_a).

**COROLLARY 2.5.** (1) If S is combinatorial (that is, if \mathcal{H} is trivial) and \rho \subset \mathcal{D} is rectangular, then S/\rho is combinatorial.

(2) If S/\rho is combinatorial, then \rho is rectangular.

(3) If \rho and \sigma are rectangular and \sigma \subset \rho \subset \mathcal{D}, then \rho/\sigma is rectangular.

(4) If \sigma \subset \rho \subset \mathcal{D} and \sigma and \rho/\sigma are rectangular, then \rho is rectangular.

We omit the straightforward arguments.

**PROPOSITION 2.6.** If \rho is a congruence on a regular semigroup S and \rho \subset \mathcal{L}, then \rho is rectangular.

**PROOF.** From the definition, if (a, b) \in \rho then aLb and a(\rho \cap \mathcal{L})b(\rho \cap \mathcal{R})b.

We recall that a semigroup S is stable if and only if Sa \subset Sab implies Sa = Sab and dS \subset cdS implies dS = cdS. It is known [1] that S is stable if and only if S does not contain a bicycle semigroup. Stable semigroups include finite (or compact) semigroups.

**THEOREM 2.7.** Let \rho \subset \mathcal{D} be a congruence on the stable regular semigroup S. Then

(1) \rho is rectangular;

(2) if I^2 = I \in S/\rho, then I is a completely simple subsemigroup of S;

(3) S/\rho is stable.

**PROOF.** We first show that \rho is rectangular. Let (x, y) \in \rho and choose e^2 = eRx. Let I = \phi(e). Then there exists t \in S so that xt = e. Since \rho \subset \mathcal{D}, we know there is a \mathcal{D}-class D so that I \cup \{x, y\} \subset D. We can assume t \in D by replacing t by txt if necessary. In fact, xt = (xt)(xt) \in Stxt and txt \in Sxt yield xtLtxt. Hence we can assume xt, t in S. Since xt\rho yt and S is stable yte \in R_{yt} \cap L_{gt} \cap I [1, Corollary 1.1(5)]. Hence since e = e^2 \in R_{x} \cap L_{yxe}, R_{yt} = R_{yte}, and yt \in R_{y} \cap L_{t} [1, Corollary 1.1(5)], we have ytx = ytx \in R_{y} \cap L_{x} \cap \phi^{-1}\phi(x). Thus \rho is rectangular.

Suppose S/\rho is not stable; then S/\rho contains a bicyclic semigroup C(P, Q) with QP = I, the identity. Since P \perp Q in S/\rho there exists p \in P and q \in Q with

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\( p \perp q \) in \( S \). If \( qp = e \), then \( \phi(qe) = QI = Q = \phi(q) \). Thus, \( qepq \). Since \( \rho \subset \varnothing \), \( qe \in D_q \), and by stability, \( qe \in R_q \cap L_e = R_e \cap L_e = H(e) \) [1]. Thus \( Q = \phi(qe) \) belongs to a group, a contradiction.

### 3. V-commuting congruences

Let \( \rho \) be a congruence on a regular semigroup \( S \) and \( V = \{(a, b) \in S \times S | a \perp b\} \). Recall that \( \rho \) is \( V \)-commuting provided \( \rho \circ V = V \circ \rho \) and that \( \rho \) is idempotent-regular provided each idempotent \( \rho \)-class is a regular subsemigroup of \( S \). We note that either containment, \( \rho \circ V \subset V \circ \rho \) or \( V \circ \rho \subset \rho \circ V \), implies the other. We will use this without comment. If \( A, B \in S/\rho \) with \( AAB \), we say \( ht(A) \leq ht(B) \) if for each \( \mathcal{R} \)-class \( R \) of \( S \) with \( R \cap A \neq \varnothing \), we have \( R \cap B \neq \varnothing \). Similarly if \( ALC \) in \( S/\rho \), we say \( w(A) \leq w(C) \) if for each \( \mathcal{L} \)-class \( L \) of \( S \) with \( L \cap A \neq \varnothing \) we have \( L \cap B \neq \varnothing \). The following results give several characterizations of \( V \)-commuting congruences and point out relationships between \( V \)-commuting and idempotent-regular congruences.

In the following, \( \rho_E = \rho \cap E \times E, \mathcal{R}_E = \mathcal{R} \cap E \times E \), and \( \mathcal{L}_E = \mathcal{L} \cap E \times E \).

**Theorem 3.1.** Let \( \rho \) be a congruence on a regular semigroup \( S \). The following (1)–(8) are equivalent.

1. \( \rho \circ V = V \circ \rho \).
2. If \( A \perp B \) in \( S/\rho \) and \( a \in \overline{A} \), then there is \( b \in \overline{B} \) so that \( a \perp b \).
3. \( \rho(V_x) = V_{\rho(x)} \) for each \( x \in S \).
4. (a) If \( AAL = I^2 \) in \( S/\rho \) and \( a \in \overline{A} \) then there is \( e^2 = e \in \overline{I} \) so that \( a \mathcal{R} e \) in \( S \).
   (b) If \( ALE = F^2 \) in \( S/\rho \) and \( a \in \overline{A} \) then there is \( f^2 = f \in \overline{F} \) so that \( a \mathcal{L} f \) in \( S \).
5. (a) If \( EAL \mathcal{R} I, I^2 = I, F^2 = F \) in \( S/\rho \) then \( \overline{I} \overline{A} = \overline{A} \overline{F} = \overline{A} \).
   (b) \( \rho \) is idempotent-regular.
6. (a) If \( I^2 = I \mathcal{R} J = J^2 \) in \( S/\rho \) then \( ht(I) = ht(J) \), and if \( I^2 = I \mathcal{L} J = J^2 \) in \( S/\rho \) then \( w(I) = w(J) \).
   (b) \( \rho \) is idempotent-regular.
7. (a) \( \rho \circ \mathcal{R} \cap E \times E = \mathcal{R} \circ \rho \cap E \times E \), and \( \rho \circ \mathcal{L} \cap E \times E = \mathcal{L} \circ \rho \cap E \times E \).
   (b) \( \rho \) is idempotent-regular.
8. \( \rho_E \circ \mathcal{R}_E = \mathcal{R}_E \circ \rho_E \) and \( \rho_E \circ \mathcal{L}_E = \mathcal{L}_E \circ \rho_E \).

**Proof.** (1) \( \Rightarrow \) (2). Let \( A \perp B \) in \( S/\rho \) and choose \( a \in \overline{A} \). By Hall [4], there are \( a_0 \in \overline{A} \) and \( b_0 \in \overline{B} \), so that \( a_0 \perp b_0 \). Since \( a \rho a_0 \perp b_0 \), by (1) there is \( b \) so that \( a \perp b \rho b_0 \). Hence \( b \in \overline{B} \) and \( a \perp b \).

(2) \( \Rightarrow \) (3). That \( \rho(V_x) \subset V_{\rho(x)} \) is clear. To show the reverse, let \( \rho(x) = A \) and \( A \perp B \). Then \( x \in \overline{A} \) and by (2), there is \( b \in \overline{B} \) with \( x \perp b \). Hence \( B = \rho(b) \) and \( b \in V_x \).
(3) \rightarrow (4). Let FLA \mathcal{R} I and \( a \in \bar{A} \). If \( A' \) is an inverse of \( A \) relative to \( I \) and \( F \), then \( A' \in V_{\rho(a)} = \rho(V_a) \). So \( A' = \rho(x) \) for some \( x \perp a \). Then \( ax \mathcal{R} a \) and \( ax \in \bar{I} \).

Let \( e = ax \); then (4)(a) holds. Similarly (4)(b) holds. To see that \( \rho \) is idempotent-regular, note that \( I \perp I \). By (3), each element of \( \bar{I} \) has an inverse in \( \bar{I} \), so \( \bar{I} \) is a regular subsemigroup.

(4) \rightarrow (5). Let FLA \mathcal{R} I and \( I^2 = I, F^2 = F \) in \( S/\rho \). Clearly \( \bar{I} \bar{A} \subset \bar{A} \). To show \( \bar{A} \subset \bar{I} \bar{A} \), let \( a \in \bar{A} \). By (4)(a) there exists \( x \in \bar{I} \) with \( a \mathcal{R} x \). Since \( \rho \) is idempotent-regular there exists \( e = e^2 \in \bar{I} \cap R_x \); then \( a \mathcal{R} e \), so \( a = ea \). Hence \( a \in \bar{I} \bar{A} \). The other arguments go similarly.

(5) \rightarrow (6). Let \( I^2 = I\mathcal{R} J = J^2 \). We show that if \( t \in \bar{I} \), then \( R_t \cap \bar{I} \neq \emptyset \). This will give \( ht(I) \subseteq ht(J) \) and a symmetric argument will complete showing \( ht(I) = ht(J) \). Let \( t \in \bar{I} \) and using (5)(b) choose \( e^2 = eL_t \) with \( e \in \bar{I} \). By (5)(a), \( \bar{J} \bar{I} = \bar{I} \) and there are \( x \in \bar{J}, y \in \bar{I} \) so that \( e = xy \). Then \( e = e^3 = e(xy)e = (ex)(ye) \), so \( e \mathcal{R} x \mathcal{R} e \) with \( ex \in \bar{J} \) since \( \bar{I} \bar{J} = \bar{J} \). Now \( e \mathcal{R} x \mathcal{R} e \) implies \( t = te \mathcal{R} tex \in \bar{J} \). Hence \( R_t \cap J \neq \emptyset \).

(6) \rightarrow (7). We show \( \rho \circ \mathcal{R} \cap E \times E \in \mathcal{R} \circ \rho \cap E \times E \). The other containments follow similarly. If \( (e, f) \in \rho \circ \mathcal{R} \cap E \times E \), there is \( x \) so that \( e \mathcal{R} xf \mathcal{R} f \). Then \( \rho(e) \mathcal{R} \rho(f) \) and by (6)(a), \( h \rho(e) = h \rho(f) \). Thus there is \( t \) so that \( e \mathcal{R} x \mathcal{R} f \mathcal{R} f \). Thus \( (e, f) \in \rho \circ \mathcal{R} \cap E \times E \).

(7) \rightarrow (8). Let \( (e, g) \in \rho_E \circ \mathcal{R} E \). Then there exists \( f = f^2 \) with \( ep \mathcal{R} \rho g \). By (7)(a) there exists \( x \) with \( e \mathcal{R} x \mathcal{R} pg \). By (7)(b) there exists \( h = h^2 pg \) with \( x \mathcal{R} h \). Then \( e \mathcal{R} h \mathcal{R} g \), so \( (e, g) \in \mathcal{R} E \circ \rho_E \). The remaining arguments are similar.

(8) \rightarrow (1). Let \( (a, b) \in \rho \circ V \); then there exists \( x \) with \( a \mathcal{R} x \perp b \). Let \( e^2 = e \mathcal{R} a \), and let \( k = \phi(e), I = \phi(bx) \), and \( J = \phi(xb) \). Then \( J \mathcal{R} k \), so there exist idempotents \( j \in \bar{J} \) and \( h \in \bar{K} \) with \( j \mathcal{R} k \) [4]. Now ep \( \mathcal{R} j \), so by (8) there exists \( i = i^2 \) with \( e \mathcal{R} i \mathcal{R} j \). Similarly there exists \( f^2 = f \mathcal{R} a \) with \( f \in \bar{I} \). Let \( a' \) be the inverse of \( a \) relative to \( f \) and \( i \). Since \( \phi(b) \) and \( \phi(a') \) are both inverse to \( \phi(a) \) relative to \( I \) and \( J \), we have \( a \perp a' \mathcal{R} b \). Hence \( a \perp a' \mathcal{R} b \) and (1) follows.

**REMARK 1.** The equivalences of (1), (2), and (3) hold in a more general setting, using essentially similar arguments. We may replace \( V \) by any relation \( \mathcal{X} \) that enjoys the lifting property, that is, if \( A \mathcal{X} B \) in \( S/\rho \), then there are \( a \in \bar{A} \) and \( b \in \bar{B} \) so that \( A \mathcal{X} b \) in \( S \). From [4] we know that \( \mathcal{L}, \mathcal{R}, \) and \( \mathcal{D} \) satisfy this lifting property.

**REMARK 2.** Although we will not make use of it, \( V \) can be described as follows. These are equivalent: (1) \( a \perp b \) (2) \( ab \in R_a \cap L_b \cap E \) (3) \( (a, b) \in \mathcal{L} \circ \Delta_E \circ \mathcal{R} \cap m^{-1}(E) \), where \( \Delta_E = \{(e, e) : e \in E \} \) and \( m : S \times S \rightarrow S \) is the multiplication on \( S \). We omit the straightforward arguments. Thus, \( V = m^{-1}(R_a \cap L_b \cap E) = \mathcal{L} \circ \Delta_E \circ \mathcal{R} \cap m^{-1}(E) \).
REMARK 3. We see from (8) of Theorem 3.1 that a congruence $\rho$ commutes with $V$ if and only if $\rho$ induces a decomposition of $E$ in which the classes are height and width compatible. Recall [12] that two congruences $\alpha$ and $\beta$ are $\theta$-equivalent $(\alpha \theta \beta)$ if $\alpha$ and $\beta$ induce the same decomposition of $E$. We see then that If $\alpha \circ V = V \circ \alpha$ and $\alpha \theta \beta$ then $\beta \circ V = V \circ \beta$.

REMARK 4. A regular semigroup $S$ is said to be $V$-regular, [10] or [11], if for all $a, b \in S$ we have $V_{ab} \subseteq V_b V_a$, and weakly $V$-regular, [8], if for all $a, b \in S$ we have $V_{ab} \subseteq V_a S V_b$. Theorem 1 of [8] and Proposition 2.4 of [9] show that if $S$ is weakly $V$-regular and $\rho$ is a congruence on $S$ then (2) of Theorem 3.1 holds. Hence $\rho \circ V = V \circ \rho$.

REMARK 5. If $\rho \subseteq \mathcal{L}$, then $\rho \circ V = V \circ \rho$ [6].

COROLLARY 3.2. If $\rho$ is rectangular, idempotent-regular, and under $\mathcal{D}$, then $\rho \circ V = V \circ \rho$.

PROOF. This follows from (7) $\rightarrow$ (1) and Proposition 2.2.

The property $\rho \circ V = V \circ \rho$ holds for a rather large class of congruences on regular semigroups. For example, every congruence $\rho$ on an inverse semigroup satisfies $\rho \circ V = V \circ \rho$. The following more general result holds.

THEOREM 3.3. If $\rho$ is a congruence on a regular semigroup $S$ and $S/\rho$ is an inverse semigroup, then $\rho \circ V = V \circ \rho$.

PROOF. If $(a, b) \in \rho \circ V$, then there is $x \in S$ so that $a \rho x \perp b$. Choose $y \in S$ so that $a \perp y$. Hence $\rho(a) \perp \rho(y)$ and $\rho(a) = \rho(x) \perp \rho(b)$. Since $S/\rho$ is inverse, then $\rho(b) = \rho(y)$ and $a \perp y \rho b$. Hence $(a, b) \in V \circ \rho$.

EXAMPLE 3.4. Let $S$ be the band $\{e, f, g\}$ with binary operation as follows:

$\begin{array}{ccc}
  e & f & g \\
  e & e & f \\
  f & e & f \\
  g & e & g \\
\end{array}$

Note that $D_g = \{g\}$, $D_e = D_f = R_e = R_f = \{e, f\}$, $L_e = \{e\}$, and $L_f = \{f\}$. The relation $\rho = \{(f, g), (g, f)\} \cup \Delta$ is a congruence on $S$. We observe that $\rho$ is rectangular and idempotent-regular, but $\rho \not\subseteq \mathcal{D}$ and $\rho$ commutes with none of $\mathcal{R}$, $\mathcal{V}$, and $\mathcal{D}$.
The examples given in 5.2 are regular completely semisimple semigroups which have congruences that do not commute with \( V \). The example given in Example 5.3 has a congruence \( \rho \in \mathcal{D} \) but \( \rho \circ V \neq V \circ \rho \); in fact, \( \rho \) is not idempotent-regular.

As was stated in Remark 4 above, if \( S \) is \( V \)-regular, then each congruence \( \rho \) on \( S \) commutes with \( V \). For \( X \) any set, it is known, [10] or [11], that the full transformation semigroup \( \mathcal{T}_X \) is \( V \)-regular. We present the following straightforward argument that any congruence \( \rho \) on \( \mathcal{T}_X \) commutes with \( V \). We give five lemmas, omitting the routine proofs of the first four.

**Lemma 3.5.** If \( a \in \mathcal{T}_X \) then \( a' \perp a \) if and only if (1) \( a' \) is a section for \( a \) on \( aX \), that is, \( aa'(x) = x \) for each \( x \in aX \) and (2) \( a'X \subset a'aX \).

**Lemma 3.6.** If \( \rho \) and \( \tau \) are \( V \)-commuting congruences on a regular semigroup \( S \) then \( (\rho \cup \tau) \circ V = V \circ (\rho \cup \tau) \).

**Lemma 3.7.** If \( I \) is an ideal of a regular semigroup \( S \) and \( \rho_I \) is the associated Rees congruence, then \( \rho_I \circ V = V \circ \rho_I \).

**Lemma 3.8.** If \( \rho_I \) is a Rees congruence and \( \rho \) is a \( V \)-commuting congruence on a regular semigroup \( S \), then \( (\rho \cap \rho_I) \circ V = V \circ (\rho \cap \rho_I) \).

**Lemma 3.9.** If \( \rho \) is a difference rank congruence on \( \mathcal{T}_X \), then \( \rho \circ V = V \circ \rho \).

**Proof.** Let \( |X| = p > q > \aleph_0 \) and \( \rho = \{(a, b) \mid |dr(a, b) < q \} \), where \( D(a, b) = \{x \in X \mid a(x) \neq b(x)\} \) and \( dr(a, b) = \max\{|a(D(a, b))|, |b(D(a, b))|\} \). Let \( (a, b) \in V \circ \rho \) and choose \( a' \) so that \( a \perp a'\rho b \) and set \( D = D(a', b) \). Define \( b' \in \mathcal{T}_X \) as follows. For \( y \in bX \setminus (bD \cup a'D) \), set \( b'(y) = a(y) \). Since \( a \perp a' \), using Lemma 3.5, we have \( y = a'a(y) \) and since \( y \notin a'D \) it follows that \( a(y) \notin D \).

Hence \( y = a'a(y) = ba(y) = bb'(y) \). Thus \( b' \) is a section for \( b \) on \( bX \setminus (bD \cup a'D) \). Hence \( y = a'a(y) = ba(y) = bb'(y) \). Thus \( b' \) is a section for \( b \) on \( bX \setminus (bD \cup a'D) \). For \( y \in bD \cup (bX \cap a'D) \), choose \( b'(y) \) so that \( bb'(y) = y \). For \( y \in a'D \setminus bX \), choose \( x \in D \) with \( y = a'(x) \) and set \( b'(y) = b'b(x) \). Now, \( b' \) satisfies the conditions of Lemma 3.5 on \( bX \cup a'D = a'X \cup bD \). Since \( dr(a', b) < q \), the symmetric difference of \( a'X \) and \( bX \) has cardinality less than \( q \); notionally we use \( a'X = qabX \) and \( bX = qabX \). From the above defining of \( b' \) on \( bX \), we have \( b'bX = qabX \). Define \( b' \) on \( X \setminus (bX \cup a'X) \) by \( b'(x) = a(x) \) if \( a(x) \in b'bX \) and \( b'(x) = z_0 \) for some \( z_0 \in b'bX \) otherwise. Now \( b' \perp b \) since \( b' \) is a section for \( b \) on \( bX \) and \( b'X \subset b'bX \). Further, since \( |a'D| < q \) and \( aX \subset aa'X = qabX \) we know \( dr(b', a) < q \). Thus \( aap' \perp b \) and \( \rho \circ V = V \circ \rho \) follows.
THEOREM 3.10. If \( \rho \) is any congruence on \( \mathcal{F}_X \), then \( \rho \circ V = V \circ \rho \).

PROOF. If the primary cardinal number of \( \rho \) is infinite, then by a result of Malcev [3, Theorem 10.72] there are ideals \( I_0, \ldots, I_n \) and difference rank congruences \( \delta_0, \ldots, \delta_{n+1} \) so that \( \rho = \rho I_0 \cup (\rho I_1 \cap \delta_1) \cup \cdots \cup (\rho I_n \cap \delta_n) \cup \delta_{n+1} \).

The result follows from this case from the previous lemmas.

If the primary cardinal number of \( \rho \) is finite, namely \( n \), then [3, Theorem 10.68] we can write \( \rho = I \cup [\sigma \cap (D_n \times D_n)] \cup [I_n \times I_n] \) where \( D_n \) is the \( \mathcal{D} \)-class of all elements of \( \mathcal{F}_X \) with rank \( n \), \( I_n \) is the principal ideal of \( \mathcal{F}_X \) consisting of elements of rank less than \( n \), \( \sigma \) is a congruence on \( I_{n+1}/I_n \), and \( I \) is the identity congruence. Again, due to a result of Malcev [3, Theorem 10.58], we know that \( \sigma \subset \mathcal{H} \) and the \( \mathcal{H} \)-classes in \( I_{n+1}/I_n \) coincide with the \( \mathcal{H} \)-classes of \( \mathcal{F}_X \). Let \( (a, b) \in \rho \circ V \). There exists \( c \in \mathcal{F}_X \) so that \( apc \perp b \). Now \( apc \) implies one of \( a = c, (a, c) \in I_n \times I_n \), or \( (a, c) \in \sigma \subset \mathcal{H} \). In case \( a = c \) or \( (a, c) \in I_n \times I_n \) we easily see \( (a, b) \in V \circ \rho \). If \( (a, c) \in \sigma \cap (D_n \times D_n) \) then in \( I_{n+1}/I_n \) since \( \sigma \subset \mathcal{H} \), we know there is \( d \in I_{n+1}/I_n \) so that \( a \perp d \rho b \) [6, Corollary 2.3]. Hence \( (a, b) \in V \circ \rho \) and \( \rho \circ V = V \circ \rho \).

Let \( VC \) denote the class of regular semigroups with the property that if \( S \) is in \( VC \) and \( \rho \) is a congruence on \( S \) then \( \rho \circ V = V \circ \rho \). From the above, it follows that \( VC \) contains inverse semigroups and full transformation semigroups. More generally, using Theorem 1 of [8] and Theorem 3.1 we see that any weakly \( V \)-regular semigroup is in \( VC \). Example 3.4 shows that \( VC \) does not contain all bands and hence not all orthodox semigroups. The examples in 5.2 are completely semisimple semigroups (= stable semigroups) that are not in \( VC \). Example 5.3 shows that a bisimple regular semigroup may not be in \( VC \).

THEOREM 3.11. The class \( VC \) is closed under quotients.

PROOF. Let \( \sigma \) be a congruence on a semigroup \( S \) in \( VC \), and let \( \bar{\sigma} \) be a congruence on \( S/\sigma \). Then \( \rho = \{(a, b) \in S \times S | (\sigma(a), \sigma(b)) \in \bar{\sigma}\} \) is a congruence on \( S \) with \( \sigma \subset \rho \). Further \( \bar{\rho} = \rho/\sigma = \{(\sigma(a), (a, b)) | (a, b) \in \rho\} \). Let \( (\sigma(a), (a, b)) \in V \circ \bar{\rho} \). Then \( \sigma(a) \perp \sigma(x) \bar{\rho}(b) \) for some \( \sigma(x) \in \sigma \circ S \). Using 3.1, we have that \( \sigma \circ V = V \circ \sigma \) and \( \sigma(a) \perp \sigma(x) \) imply there exists \( x' \in S \) with \( x \sigma x' \) and \( a \perp x' \). Now \( \sigma(x) = (x') \bar{\rho} \sigma(b) \) implies \( x' \rho b \), and \( a \perp x' \rho b \) implies there is \( y \in S \) so that \( a \rho y \perp b \). Thus \( \sigma(a) \bar{\rho} \sigma(y) \perp \sigma(b) \) and \( (\sigma(a) \sigma(b)) \in \bar{\rho} \circ V \). Hence \( \bar{\rho} \circ V = V \circ \bar{\rho} \).

THEOREM 3.12. Let \( \sigma \) and \( \rho \) be congruences on a regular semigroup \( S \) with \( \sigma \subset \rho \).

1. If \( V \circ \sigma = \sigma \circ V \) and \( V \circ \rho/\sigma = \rho/\sigma \circ V \) then \( \rho \circ V = V \circ \rho \).
2. If \( V \circ \sigma = \sigma \circ V \) and \( V \circ \rho = \rho \circ V \) then \( \rho/\sigma \circ V = V \circ \rho/\sigma \).
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Proof. To prove (1), let \((a, b) \in \rho \circ V\). There is \(x \in S\) so that \(a \rho x \perp b\). Thus \(\sigma(a)(\rho/\sigma)\sigma(x) \perp \sigma(b)\). Since \(\rho/\sigma\) commutes with \(V\), there is \(\sigma(y)\) so that \(\sigma(a) \perp \sigma(y)(\rho/\sigma)\sigma(b)\). By Theorem 3.1 there is \(z\) with \(\rho y \sigma z\) such that \(a \perp z\). Since \(\sigma(y)(\rho/\sigma)\sigma(b)\) and \(b \rho y \sigma z\), we have \(b \rho z\) and \(a \perp z\). Hence \((a, b) \in V \circ \rho\).

To prove (2) let \((\sigma(a), \sigma(b)) \in \rho/\sigma \circ V\) and choose \(x\) so that \(\sigma(a)(\rho/\sigma)\sigma(x) \perp \sigma(b)\). Thus \(a \rho x\) and there is \(z\) with \(x \sigma z\) and \(z \perp b\). Hence \(a \rho z \perp b\) and there is \(w\) so that \(a \perp w \rho b\). Thus \(\sigma(a) \perp \sigma(w)(\rho/\sigma)\sigma(b)\) and \((\sigma(a), \sigma(b)) \in V \circ \rho/\sigma\).

We finish this section by giving a result that is related to Theorem 2.7. It gives a class of congruences that commute with \(V\) and are more restrictive than idempotent-regular congruences.

**Theorem 3.13.** Let \(\rho\) be a congruence on a regular semigroup \(S\) so that if \(I^2 = I \in S/\rho\) then \(\overline{I}\) is a completely simple subsemigroup of \(S\). Then \(\rho \subset \mathcal{D}\) and \(\rho\) is rectangular.

Proof. We first show that \(\rho \circ V = V \circ \rho\) by verifying that part (6) of Theorem 3.1 holds. It is clear that \(\rho\) is idempotent-regular, and we show that if \(I^2 = I \in \mathcal{R}\) then \(ht(I) = ht(J)\). The dual condition follows in a similar manner. Let \(e^2 = e \in \overline{I}\). From [4] we know that there is \(f \in \overline{I}\) and \(g = g^2 \in \overline{J}\) with \(f \leq e\) (that is, \(fe = ef = f\)) and \(f \mathcal{R} g\). Since \(\overline{I}\) is completely simple, \(e = f\) and \(e \mathcal{R} g\). By this and the symmetric argument, \(ht(I) = ht(J)\).

To see that \(\rho \subset \mathcal{D}\), let \(A \in S/\rho\) and choose \(I^2 = I \in \mathcal{R} A\). Then from part 4(a) of Theorem 3.1, there is \(e^2 = e \in \overline{I}\) so that \(a \mathcal{R} e\). Since \(\overline{I}\) is completely simple, there is a \(\mathcal{D}\)-class \(D\) so that \(\overline{I} \subset D\). It follows that \(A \subset D\).

To see that \(\rho\) is rectangular, let \(a, b \in \overline{A}\) where \(\mathcal{A} R I = I^2 \in S/\rho\). There are \(e, f \in \overline{I}\) with \(e^2 = e \mathcal{R} a\) and \(f^2 = f \mathcal{R} b\). Since \(\overline{I}\) is completely simple there exists \(g^2 = g \in \overline{I} \cap R_f \cap L_e\). Thus since \(e \in L_g \cap R_a\) and \(g \in L_e \cap R_b\), we have \(ga \in R_b \cap L_a\) and \(be \in R_e \cap L_b\). Now \(a \rho b\) and \(g \mathcal{R} e\) imply \(gap \mathcal{R} be\) and thus \(\rho\) is rectangular.

**4. \((\mathcal{L}, \mathcal{R})\)-commuting congruences**

A congruence \(\rho\) on a semigroup \(S\) is said to be \((\mathcal{L}, \mathcal{R})\)-commuting provided \(\rho \circ \mathcal{L} = \mathcal{L} \circ \rho\) and \(\rho \circ \mathcal{R} = \mathcal{R} \circ \rho\). From Proposition 2.2, if \(\rho \subset \mathcal{D}\) and \(\rho\) is rectangular, then \(\rho\) is \((\mathcal{L}, \mathcal{R})\)-commuting, and Example 3.4 shows that without \(\rho \subset \mathcal{D}\) this is not true. An example below (4.2) shows that \((\mathcal{L}, \mathcal{R})\)-commutativity does not imply rectangularity. The following theorem gives four characterizations of \((\mathcal{L}, \mathcal{R})\)-commutativity among idempotent-regular congruences, that is, those where idempotent classes are regular subsemigroups.
Theorem 4.1. Let $\rho$ be an idempotent-regular congruence on a regular semigroup $S$. These are equivalent.

1. If $A \perp B$, $I = AB$, $F = BA$, and $A^R L B^R \subseteq A^L E B^L$ in $S/\rho$, then $A B = \bar{I}$ and $B A = \bar{F}$.

2. If $K = K^2$ and $A^R B^L \subseteq B^R A^L$ in $S/\rho$, then $A \cong B = \bar{A}$.

3. If $I^2 = I$, $F^2 = F$, and $F A \subseteq A F$ in $S/\rho$, then $ht(I) = ht(A)$ and $w(F) = w(A)$.

4. $\rho \circ R = R \circ \rho$ and $\rho \circ L = L \circ \rho$.

5. $\phi(R_x) = R_{\phi(x)}$ and $\phi(L_x) = L_{\phi(x)}$ for each $x \in S$.

Proof. (1) $\Rightarrow$ (3). We first show that $ht(I) \leq ht(A)$. Choose $B \in S/\rho$ so that $A \perp B$, $AB = I$, and $BA = F$. Let $x \in \bar{I}$. Since $\rho$ is idempotent-regular there is $e$ so that $eRx$ and $e^2 = e \in \bar{I} = \bar{A} \bar{B}$. Then $e = ab = (ea)(be)$ for some $a \in \bar{A}$ and $b \in \bar{B}$. Since $A^R I$ we have $IA = A$ and $ea \in \bar{I} \bar{A} \subseteq \bar{A}$. Further, $e = (ea)(be)$ implies that $eaR \bar{e}$. Thus $eaR \bar{e}x$ and $ea \in \bar{A}$ yields $ht(I) \leq ht(A)$. Similarly, one sees that $w(F) \leq w(A)$. In particular, if $I^2 = I \cong J^2$, then $ht(I) = ht(J)$ and it follows from Theorem 3.1 that $\rho \circ V = V \circ \rho$ and $ht(A) \leq ht(A)$. Similarly, one shows that $w(F) = w(A)$.

(3) $\Rightarrow$ (4). Note first that if $A^R B^L = I^2$ in $S/\rho$, then $ht(A) = ht(B) = ht(I)$. Let $(a, b) \in \rho \circ R$. Then there is $x$ so that $aRx$ and $b = (ea)(be)$ for some $a \in \bar{A}$ and $b \in \bar{B}$. Since $A^R I$ we have $IA = A$ and $ea \in \bar{I} \bar{A} \subseteq \bar{A}$. Further, $e = (ea)(be)$ implies that $eaR \bar{e}$. Thus $eaR \bar{e}x$ and $ea \in \bar{A}$ yields $ht(I) \leq ht(A)$. Similarly, one sees that $w(F) \leq w(A)$. In particular, if $I^2 = I \cong J^2$, then $ht(I) = ht(J)$ and it follows from Theorem 3.1 that $\rho \circ V = V \circ \rho$ and $ht(A) \leq ht(A)$. Similarly, one shows that $w(F) = w(A)$.

(4) $\Rightarrow$ (1). We show $\bar{I} \in A \bar{B}$. Let $x \in \bar{I}$ and choose $e = e^2 \in \bar{I}$ so that $eRx$. Using the Remark 1 following Theorem 3.1 with (4), we see that there are $a \in \bar{A}$ and $b \in \bar{B}$ with $aRx \in \bar{A}$. Thus $ba \in \bar{F}$ and there is $h^2 = h \in F$ so that $ba \in \bar{F}$. Since $a \in L_h \cap R_e$, let $a'$ be the inverse of $a$ with respect to $h$ and $e$. If $\phi(a') = A'$ then $A' \subseteq A$ and $A' \subseteq B$. Hence $AA' = I = AB$ implies $A' = B$. Hence $x = e x = (aa')x = a(a'x) \in A \bar{B}$ since $a'x \in \bar{B} \subseteq \bar{B}$.

(1) $\Rightarrow$ (2). Let $I^2 = I \cong A$ and let $A'$ be an inverse of $A$ relative to $K$ and $I$. Note that $A'(AB) = B$ implies $A' \subseteq AB \subseteq \bar{B}$. As in the proof that (1) $\Rightarrow$ (3) we argue that $ht(AB) \leq ht(I)$ and if $a \in AB$ there exists $e^2 = e \in \bar{I}$ with $eRx$. Thus $a = ea \in \bar{I} \bar{A}$. Hence $AB = \bar{I} \bar{A} = AA' \bar{A} = A B \subseteq \bar{B} \subseteq AB$. Hence (2) is established.

(2) $\Rightarrow$ (1). This is clear.

(4) $\iff$ (5). This follows as (1) $\iff$ (3) in Theorem 3.1 using Remark 1 after 3.1.

Example 4.2. Let $C(p, q)$ be the bicyclic semigroup and let $\rho(p, q) = n - m$. Then $\rho: C(p, q) \to Z$ is a homomorphism. Note that $\rho$ is not rectangular since $\rho(p) = \rho(p^2q)$ but $\rho(p) \neq \rho(pq)$. Further, $\rho$ is not $(L, R)$-commuting, since $1p p q L q$ but $1 L x p q$ does not hold for any $x \in C(p, q)$. However, the congruence...
of the homomorphism \( \alpha \rho \) where \( \alpha : \mathbb{Z} \to \mathbb{Z} \) (modulo \( n \)) does commute with \( \mathcal{L} \) and \( \mathcal{R} \) and is not rectangular. Both these congruences on \( C(p, q) \) are contained in \( \mathcal{D} \) and commute with \( \mathcal{V} \).

### 5. Idempotent-regular congruences

Nambooripad [9] has posed the problems of characterizing idempotent-regular congruences, and of characterizing regular semigroups which have only idempotent-regular congruences. Results from previous sections touch on these problems, and we present some further information.

**Theorem 5.1.** Let \( \sigma \) and \( \rho \) be congruences on a regular semigroup \( S \) with \( \sigma \subseteq \rho \).

1. If \( \rho \) is idempotent-regular, then so is \( \rho/\sigma \).
2. If \( \sigma \circ \mathcal{V} = \mathcal{V} \circ \rho \) and \( \rho/\sigma \) is idempotent-regular, then \( \rho \) is idempotent-regular.

**Proof.** Let \( I = I^2 \) be in \((S/\sigma)/(\rho/\sigma)\) and \( \sigma(a) \) be in \( \bar{I} \), the pre-image of \( I \) in \( S/\sigma \). Then \( a \) is in \( I^* \), the pre-image of \( \bar{I} \) in \( S \). Now \( I^* \) is an idempotent \( \rho \)-class and is hence a regular subsemigroup. Thus there is \( b \in I^* \) so that \( a \perp b \); then \( \sigma(a) \perp \sigma(b) \) and \( \bar{I} \) is regular. This proves (1). To prove (2) let \( I^2 = I \in S/\rho \) with \( x \in I^* \) the \( \rho \)-class mapping to \( I \). Then \( \sigma(x) \) has an inverse \( \sigma(y) \) in \( \sigma(I^*) \). Since \( \sigma \circ \mathcal{V} = \mathcal{V} \circ \sigma \), using Theorem 3.1, we see that there is \( z \) with \( z \sigma y \) and \( z \perp x \). Hence \( z \in I^* \) and it follows that \( \rho \) is idempotent-regular.

**Example 5.2.** This is a class of examples of regular semigroups with congruences which are not idempotent-regular but which are rectangular and \((\mathcal{L}, \mathcal{R})\)-commuting. These constructions extend an example suggested to the authors by J. D. Lawson. We give a special case and illustrate how it generalizes.

Let \( S \) be the rectangular band \( f_1, f_2, Lf_3, Rf_4, Lf_1 \) and \( T \) the partial semigroup \( e_1, Le_2, Re_3, e_4 \) with \( e_1^2 = e_1, e_2^2 = e_2, xy = e_3, yx = e_1, xe_3 = x, \) and \( e_3 y = y \). The mapping \( e_1 \to f_1, x \to f_2, e_3 \to f_3, \) and \( y \to f_4 \) is a partial homomorphism, and the semigroup \( T^* \) formed by extending [3, p. 138] \( S \) by \( T \cup \{0\} \) is a regular semigroup. The congruence with idempotent classes \( \{e_1, f_1\}, \{x, f_2\}, \{e_3, f\}, \) and \( \{y, f_4\} \) is not idempotent-regular since \( \{x, f_2\} \) and \( \{y_1, f_4\} \) are not regular subsemigroups of \( T^* \).

A generalization of the above is obtained as follows. Let \( S_0, S_1, \ldots, S_n \) be (for simplicity, non-trivial) rectangular stable regular \( \mathcal{L} \)-classes (partial semigroups) so that \( S_i \) is a sub-rectangle of \( S_{i-1} \) and if the \( xy \)-entry of \( S_i \) is an idempotent, then so is the \( xy \)-entry of \( S_{i-1} \). Assume that \( S_0 \) is a band. Then the inclusion mappings

\[
S_n \to S_{n-1} \to \cdots \to S_2 \to S_1 \to S_0
\]
are partial homomorphisms. Let $S'_i = S_i \cup \{0\}$ with undefined products set equal to 0, $i = 1, \ldots, n$. Let $T_1$ be the ideal extension of $S_0$ by $S'_1$, and for $i = 2, 3, \ldots, n$, let $T_i$ be the ideal extension of $S'_i$ by $T_{i-1}$ using the partial homomorphism induced by the one from $S_i$ to $S_{i-1}$. Then $T_n$ is a regular stable (completely semisimple) semigroup. If $\rho_0$ is a congruence on $S_0$, then define $\rho$ on $T_n$ so that $C$ is a $\rho$-class if and only if there is a $\rho_0$-class $C_0$ in $S$ so that $C = (i_1i_2 \cdots i_n)^{-1}(C_0)$. It is easy to see that $\rho$ is a congruence on $T_n$ that might not be idempotent-regular.

**Example 5.3.** In [2] the four-spiral semigroup $Sp_4$ is presented as the free semigroup $F_X$ on $X = \{a, b, c, d\}$ modulo the congruence generated by $\{(a, a^2), (b, b^2), (c, c^2), (d, d^2), (a, ba), (b, ab), (b, bc), (c, cb), (c, dc), (d, cd), (d, da)\}$. The semigroup $Sp_4$ is a bisimple, fundamental (in fact, combinatorial) regular idempotent-generated semigroup which is isomorphic to a rectangular band of four subsemigroups, one of which is not regular [2, Theorems 2.4 and 2.7]. Hence $S$ has a congruence $\rho$ (of course $\rho \subset \mathcal{D}$) that is not idempotent-regular.

**References**


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