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MULTIPLIERS FOR HARDY SPACES ON LOCALLY COMPACT VILENKIN GROUPS

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Abstract

In a recent paper the authors proved a multiplier theorem for Hardy spaces $H^p(G)$, $0 < p \leq 1$, defined on a locally compact Vilenkin group G. The assumptions on the multiplier were expressed in terms of the "norms" of certain Herz spaces $K(1/p - 1/r, r, p)$ with r restricted to $1 \leq r < \infty$ and $p < r$. In the present paper we show how this restriction on r may be weakened to $p \leq r < \infty$. Furthermore, we present two modifications of our main theorem and compare these with certain results for multipliers on $L^p(\mathbf{R}^n)$-spaces, $1 < p < \infty$, due to Seeger and to Cowling, Fendler and Fournier. We also discuss the sharpness of some of our results.

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1. Introduction

Throughout this paper G will denote a locally compact Vilenkin group, that is to say, G is a locally compact Abelian topological group containing a strictly decreasing sequence of open compact subgroups $(G_n)_{-\infty}^{\infty}$ such that

(i) $\sup\{\text{order}(G_n/G_{n+1}) : n \in \mathbb{Z}\} < \infty$,

(ii) $\bigcup_{-\infty}^{\infty} G_n = G$ and $\bigcap_{-\infty}^{\infty} G_n = \{0\}$.

The dual group of G is denoted by Γ and for each $n \in \mathbb{Z}$ we set

$$\Gamma_n = \{\gamma \in \Gamma : \gamma(x) = 1 \text{ for all } x \in G_n\}.$$

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We choose Haar measures μ on G and λ on Γ such that $\mu(G_0) = \lambda(\Gamma_0) = 1$. Then $(\mu(G_n))^{-1} = \lambda(\Gamma_n) {:=} m_n$ for each $n \in \mathbb{Z}$. It is an easy consequence of condition (i) for G that for every $\alpha > 0$ there exists a constant $C > 0$, C depending only on α, such that for every $k \in \mathbb{Z}$, both

$$\sum_{j=k}^{\infty} (m_j)^{-\alpha} \leq C(m_k)^{-\alpha}, \tag{1.1}$$

and

$$\sum_{j=-\infty}^{k} (m_j)^{\alpha} \leq C(m_k)^{\alpha}. \tag{1.2}$$

The metric d on $G \times G$ defined by $d(x,x) = 0$ and $d(x,y) = (m_n)^{-1}$ if $x - y \in G_n \backslash G_{n+1}$ generates the original topology on G. For $x \in G$ we set $|x| = d(x,0)$. If A is any set then χ_A will denote the characteristic function of A. Also, for each $n \in \mathbb{Z}$ we set $\Delta_n = m_n \chi_{G_n}$. It is easy to see that the Fourier transform of Δ_n is given by $(\Delta_n)^\wedge = \chi_{\Gamma_n}$. In [5] the definition and a brief summary of the basic properties of the spaces of test functions $\mathscr{S}(G)$ and distributions $\mathscr{S}'(G)$ are given. We now present the definition of the Herz spaces and the Hardy spaces on G.

DEFINITION 1.1. Let $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty$. A measurable function $f : G \to \mathbb{C}$ belongs to the Herz space $K(\alpha, p, q)$ if

$$\|f\|_{K(\alpha,p,q)} := \left(\sum_{l=-\infty}^{\infty} ((m_l)^{-\alpha} \| f \chi_{G_l \backslash G_{l+1}} \|_p)^q \right)^{1/q} < \infty,$$

with the usual modification if $q = \infty$.

DEFINITION 1.2. Let $0 < p \leq 1$. A distribution $f \in \mathscr{S}'(G)$ belongs to the Hardy space $H^p(G)$ if the function $f^* : G \to \mathbb{C}$ defined by $f^*(x) = \sup_l |f * \Delta_l(x)|$ belongs to $L^p(G)$. We set $\|f\|_{H^p} = \|f^*\|_p$.

DEFINITION 1.3. A function $a : G \to \mathbb{C}$ is a (p, ∞) atom, $0 < p \leq 1$, if there exists a set I of the form $x + G_n$ such that (i) supp $a \subset I$, (ii) $\|a\|_\infty \leq (\mu(I))^{-1/p} = (m_n)^{1/p}$, and (iii) $\int_G a(x)\, d\mu(x) = 0$.

In [5] it was shown that the Hardy spaces $H^p(G)$ can also be characterized in the usual way in terms of (p, ∞) atoms on G.

The space of (Fourier) multipliers of $H^p(G)$ will be denoted by $\mathscr{M}(H^p)$; thus $\phi \in \mathscr{M}(H^p)$ if $\phi \in L^\infty(\Gamma)$ and if there exists a constant $C > 0$ such that for all $f \in H^p(G)$ we have $\|(\phi \hat{f})^\vee\|_{H^p} \leq C\|f\|_{H^p}$. We mention here that in order to show that a function $\phi \in L^\infty(\Gamma)$ belongs to $\mathscr{M}(H^p)$ it is sufficient to show the existence of a constant $C > 0$ such that for all $k \in \mathbb{Z}$ and every (p, ∞) atom a with supp $a \subset G_n$ for some $n \in \mathbb{Z}$ and $\|a\|_\infty \leq (m_n)^{1/p}$ we have $\|(\phi_k \hat{a})^\vee\|_{H^p} = \|(\phi_k)^\vee * a\|_{H^p} \leq C$, where $\phi_k = \phi\chi_{\Gamma_k}$; see Remark (4.2) in [5] for further details.

2. Multipliers on Hardy spaces $H^p(G)$

Throughout this section we shall use the notation $\phi_k = \phi\chi_{\Gamma_k}$ and $\phi^k = \phi\chi_{\Gamma_{k+1}\setminus\Gamma_k}$, where $\phi \in L^\infty(\Gamma)$ and $k \in \mathbb{Z}$.

Before stating the main result of the paper, Theorem 2.1, we first prove two simple lemmas.

LEMMA 2.1. *Let* $0 < p \leq 1$. *Let* f, g *be measurable functions on* G *such that* supp $g \subset G_n$ *for some* $n \in \mathbb{Z}$ *and both* f *and* g *are constant on the cosets of* G_k *in* G *for some* $k \geq n$. *Then we have for every* $x \in G$, $|f * g(x)|^p \leq (m_k)^{1-p}|f|^p * |g|^p(x)$.

PROOF. Let $\{z_\alpha + G_k\}$ denote the collection of different cosets of G_k in G_n; thus $G_n = \bigcup_\alpha z_\alpha + G_k$. For every $x \in G$ we have

$$\begin{aligned} f * g(x) &= \sum_\alpha \int_{z_\alpha + G_k} f(x-t)g(t)\,d\mu(t) \\ &= \sum_\alpha f(x - z_\alpha)g(z_\alpha)(m_k)^{-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} |f * g(x)|^p &\leq \sum_\alpha (m_k)^{-p}|f(x-z_\alpha)|^p|g(z_\alpha)|^p \\ &= (m_k)^{1-p}\int_G |f(x-t)|^p|g(t)|^p\,d\mu(t) \\ &= (m_k)^{1-p}|f|^p * |g|^p(x). \end{aligned}$$

LEMMA 2.2. *Let $\alpha > 0$, let $p, r > 0$ and let $(a_j)_{-\infty}^{\infty}$ be any sequence of real numbers. Consider the following conditions:*

$$\sup_k (m_k)^{1-p/r}\left(\sum_{j=k}^{\infty}((m_k)^{p/r-p}|a_j|^p)^\alpha\right)^{1/\alpha} < \infty, \tag{2.1}$$

$$\sup_k (m_k)^{1-p}\left(\sum_{j=k}^{\infty}(|a_j|^p)^\alpha\right)^{1/\alpha} < \infty, \tag{2.2}$$

$$\sup_k (m_k)^{1/p-1}|a_k| < \infty. \tag{2.3}$$

Then

(i) *For* $0 < p < r$, (2.1) *is equivalent to* (2.3).

(ii) *For* $0 < p < 1$, (2.2) *is equivalent to* (2.3).

(iii) *For* $p = 1$ *and* $1 < r$, (2.2) *implies* (2.1) *and, hence,* (2.3).

PROOF. (i) Clearly, (2.1) implies (2.3) for all $p, r > 0$. Conversely, if (2.3) holds then there exists $C > 0$ so that for all $j \in \mathbb{Z}$, $|a_j|^p < C(m_j)^{p-1}$. Therefore, for $0 < p < r$,

$$(m_k)^{1-p/r}\left(\sum_{j=k}^{\infty}((m_j)^{p/r-p}|a_j|^p)^\alpha\right)^{1/\alpha} \le C(m_k)^{1-p/r}\left(\sum_{j=k}^{\infty}(m_j)^{(p/r-1)\alpha}\right)^{1/\alpha} \le C,$$

where the last inequality follows from (1.1).

(ii) Clearly, (2.2) implies (2.3) whenever $p > 0$. If (2.3) holds we see, like in the proof of (i), that

$$(m_k)^{1-p}\left(\sum_{j=k}^{\infty}|a_j|^{p\alpha}\right)^{1/\alpha} \le C(m_k)^{1-p}\left(\sum_{j=k}^{\infty}(m_j)^{(p-1)\alpha}\right)^{1/\alpha} < C,$$

because $p - 1 < 0$.

(iii) For $j \ge k$ and $1 < r$ we have $(m_j)^{1/r-1} \le (m_k)^{1/r-1}$. Thus, assuming (2.2) with $p = 1$ we immediately obtain (2.1).

THEOREM 2.1. *Let $0 < p \le 1$ and $\phi \in L^\infty(\Gamma)$.*

(a) *If $p \le r \le 1$ and if*

$$\sup_k (m_k)^{1-p/r}\left(\sum_{j=k}^{\infty}((m_j)^{p/r-p}\|(\phi^j)^\vee\|^p_{K(1/p-1/r,r,p)})^{2/(2-p)}\right)^{(2-p)/2} < \infty,$$

then $\phi \in \mathcal{M}(H^p)$.

(b) *If* $1 \le r < \infty$ *and*

$$\sup_k (m_k)^{1-p} \left(\sum_{j=k}^{\infty} (\|(\phi^j)^\vee\|^p_{K(1/p-1/r,r,p)})^{2/(2-p)} \right)^{(2-p)/2} < \infty,$$

then $\phi \in \mathcal{M}(H^p)$.

PROOF. Let a be a (p, ∞) atom with supp $a \subset G_n$ and

$$\|a\|_\infty \le (\mu(G_n))^{-1/p}$$

for some $n \in \mathbb{Z}$. Fix $k \in \mathbb{Z}$, let $f = (\phi_k \hat{a})^\vee$ and $f^* = \sup_l |f * \Delta_l|$. Then

$$\begin{aligned} \|f\|^p_{H^p} &= \int_{G_n} (f^*(x))^p \, d\mu(x) + \int_{G \setminus G_n} (f^*(x))^p \, d\mu(x) \\ &= A + B, \quad \text{say}. \end{aligned}$$

We have

$$\begin{aligned} A &\le \left(\int_{G_n} (f^*(x))^2 \, d\mu(x) \right)^{p/2} (\mu(G_n))^{1-p/2} \le C \|f\|_2^p (m_n)^{p/2-1} \\ &\le C \|\phi\|^p_\infty \|a\|^p_2 (m_n)^{p/2-1} \le C, \end{aligned}$$

because a is a (p, ∞) atom and $\phi \in L^\infty(\Gamma)$. To find a similar inequality for B we first observe that Kitada proved in [3] that

$$f^*(x) \le \sum_{j=n}^{\infty} |(\phi^j)^\vee * a_j(x)|,$$

where $a_j = a * (\Delta_{j+1} - \Delta_j)$. Therefore,

$$B \le \sum_{k=-\infty}^{n-1} \sum_{j=n}^{\infty} \int_{G_k \setminus G_{k+1}} |(\phi^j)^\vee * a_j(x)|^p \, d\mu(x).$$

In [3] Kitada also showed that for $x \in G_k \setminus G_{k+1}$ with $k \le n - 1$ we have

$$(\phi^j)^\vee * a_j(x) = (\phi^j)^\vee \chi_{G_k \setminus G_{k+1}} * a_j(x),$$

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so that, after an application of Hölder's inequality, we obtain

$$(2.4)\quad B \le \sum_{j=n}^{\infty}\sum_{k=-\infty}^{n-1}\left(\int_{G_k\setminus G_{k+1}} |(\phi^j)^\vee \chi_{G_k\setminus G_{k+1}} * a_j(x)|^r \, d\mu(x)\right)^{p/r} \cdot \mu\big(G_k\setminus G_{k+1}\big)^{1-p/r}.$$

(a) Now we assume that $p \le r \le 1$. Since $\operatorname{supp}\phi^j \subset \Gamma_{j+1}$ we see that $(\phi^j)^\vee \chi_{G_k\setminus G_{k+1}}$ is constant on the cosets of G_{j+1} in G whenever $k+1 \le j+1$. Also, a_j is constant on the cosets of G_{j+1} in G and $\operatorname{supp} a_j \subset G_n$ for $j \ge n$. Thus it follows from Lemma 2.1 that

(2.5)

$$\begin{aligned} B &\le C\sum_{j=n}^{\infty}\sum_{k=-\infty}^{n-1}\left((m_j)^{1-r}\int_{G_k\setminus G_{k+1}} |(\phi^j)^\vee \chi_{G_k\setminus G_{k+1}}|^r * |a_j|^r(x)\, d\mu(x)\right)^{p/r}\Big(m_k\Big)^{p/r-1} \\ &\le C\sum_{j=n}^{\infty}(m_j)^{p/r-p}\|a_j\|_r^p \sum_{k=-\infty}^{n-1}(m_k)^{p/r-1}\|(\phi^j)^\vee\chi_{G_k\setminus G_{k+1}}\|_r^p. \end{aligned}$$

Thus,

$$\begin{aligned} B &\le C\sum_{j=n}^{\infty}(m_j)^{p/r-1}\|a_j\|_r^p\|(\phi^j)^\vee\|_{K(1/p-1/r,r,p)}^p \\ &\le C\left(\sum_{j=n}^{\infty}\|a_j\|_r^2\right)^{p/2}\left(\sum_{j=n}^{\infty}((m_j)^{p/r-p}\|(\phi^j)^\vee\|_{K(1/p-1/r,r,p)}^p)^{2/(2-p)}\right)^{(2-p)/2}. \end{aligned}$$

Since $a_j = a * (\Delta_{j+1} - \Delta_j)$ implies that $\hat a_j = \hat a \chi_{\Gamma_{j+1}\setminus\Gamma_j}$, we see that

$$\begin{aligned} \sum_{j=n}^{\infty}\|a_j\|_r^2 \le \sum_{j=n}^{\infty}\|a_j\|_2^2(\mu(G_n))^{2/r-1} &= (m_n)^{1-2/r}\sum_{j=n}^{\infty}\|\hat a_j\|_2^2 \\ &\le (m_n)^{1-2/r}\|\hat a\|_2^2 \le (m_n)^{2/p-2/r}. \end{aligned}$$

Therefore, using the assumption of the theorem, we see that $B \le C$ and we may conclude that $\phi \in \mathscr{M}(H^p)$.

(b) Next, assume that $1 < r < \infty$. Applying Young's inequality in inequality (2.4) we see that

$$B \le C\sum_{j=n}^{\infty}\sum_{k=-\infty}^{n-1}(m_k)^{p/r-1}\|(\phi^j)^\vee\chi_{G_k\setminus G_{k+1}}\|_r^p\|a_j\|_1^p$$

$$\begin{aligned}&\leq C\sum_{j=n}^{\infty}\|a_j\|_1^p\|(\phi^j)^\vee\|_{K(1/p-1/r,r,p)}^p\\&\leq C\left(\sum_{j=n}^{\infty}\|a_j\|_1^2\right)^{p/2}\left(\sum_{j=n}^{\infty}\left(\|(\phi^j)^\vee\|_{K(1/p-1/r,r,p)}^p\right)^{2/(2-p)}\right)^{(2-p)/2}\\&\leq C(m_n)^{1-p}\left(\sum_{j=n}^{\infty}\left(\|(\phi^j)^\vee\|_{K(1/p-1/r,r,p)}^p\right)^{2/(2-p)}\right)^{(2-p)/2}.\end{aligned}$$

Thus, by assumption, $B \leq C$ and we may again conclude that $\phi \in \mathcal{M}(H^p)$.

Our first observation is that Theorem 2.1 combined with Lemma 2.2 immediately implies Corollary 2.1 below. Corollaries 2.2 and 2.3 are simply restatements of Theorem 2.1 in case $p = 1 \leq r$ and in case $0 < p = r \leq 1$, respectively.

COROLLARY 2.1. *Let* $0 < p < 1$ *and* $p < r < \infty$. *If* $\phi \in L^\infty(\Gamma)$ *and if*

$$\sup_k (m_k)^{1/p-1}\|(\phi^k)^\vee\|_{K(1/p-1/r,r,p)} < \infty,$$

then $\phi \in \mathcal{M}(H^p)$.

COROLLARY 2.2. *If* $\phi \in L^\infty(\Gamma)$ *satisfies*

$$\sum_{j=-\infty}^{\infty}\|(\phi^j)^\vee\|_{K(1-1/r,r,1)}^2 < \infty$$

for some $r \geq 1$, *then* $\phi \in \mathcal{M}(H^1)$.

COROLLARY 2.3. *Let* $0 < p \leq 1$. *If* $\phi \in L^\infty(\Gamma)$ *and if*

$$\sum_{j=-\infty}^{\infty}\left((m_j)^{1/p-1}\|(\phi^j)^\vee\|_p\right)^{2p/(2-p)} < \infty,$$

then $\phi \in \mathcal{M}(H^p)$.

REMARK 1. Combining the techniques used in the proof of Theorem 2.1 with those used to prove Theorem (4.7) in [5] we can actually show that under the assumptions of Corollary 2.1 the function ϕ is a multiplier on the power-weighted Hardy spaces $H_\alpha^p(G)$ for α satisfying $-1 + p/r < \alpha \leq 0$. Thus, we can extend Corollary (4.8) in [5] from $0 < p < 1$ and $1 \leq r < \infty$ to $0 < p < 1$ and $p < r$.

REMARK 2. Corollary 2.1 with $0 < p < 1$ and $r = 1$ may be considered as the analogue on G of Theorem 3a in [1], in which Baernstein and Sawyer obtained a comparable result for multipliers on Hardy spaces defined on $\mathbb{R}^n$.

We now turn to a discussion of the sharpness of the preceding corollaries. We first consider Corollaries 2.1 and 2.2.

THEOREM 2.2. (a) *Let* $0 < p < 1$ *and* $p < r < \infty$. *There exists* $\phi \in L^\infty(\Gamma)$ *such that*

$$\sup_k (m_k)^{1/p-1} \|(\phi^k)^\vee\|_{K(1/p-1/r,r,q)} < \infty$$

for every $q > p$ *and* $\phi \notin \mathscr{M}(H^p)$.

(b) *Let* $1 < r < \infty$. *There exists* $\phi \in L^\infty(\Gamma)$ *such that*

$$\sum_{j=-\infty}^{\infty} \|(\phi^j)^\vee\|^2_{K(1-1/r,r,q)} < \infty \tag{2.6}$$

for every $q > 1$ *and* $\phi \notin \mathscr{M}(H^1)$.

PROOF. For part (a) we may use the example described in the proof of Theorem (4.9) in [5]. A careful reading of this proof shows that the restriction $1 \le r$ in that theorem can be relaxed to $p < r$. To prove (b), choose a sequence $(\varepsilon_l)_1^\infty$ with each $\varepsilon_l = \pm 1$ such that

$$\sum_{l=1}^{\infty} \frac{\varepsilon_l}{l}\left(1 - \frac{\mu(G_{-l+1})}{\mu(G_{-l})}\right)$$

converges. Define $\psi : \Gamma \to \mathbb{C}$ by

$$\psi(\gamma) = \sum_{l=1}^{\infty} \varepsilon_l l^{-1} \lambda(\Gamma_{-l})(F_{-l+1} - F_{-l})(\gamma),$$

where $F_k = (\chi_{G_k})^\wedge = (\lambda(\Gamma_k))^{-1}\chi_{\Gamma_k}$. If $\gamma \notin \Gamma_0$ then $\psi(\gamma) = 0$, whereas if $\gamma \in \Gamma_{-k+1} \setminus \Gamma_{-k}$ for some $k \ge 1$ then

$$|\psi(\gamma)| \le |\varepsilon_k k^{-1} \lambda(\Gamma_{-k})(\lambda(\Gamma_{-k+1}))^{-1}| + \left|\sum_{l=k+1}^{\infty} \varepsilon_l l^{-1} \lambda(\Gamma_{-l})\left((\lambda(\Gamma_{-l+1})^{-1} - (\lambda(\Gamma_{-l}))^{-1}\right)\right|.$$

Thus, $\psi \in L^\infty(\Gamma) \cap L^1(\Gamma)$. Next, choose $\gamma_1 \in \Gamma_1 \backslash \Gamma_0$ and define $\phi : \Gamma \to \mathbb{C}$ by $\phi(\gamma) = \psi(\gamma - \gamma_1)$. Then $\phi \in L^\infty(\Gamma) \cap L^1(\Gamma)$ and $\operatorname{supp} \phi \subset \Gamma_1 \backslash \Gamma_0$ so that $\phi^1 = \phi$ and $\phi^j = 0$ for $j \neq 1$. Furthermore,

$$\phi^\vee(x) = \sum_{l=1}^{\infty} \varepsilon_l l^{-1} \lambda(\Gamma_{-l})(\chi_{G_{-l+1}} - \chi_{G_{-l}})(x)\gamma_1(x)$$

and for any $q > 1$ and any r with $1 \leq r < \infty$ we have

$$\begin{aligned}\|\phi^\vee\|^q_{K(1-1/r,r,q)} &= \sum_{l=1}^{\infty} \left((m_{-l})^{1/r-1} l^{-1} \lambda(\Gamma_{-l}) \|\chi_{G_{-l+1}} - \chi_{G_{-l}}\|_r\right)^q \\ &\leq \sum_{l=1}^{\infty} l^{-q} (m_{-l})^{(1/r-1+1-1/r)q} < \infty.\end{aligned}$$

Thus (2.6) holds. Moreover,

$$\|\phi^\vee\|_1 = \sum_{l=1}^{\infty} l^{-1} \lambda(\Gamma_{-l})(\mu(G_{-l+1}) - \mu(G_{-l})) = \infty,$$

that is $\phi^\vee \notin L^1(G)$. Next, if we define $g : G \to \mathbb{C}$ by $g(x) = \Delta_1(x) - \Delta_0(x)$ then g is a multiple of a $(1, \infty)$ atom so that $g \in H^1(G)$. Also, $\hat{g} = \chi_{\Gamma_1 \backslash \Gamma_0}$ and this implies that $(\phi\hat{g})^\vee = \phi^\vee$ with $\phi^\vee \notin H^1(G)$. This proves that $\phi \notin \mathscr{M}(H^1)$.

In the following theorem we consider the sharpness of Corollary 2.3.

THEOREM 2.3. *For every p with $0 < p \leq 1$ and every q with $2p/(2-p) < q \leq \infty$ there exists $\phi \in L^\infty(\Gamma)$ such that*

(i) $\sum_{j=-\infty}^{\infty} ((m_j)^{1/p-1} \|(\phi^j)^\vee\|_p)^q < \infty$,

(ii) $\phi \notin \mathscr{M}(H^p)$.

PROOF. For each $j \in \mathbb{N}$ decompose G_0 into the mutually disjoint cosets of G_j in G_0, say,

$$G_0 = \bigcup_{i=1}^{m_j} b_{j,i} + G_j.$$

Define $g_j : G \to \mathbb{C}$ by

$$g_j(x) = \sum_{i=1}^{m_j} \left(\frac{m_{j+1}}{m_j} \chi_{b_{j,i}+G_{j+1}} - \chi_{b_{j,i}+G_j} \right)(x).$$

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Clearly, supp $g_j \subset G_0$, $\int_G g_j(x)\,d\mu(x) = 0$ and $\|g_j\|_2 \le P^{1/2}$, where $P = \sup_j(m_{j+1}/m_j)$. Moreover, since

$$(g_j)^\wedge(\gamma) = \sum_{i=1}^{m_j} \overline{\gamma(b_{j,i})}\frac{1}{m_j}(\chi_{\Gamma_{j+1}} - \chi_{\Gamma_j})(\gamma), \tag{2.7}$$

we see that supp $(g_j)^\wedge \subset \Gamma_{j+1}\backslash\Gamma_j$. Next, for each $n \in \mathbb{N}$ define $h_n : G \to \mathbb{C}$ by

$$h_n(x) = \sum_{j=1}^{n} g_j(x).$$

Then supp $h_n \subset G_0$, $\int_G h_n(x)\,d\mu(x) = 0$ and $\|h_n\|_2 \le P^{1/2}n^{1/2}$. Thus h_n is a multiple of a $(p, 2)$ atom and $\|h_n\|_{H^p} \le P^{1/2}n^{1/2}$.

We now turn to the definition of the function $\phi \in L^\infty(\Gamma)$ satisfying conditions (i) and (ii). For each $j \in \mathbb{N}$ choose an element $z_j \in G_{-j}\backslash G_{-j+1}$ and define $f_j : G \to \mathbb{C}$ by

$$f_j(x) = j^{-\alpha}(m_{j+1}\chi_{z_j+G_{j+1}} - m_j\chi_{z_j+G_j})(x),$$

where $\alpha = \frac{1}{2}((2-p)/2p + 1/q)$. Then $\|f_j\|_p \le Cj^{-\alpha}(m_j)^{1-1/p}$ and

$$(f_j)^\wedge(\gamma) = j^{-\alpha}\overline{\gamma(z_j)}(\chi_{\Gamma_{j+1}} - \chi_{\Gamma_j})(\gamma), \tag{2.8}$$

so that supp $(\hat{f}_j)^\wedge \subset \Gamma_{j+1}\backslash\Gamma_j$ and $\|\hat{f}_j\|_\infty \le j^{-\alpha} \le 1$. Define $\phi : \Gamma \to \mathbb{C}$ by

$$\phi(\gamma) = \sum_{j=1}^{\infty}(f_j)^\wedge(\gamma).$$

Clearly, $\phi \in L^\infty(\Gamma)$, $\phi^j = 0$ for $j \le 0$ and $\phi^j = (f_j)^\wedge$ for $j \ge 1$; moreover, ϕ satisfies condition (i). Furthermore, for each $n \in \mathbb{N}$ and $x \in G$ we have

$$\begin{aligned}(\phi(h_n)^\wedge)^\vee(x) &= \left(\sum_{j=1}^{\infty}(f_j)^\wedge \sum_{j=1}^{n}(g_j)^\wedge\right)^\vee(x)\\ &= \sum_{j=1}^{n}(f_j * g_j)(x).\end{aligned}$$

Thus, $(\phi(h_n)^\wedge)^\vee \in L^1(G)$, and it follows immediately from (2.7) and (2.8) that for every $j \ge 1$,

$$(f_j * g_j)(x) = j^{-\alpha}\sum_{i=1}^{m_j}\left(\frac{m_{j+1}}{m_j}\chi_{z_j+b_{j,i}+G_{j+1}} - \chi_{z_j+b_{j,i}+G_j}\right)(x).$$

Finally, assume $\phi \in \mathscr{M}(H^p)$. Then there exists $C > 0$ so that

$$\begin{aligned} C\|h_n\|_{H^p}^p &\geq \|(\phi(h_n)^\wedge)^\vee\|_{H^p}^p \geq \|\phi(h_n)^\wedge)^\vee\|_p^p \\ &\geq \sum_{j=1}^n j^{-\alpha p}\mu\left(z_j + \bigcup_{i=1}^{m_j}(b_{j,i} + G_j)\right) = \sum_{j=1}^n j^{-\alpha p} \\ &\geq Cn^{1-\alpha p}, \end{aligned}$$

that is, $\|h_n\|_{H^p} \geq Cn^{1/p-\alpha}$. Since $q > 2p/(2-p)$ implies $1/p - \alpha > 1/2$, we have a contradiction of the inequality $\|h_n\|_{H^p} \leq P^{1/2}n^{1/2}$. This shows that $\phi \notin \mathscr{M}(H^p)$, which completes the proof of Theorem 2.3.

REMARK 3. In Section 4 of [4] it was shown that if $\phi \in L^\infty(\Gamma)$ satisfies $\sum_{-\infty}^{\infty} \|(\phi^j)^\vee\|_1 < \infty$ then $\phi \in \mathscr{M}(H^1)$, and that there exists $\phi \in L^\infty(\Gamma)$ such that $\sup_j \|(\phi^j)^\vee\|_1 < \infty$ and $\phi \notin \mathscr{M}(H^1)$. Clearly, the case $p = 1$ of Corollary 2.3 and of Theorem 2.3 sharpen these results from [4].

In view of the fact that condition (i) in Theorem 2.3 is not sufficient to guarantee that $\phi \in \mathscr{M}(H^p)$ it is of some interest to determine what kind of additional condition would be sufficient to obtain $H^p(G)$-multipliers. The following theorem gives one type of solution for this problem.

THEOREM 2.4. *Let* $0 < p \leq 1$. *Let* $\phi \in L^\infty(\Gamma)$ *satisfy*

$$\sum_{j=-\infty}^{\infty} ((m_j)^{1/p-1}\|(\phi^j)^\vee\|_p)^q < \infty$$

for some q *with* $2p/(2-p) \leq q \leq \infty$. *Define* β *by* $\beta = 2pq/((2-p)q - 2p)$ *(if* $q = \infty$ *we take* $\beta = 2p/(2-p)$, *if* $q = 2p/(2-p)$ *we take* $\beta = \infty$). *Let* $(\alpha_j)_{-\infty}^{\infty} \in l^\beta(\mathbb{Z})$ *and define* $\psi : \Gamma \to \mathbb{C}$ *by* $\psi(\gamma) = \Sigma_{-\infty}^{\infty}\alpha_j\phi^j(\gamma)$. *Then* $\psi \in \mathscr{M}(H^p)$.

PROOF. We have

$$\begin{aligned} &\sum_{j=-\infty}^{\infty} \left((m_j)^{1/p-1}\|(\psi^j)^\vee\|_p\right)^{2p/(2-p)} \\ &\quad\leq \left(\sum_{j=-\infty}^{\infty}((m_j)^{1/p-1}\|(\phi^j)^\vee\|_p)^q\right)^{2p/(2-p)q} \left(\sum_{j=-\infty}^{\infty}|\alpha_j|^{2pq/(q(2-p)-2p)}\right)^{1-2p/(2-p)q} \\ &\quad< \infty. \end{aligned}$$

Thus it follows from Corollary 2.4 that $\psi \in \mathcal{M}(H^p)$.

We explicitly state the most interesting case of Theorem 2.4, namely the case when $q = \infty$.

COROLLARY 2.4. *Let* $0 < p \leq 1$. *Let* $\phi \in L^\infty(\Gamma)$ *satisfy*

$$\sup_j (m_j)^{1/p-1} \|(\phi^j)^\vee\|_p < \infty$$

and let $(\alpha_j)_{-\infty}^{\infty} \in l^{2p/(2-p)}(\mathbb{Z})$. *If* $\psi = \sum_{-\infty}^{\infty} \alpha_j \phi^j$ *then* $\psi \in \mathcal{M}(H^p)$.

REMARK 4. Corollary 2.4 is an extension to H^p-spaces, $0 < p \leq 1$, (and on locally compact Vilenkin groups instead of on $\mathbb{R}^n$) of Theorem 2 in [2], in which a similar result was obtained for multipliers on $L^p(\mathbb{R}^n)$-spaces, $1 < p < \infty$.

In the next theorem we show, at least for the case $p = 1$, the sharpness of Corollary 2.4.

THEOREM 2.5. *Let* $(\alpha_j)_{-\infty}^{\infty} \in l^\infty(\mathbb{Z}) \backslash l^2(\mathbb{Z})$. *Then there exists* $\phi \in L^\infty(\Gamma)$ *such that* $\sup_j \|(\phi^j)^\vee\|_1 < \infty$ *and* $\psi = \sum_{-\infty}^{\infty} \alpha_j \phi^j \notin \mathcal{M}(H^1)$.

PROOF. We consider the case when $\sum_1^\infty |\alpha_j|^2 = \infty$. Then there exists a sequence $(\lambda_j)_1^\infty$ in $l^2(\mathbb{N})$ such that $\sum_1^\infty |\alpha_j \lambda_j| = \infty$. Assume $|\alpha_1 \lambda_1| > 0$. We define a sequence $(N_k)_0^\infty$ inductively. Let $N_0 = 1$ and, assuming $N_k \in \mathbb{N}$ has been defined, define $N_{k+1} \in \mathbb{N}$ so that $N_{k+1} > N_k$ and

$$\sum_{j=N_k+1}^{N_{k+1}} |\alpha_j \lambda_j| > 2^{k+1} |\alpha_1 \lambda_1|.$$

Next, for each $j \in \mathbb{N}$ choose a character $\gamma_j \in \Gamma_{j+1} \backslash \Gamma_j$ and define $\phi : \Gamma \to \mathbb{C}$ by

$$\phi(\gamma) = \sum_{j=1}^{\infty} (A_{-j})^\wedge (\gamma - \gamma_j),$$

where, for $n \in \mathbb{Z}$, we set

$$A_n(x) = (\mu(G_n \backslash G_{n+1}))^{-1} \chi_{G_n \backslash G_{n+1}}(x).$$

Then $\phi \in L^\infty(\Gamma)$ and $(\phi^j)^\vee(x) = 0$ if $j \leq 1$ and $(\phi^j)^\vee(x) = \gamma_j(x) A_{-j}(x)$ if $j \geq 1$. Clearly, $\sup_j \|(\phi^j)^\vee\|_1 = 1$.

Next, for $k \in \mathbb{N}$ define $g_k : \Gamma \to \mathbb{C}$ by

$$g_k(\gamma) = \sum_{j=N_k+1}^{N_{k+1}} \lambda_j \chi_{\Gamma_{-N_k}}(\gamma - \gamma_j).$$

If $h_k = (g_k)^\vee$ then

$$h_k(x) = \sum_{j=N_k+1}^{N_{k+1}} \lambda_j \gamma_j(x) \Delta_{-N_k}(x)$$

and we have $\int_G h_k(x)\, d\mu(x) = 0$ and

$$\|h_k\|_2^2 = \sum_{j=N_k+1}^{N_{k+1}} |\lambda_j|^2 m_{-N_k}.$$

Also, since $\operatorname{supp} h_k \subset G_{-N_k}$, we see that

$$\|(|\cdot|h_k)\|_2^2 \le (m_{-N_k})^{-2} \|h_k\|_2^2.$$

Therefore,

$$\|h_k\|_2^{1/2} \|(|\cdot|h_k)\|_2^{1/2} \le \left(\sum_{j=1}^{\infty} |\lambda_j|^2 \right)^{1/2}.$$

Consequently h_k is a $(1, 2, 1)$ molecule centered at $0 \in G$ (see [4] for a definition of $(1, 2, 1)$ molecules on G); this implies that $h_k \in H^1(G)$ and that there exists a constant $C_1 > 0$, C_1 independent of h_k, so that $\|h_k\|_{H^1} \le C_1 (\sum_1^\infty |\lambda_j|^2)^{1/2}$.

Now define $\psi : \Gamma \to \mathbb{C}$ by $\psi = \sum_1^\infty \alpha_j \phi^j$ and assume that $\psi \in \mathscr{M}(H^1)$. Then there exists a constant $C_2 > 0$ so that for every $h \in H^1(G)$ we have $\|(\psi \hat{h})^\vee\|_{H^1} \le C \|h\|_{H^1}$. Choose $k_0 \in \mathbb{N}$ so that

$$2^{k_0+1} |\alpha_1 \lambda_1| \ge 2 C_1 C_2 \left(\sum_1^\infty |\lambda_j|^2 \right)^{1/2}.$$

Now

$$(\psi \hat{h}_{k_0})^\vee(x) = \sum_{j=N_{k_0}+1}^{N_{k_0+1}} \alpha_j \lambda_j \gamma_j(x) A_{-j}(x)$$

and we see that

$$\|(\psi \hat{h}_{k_0})^\vee\|_{H^1} \ \ge \ \|(\psi \hat{h}_{k_0})^\vee\|_1 \ = \sum_{j=N_{k_0+1}}^{N_{k_0}+1} |\alpha_j \lambda_j| \ \ge \ 2C_2 \|h_{k_0}\|_{H^1},$$

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a contradiction. Thus we have shown that $\psi \notin \mathcal{M}(H^1)$.

Finally, if $\sum_{-\infty}^{-1} |\alpha_j|^2 = \infty$, then except for some minor changes, an argument like the preceding one leads again to functions ϕ and ψ with the required properties. This completes the proof of Theorem 2.5.

As our final result we present a theorem whose proof is a minor variation of the proof of Theorem 2.1. We then briefly indicate how this theorem is related to a result of Seeger in [6] about multipliers for $L^p(\mathbb{R}^n)$-spaces.

THEOREM 2.6. *Let* $0 < p \le 1$. *Assume* $\phi \in L^\infty(\Gamma)$ *satisfies*

$$\sup_k \sum_{j=k}^{\infty} ((m_j)^{1/p-1} \|(\phi^j)^\vee \chi_{G\setminus G_k}\|_p)^{2p/(2-p)} < \infty.$$

Then $\phi \in \mathcal{M}(H^p)$.

PROOF. We use the same notation as in the proof of Theorem 2.1(a) and we consider the case $r = p$. Then $A \le C$ and we have, according to (2.5),

$$\begin{aligned} B &\le C \sum_{j=n}^{\infty} (m_j)^{1-p} \|a_j\|_p^p \sum_{k=-\infty}^{n-1} \|(\phi^j)^\vee \chi_{G_k\setminus G_{k+1}}\|_p^p \\ &= C \sum_{j=n}^{\infty} (m_j)^{1-p} \|a_j\|_p^p \|(\phi^j)^\vee \chi_{G\setminus G_n}\|_p \\ &\le C \left(\sum_{j=n}^{\infty} \|a_j\|_p^2\right)^{p/2} \left(\sum_{j=n}^{\infty} ((m_j)^{1-p} \|(\phi^j)^\vee \chi_{G\setminus G_n}\|_p^p)^{2/(2-p)}\right)^{(2-p)/2} \le C. \end{aligned}$$

Thus $\phi \in \mathcal{M}(H^p)$.

COROLLARY 2.5. *Let* $0 < p \le 1$ *and* $\phi \in L^\infty(\Gamma)$. *If there exists* $\varepsilon > 0$ *such that for every* $n \in \mathbb{Z}$,

$$\sup_j (m_j)^{\varepsilon+1/p-1} \|(\phi^j)^\vee \chi_{G\setminus G_n}\|_p \le C(m_n)^\varepsilon,$$

then $\phi \in \mathcal{M}(H^p)$.

PROOF. For each $n \in \mathbb{Z}$ we have

$$\sum_{j=n}^{\infty} ((m_j)^{1/p-1} \|(\phi^j)^\vee \chi_{G\setminus G_n}\|_p)^{2p/(2-p)} \le C(m_n)^{\varepsilon 2p/(2-p)} \sum_{j=n}^{\infty} (m_j)^{-\varepsilon 2p/(2-p)} \le C,$$

by inequality (1.1), because $\varepsilon 2p/(2-p) > 0$. Thus we may conclude that $\phi \in \mathscr{M}(H^p)$.

REMARK 5. In [6, Theorem 1] Seeger used a restriction on

$$\sup_{t>0} |(\phi m(t\cdot))^\vee \chi_{|x|\geq\omega}\|_1 \tag{2.9}$$

to prove that certain $m \in L^\infty(\mathbb{R}^n)$ are multipliers for $L^p(\mathbb{R}^n)$, $1 < p < \infty$, see [6, Section 1] for details. On G the analogue of (2.9) is

$$\sup_n \|(\phi^j)^\vee \chi_{G\setminus G_n}\|_1.$$

Thus, Corollary 2.5 may be considered as a version on locally compact Vilenkin groups G of an extension to Hardy spaces H^p, $0 < p \leq 1$, of Seeger's multiplier theorem for $L^p(\mathbb{R}^n)$-spaces, $1 < p < \infty$.

CONCLUDING REMARK. At various places throughout this paper we have compared our results to certain multiplier theorems for Lebesgue or Hardy spaces defined on $\mathbb{R}^n$. The results presented here raise obvious questions and conjectures for possible additional multiplier theorems for the $H^p(\mathbb{R}^n)$-spaces. We intend to report on some of these questions elsewhere.

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