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Vector fields and admissible embeddings for quiver moduli

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Abstract

We introduce a double framing construction for moduli spaces of quiver representations. This allows us to reduce certain sheaf cohomology computations involving the universal representation, to computations involving line bundles, making them amenable to methods from geometric invariant theory. We will use this to show that in many good situations the vector fields on the moduli space are isomorphic as vector spaces to the first Hochschild cohomology of the path algebra. We also show that considering the universal representation as a Fourier–Mukai kernel in the appropriate sense gives an admissible embedding of derived categories.

Contents

1. Introduction

The universal object on a fine moduli space allows us to probe the geometry of the moduli space. We will apply this principle to moduli spaces of quiver representations, in order to describe their (infinitesimal) symmetries, and to show that the universal object provides an admissible

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embedding of the derived category of the path algebra into the derived category of the moduli space.

Let Q be an acyclic quiver, with $\mathbf{d}$ an indivisible dimension vector and θ a stability parameter, such that $\mathbf{d}$ is θ-amply stable (i.e., the unstable locus is of codimension at least 2) and stability agrees with semistability. These standing assumptions are codified in Assumption 2.4. Then the moduli space $X := \mathrm{M}^{\theta\text{-st}}(Q,\mathbf{d})$ of θ-(semi)stable representations of dimension vector $\mathbf{d}$ is a smooth projective variety, which comes equipped with a universal object $\mathcal{U} = ((\mathcal{U}_i)_{i\in Q_0}, (\mathcal{U}_a)_{a\in Q_1})$, which is a representation of Q with values in vector bundles on X. We can interpret $\mathcal{U}$ as a left $\mathcal{O}_X Q$-module by identifying it with $\bigoplus_{i\in Q_0}\mathcal{U}_i$ equipped with the structure of a left $\mathbf{k}Q$-module given by the morphisms $\mathcal{U}_a$.

We denote by e_i the idempotent associated to $i\in Q_0$, so that $e_j\mathbf{k}Qe_i$ is the vector space spanned by paths from i to j. We define the morphism

$$H^{\mathcal{U}}_{i,j}\colon e_j\mathbf{k}Qe_i \to \mathcal{H}om(\mathcal{U}_i,\mathcal{U}_j) = \mathcal{U}_i^\vee\otimes\mathcal{U}_j : p\mapsto \mathcal{U}_p, \tag{1}$$

where p is an oriented path $a_\ell\cdots a_1$ from i to j, and $\mathcal{U}_p$ is the composition $\mathcal{U}_{a_\ell}\circ\cdots\circ\mathcal{U}_{a_1}\colon\mathcal{U}_i\to\mathcal{U}_j$. Taking global sections, we obtain the morphism

$$h^{\mathcal{U}}_{i,j}\colon e_j\mathbf{k}Qe_i \to \mathrm{Hom}(\mathcal{U}_i,\mathcal{U}_j)\cong \mathrm{H}^0(X,\mathcal{U}_i^\vee\otimes\mathcal{U}_j). \tag{2}$$

The main (technical) result of this paper is the following theorem.

Theorem A. *Let Q, $\mathbf{d}$ and θ be as in* Assumption 2.4. *Denote $X=\mathrm{M}^{\theta\text{-st}}(Q,\mathbf{d})$, and consider the universal representation $\mathcal{U}$ on X. Then:*

(i) *for all $i,j\in Q_0$ the linear map $h^{\mathcal{U}}_{i,j}$ from* (2) *is an isomorphism;*

(ii) *the direct sum over $i,j\in Q_0$ of the isomorphisms $h^{\mathcal{U}}_{i,j}$ induces an isomorphism of algebras*

$$h^{\mathcal{U}}\colon \mathbf{k}Q \xrightarrow{\cong} \mathrm{End}_X(\mathcal{U}). \tag{3}$$

We will use Theorem A to describe vector fields on quiver moduli, and prove that the universal representation gives an admissible embedding of derived categories.

To obtain these applications, we will build upon the main result of [BBF+23], which requires a slightly stronger condition than just θ-ample stability, called θ-strong ample stability, which will be defined in Definition 2.3.

Vector fields. The first application of Theorem A is a recipe for computing vector fields on $X=\mathrm{M}^{\theta\text{-st}}(Q,\mathbf{d})$, that is, a description of $\mathrm{H}^0(X,\mathrm{T}_X)$, as a measure of the symmetry group of X.

Theorem B. *Let Q, $\mathbf{d}$ and θ be as in* Assumption 2.4, *and assume in addition that $\mathbf{d}$ is θ-strongly amply stable. Denote $X=\mathrm{M}^{\theta\text{-st}}(Q,\mathbf{d})$. There exists the exact sequence*

$$0\to\mathbf{k}\xrightarrow{\phi}\bigoplus_{i\in Q_0} e_i\mathbf{k}Qe_i\xrightarrow{\psi}\bigoplus_{a\in Q_1} e_{\mathrm{t}(a)}\mathbf{k}Qe_{\mathrm{s}(a)}\to\mathrm{H}^0(X,\mathrm{T}_X)\to 0, \tag{4}$$

where the maps ϕ and ψ are defined as

$$\phi(z) = z\sum_{i\in Q_0} e_i, \tag{5}$$

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$$\psi\left(\sum_{i\in Q_0} z_i e_i\right)=\sum_{a\in Q_1}(z_{\mathrm{t}(a)}-z_{\mathrm{s}(a)})a, \tag{6}$$

for $z, z_i \in \mathbf{k}$.

After the statement of Theorem 4.5 we explain how the sequence (4) is similar to a presentation of the first Hochschild cohomology of the path algebra $\mathbf{k}Q$, which is also a measure of a symmetry group, leading to an isomorphism of vector spaces

$$\mathrm{H}^0(X,\mathrm{T}_X)\cong \mathrm{HH}^1(\mathbf{k}Q). \tag{7}$$

This isomorphism has a precursor in the relationship between (infinitesimal) symmetries of a variety and (infinitesimal) symmetries of a moduli space of sheaves on the variety. The first example is given by the moduli space $\mathrm{M}_C(r,\mathcal{L})$ of stable vector bundles of rank $r\geq 2$ and determinant $\mathcal{L}$, on the smooth projective curve C of genus $g\geq 2$, such that $\gcd(r,\deg\mathcal{L})=1$, for which there exists an isomorphism

$$\mathrm{H}^0(\mathrm{M}_C(r,\mathcal{L}),\mathrm{T}_{\mathrm{M}_C(r,\mathcal{L})})\cong \mathrm{H}^0(C,\mathrm{T}_C), \tag{8}$$

both sides being zero by [NR75, Theorem 1(a)]. The second example is given by the Hilbert scheme $\mathrm{Hilb}^n S$ of n points on a smooth projective surface S, for which there exists an isomorphism

$$\mathrm{H}^0(\mathrm{Hilb}^n S,\mathrm{T}_{\mathrm{Hilb}^n S})\cong \mathrm{H}^0(S,\mathrm{T}_S), \tag{9}$$

by [Boi12, Corollaire 1].

In (9), the isomorphism is in fact induced from an inclusion of algebraic groups $\mathrm{Aut}(S)\hookrightarrow \mathrm{Aut}(\mathrm{Hilb}^n S)$ (see more on this below the next statement), which after taking Lie algebras means that (9) is an isomorphism of Lie algebras. This brings us to the following conjecture.

Conjecture C. In the setting of Theorem B there exists a naturally induced isomorphism of Lie algebras

$$\mathrm{H}^0(X,\mathrm{T}_X)\cong \mathrm{HH}^1(\mathbf{k}Q), \tag{10}$$

where the Lie algebra structure on the left (respectively, right) is given by the Schouten–Nijenhuis bracket of vector fields (respectively, the Gerstenhaber bracket).

In Example 4.6 we give an example where (10) is manifestly not an isomorphism of Lie algebras without the ample stability condition, following the failure of (2) being an isomorphism in Theorem A.

In fact, more precise results on the automorphism groups of these two families of moduli spaces are now available. For $\mathrm{M}_C(r,\mathcal{L})$ the automorphism group is described in terms of automorphisms of C and r-torsion in the Jacobian of C [KP95]. For $\mathrm{Hilb}^n S$ the description of the automorphism group depends on the geometry of S. If S has a big and nef (anti)canonical bundle then $\mathrm{Aut}(S)\cong \mathrm{Aut}(\mathrm{Hilb}^n S)$ [BOR20, Theorem 1]; see also [Hay20, Theorem 1.3] for a similar result for rational surfaces of Iitaka dimension at least 1. If, however, S is a K3 surface, a rich theory of *non-natural* automorphisms exists, starting with [Bea83, §6].

The extent to which (10) also holds *without* taking Lie algebras (and thus on the level of algebraic groups, where discrete contributions are possible), and thus to which extent the analogue of [BOR20] holds, is not clear.

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Admissible embeddings. The second application of Theorem A should be seen in the context of Schofield's conjecture, as stated on [Hil96, page 80], which says that $\mathcal{U}$ is a partial tilting object. It is settled in [BBF+23] for a large class of quiver moduli. Theorem A gives further information about this partial tilting object, namely about the algebra structure on the (derived) endomorphisms.

We will rephrase this result using the Fourier–Mukai(-like) functor

$$\Phi_{\mathcal{U}}\colon \mathbf{D}^{\mathrm{b}}(\mathbf{k}Q) \to \mathbf{D}^{\mathrm{b}}(\mathrm{M}^{\theta\text{-st}}(Q,\mathbf{d})), \tag{11}$$

where we continue to assume Assumption 2.4, so that there exists a universal representation $\mathcal{U}$. The functor (11) is defined on objects as

$$\Phi_{\mathcal{U}}(V) = \mathbf{R}\mathcal{H}om_{\mathcal{O}_X Q}(\mathcal{U}, V \otimes_{\mathbf{k}} \mathcal{O}_X), \tag{12}$$

where $\mathcal{H}om_{\mathcal{O}_X Q}(-,-)$ is the sheafy Hom for coherent left $\mathcal{O}_X Q$-modules, which has a natural coherent $\mathcal{O}_X$-module structure in our setup; see, for example, [BF24, §3.1].

Theorem D. *Let Q, $\mathbf{d}$ and θ be as in* Assumption 2.4, *and assume in addition that $\mathbf{d}$ is θ-strongly amply stable. Consider $X = \mathrm{M}^{\theta\text{-st}}(Q,\mathbf{d})$. The functor* (11) *is fully faithful.*

There are in fact four natural Fourier–Mukai-like functors to be considered and compared. We will discuss this in §5, and show that all four are fully faithful, because they are all related to each other.

This result was only known in the thin (and thus toric) case for the canonical stability condition by Altmann–Hille [AH99, Theorem 1.3], and for quiver flag varieties by Craw, Ito, and Karmazyn [CIK18, Example 2.9]. We state the precise conditions for the toric case in Theorem 5.3 to illustrate how our methods and result generalise this setting.

As with Theorem B, this result has parallel results in the context of moduli spaces of sheaves. The first example is given by moduli spaces of vector bundles on a curve, for which the full faithfulness of

$$\Phi_{\mathcal{E}}\colon \mathbf{D}^{\mathrm{b}}(C) \to \mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(r,\mathcal{L})) \tag{13}$$

for the universal vector bundle $\mathcal{E}$ on $C \times \mathrm{M}_C(r,\mathcal{L})$ is in various levels of generality obtained in [BM19, FK18, LM23, Nar17]. The second example is given by Hilbert schemes of points on surfaces, for which the full faithfulness of

$$\Phi_{\mathcal{I}}\colon \mathbf{D}^{\mathrm{b}}(S) \to \mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}^n S), \tag{14}$$

for the universal ideal sheaf $\mathcal{I}$ on $S \times \mathrm{Hilb}^n S$ if and only if $\mathcal{O}_S$ is an exceptional object, is established in [BK24, KS15].

A link between admissible embeddings using universal objects and vector fields (and deformation theory) is explained for Hilbert schemes of points in [BFR19]. In Proposition 5.8 we will explain how the same method works for quiver moduli, and thus how Theorem B and the rigidity result of [BBF+23] (recalled in Corollary 4.4) can be obtained from the statement (and not the ingredients of the proof) of Theorem D.

2. Quiver moduli

Construction of the moduli space. We first recall the GIT construction of the moduli space of stable quiver representations, to set up the notation, as introduced by King in [Kin94]. For more

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background on this one is referred to [Rei08], and for a stacky construction one is referred to [BDF+22].

Let $Q=(Q_0,Q_1)$ be a quiver, where we write $\mathrm{s}(a)$ (respectively, $\mathrm{t}(a)$) for the source (respectively, target) of $a\in Q_1$. We will denote the path algebra by $\mathbf{k}Q$, and throughout we will work with left $\mathbf{k}Q$-modules.

Fixing a dimension vector $\mathbf{d}\in\mathbb{N}^{Q_0}$, we have the affine space

$$\mathrm{R}(Q,\mathbf{d}):=\prod_{a\in Q_1}\mathbb{A}^{d_{\mathrm{s}(a)}d_{\mathrm{t}(a)}}, \tag{15}$$

as the parameter space for representations of Q of dimension vector $\mathbf{d}$. This comes with an action of the group

$$\mathrm{G}_{\mathbf{d}}:=\prod_{i\in Q_0}\mathrm{GL}_{d_i}, \tag{16}$$

acting by conjugation in the usual way. Its orbits are the isomorphism classes, but to get a well-behaved moduli space we need to introduce one more ingredient.

Let $\theta\in\mathrm{Hom}(\mathbb{Z}^{Q_0},\mathbb{Z})$ such that $\theta(\mathbf{d})=0$, which we call a *stability parameter*. Then we say that a representation M corresponding to a point $M\in\mathrm{R}(Q,\mathbf{d})$ is *θ-semistable* if $\theta(\underline{\dim} N)\leq 0$ for all non-zero and proper subrepresentations N of M, and we say it is *θ-stable* if the inequality is strict. We will tacitly identify $\mathrm{Hom}(\mathbb{Z}^{Q_0},\mathbb{Z})$ and $\mathbb{Z}^{Q_0}$ in what follows.

This gives us the $\mathrm{G}_{\mathbf{d}}$-stable open subsets

$$\mathrm{R}^{\theta\text{-st}}(Q,\mathbf{d})\subseteq\mathrm{R}^{\theta\text{-sst}}(Q,\mathbf{d})\subseteq\mathrm{R}(Q,\mathbf{d}), \tag{17}$$

which after the GIT quotient by $\mathrm{G}_{\mathbf{d}}$ with respect to the polarisation given by θ gives

$$\mathrm{M}^{\theta\text{-st}}(Q,\mathbf{d})\subseteq\mathrm{M}^{\theta\text{-sst}}(Q,\mathbf{d})\to\mathrm{M}^{(Q,\mathbf{d})}. \tag{18}$$

We have that $\mathrm{M}^{\theta\text{-st}}(Q,\mathbf{d})$ is a smooth variety, the first morphism is an open immersion, and the second is projective. If Q is acyclic then $\mathrm{M}^{(Q,\mathbf{d})}\cong\operatorname{Spec}k$.

If $\mathbf{d}$ is indivisible then we can obtain a universal representation $\mathcal{U}=\mathcal{U}(\mathbf{a})$ on $\mathrm{M}^{\theta-\mathrm{st}}(Q,\mathbf{d})$, which depends on the choice of an $\mathbf{a}\in\mathrm{Hom}(\mathbb{Z}^{Q_0},\mathbb{Z})$ such that $\mathbf{a}(\mathbf{d})=1$; see, for example, [BF24, §2.1]. We can decompose $\mathcal{U}$ as $\bigoplus_{i\in Q_0}\mathcal{U}_i$, such that at a point $[M]\in\mathrm{M}^{\theta-\mathrm{st}}(Q,\mathbf{d})$ corresponding to an isomorphism class of θ-stable representations, the fibre of $\mathcal{U}_i$ is the vector space M_i.

We can summarise the preceding setup as follows.

Proposition 2.1. *Let Q be an acyclic quiver, $\mathbf{d}$ a dimension vector, and θ a stability parameter such that:*

- *$\mathbf{d}$ is indivisible;*
- *every θ-semistable representation of dimension vector $\mathbf{d}$ is θ-stable.*

Then $\mathrm{M}^{\theta\text{-st}}(Q,\mathbf{d})$ is a smooth projective variety, which comes equipped with a universal bundle $\mathcal{U}(\mathbf{a})$ for every $\mathbf{a}\in\mathrm{Hom}(\mathbb{Z}^{Q_0},\mathbb{Z})$ such that $\mathbf{a}(\mathbf{d})=1$.

Ample stability. We need our quiver moduli spaces to be particularly nice, beyond what is assumed in Proposition 2.1.

Definition 2.2. A dimension vector $\mathbf{d}$ is θ-amply stable if

$$\operatorname{codim}_{\mathrm{R}(Q,\mathbf{d})}(\mathrm{R}(Q,\mathbf{d})\setminus\mathrm{R}^{\theta\text{-st}}(Q,\mathbf{d}))\geq 2. \tag{19}$$

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This condition in particular ensures that $\operatorname{Pic} \mathrm{M}^{\theta\text{-st}}(Q,\mathbf{d}) \cong \mathbb{Z}^{\#Q_0-1}$ [FRS21, Proposition 3.1]. In Example 4.6 we give an example where Theorem B fails when ample stability does not hold, thus explaining why we need to assume an additional property, like the one in Definition 2.2.

We also need the following slightly stronger condition in order to apply [BBF+23]. As explained in op. cit., it is expected that this condition can be omitted from the results in op. cit.

Definition 2.3. A dimension vector $\mathbf{d}$ is θ-strongly amply stable if for every subdimension vector $\mathbf{e} \leq \mathbf{d}$ for which $\mu(\mathbf{e}) \geq \mu(\mathbf{d}-\mathbf{e})$ we have $\langle \mathbf{e}, \mathbf{d}-\mathbf{e}\rangle \leq -2$.

Here, $\mu = \mu_\theta$ denotes the slope function attached to the stability parameter θ, which for a dimension vector $\mathbf{d}$ is defined as $\mu_\theta(\mathbf{d}) := \theta(\mathbf{d})/|\mathbf{d}|$, where $|\mathbf{d}| = \sum_{i\in Q_0} d_i$, whilst $\langle -,-\rangle$ refers to the Euler form.

By [RS17, Proposition 5.1] we have that strong ample stability implies ample stability. In [BBF+23, Example 4.6] an example is given where the converse implication does not hold. For the proof of Theorem B we will need the stronger notion, because we will appeal to the main result of [BBF+23], but as in op. cit., we expect the results hold for ample stability.

Standing assumptions. To ensure all the good properties discussed above, we introduce the following conditions.

Assumption 2.4. For Q, $\mathbf{d}$, and θ we assume that:

(i) Q is acyclic;
(ii) $\mathbf{d}$ is indivisible;
(iii) every θ-semistable representation of dimension vector $\mathbf{d}$ is θ-stable;
(iv) $\mathbf{d}$ is θ-amply stable.

We will, moreover, take $\mathbf{k}$ to be algebraically closed and of characteristic 0. This is a standing assumption in some of the works we build upon, notably [BBF+23].

3. Double framing construction

In this section, we will introduce a construction to reduce the computation of sheaf cohomology of universal bundles on a quiver moduli space to sheaf cohomology of universal line bundles on another quiver moduli space. We will do this via a double framing construction which will realise a fibre product of projectivisations of universal bundles as a quiver moduli space.

On fibre products of projective bundles. Let us first collect some general facts on projective bundles. Let X be a variety and let E be a vector bundle of rank $r+1$ on X. Let

$$p\colon \mathbb{P}(E) = \mathbb{P}_X(E) = \operatorname{Proj}\operatorname{Sym}^\bullet_{\mathcal{O}_X}(E) \to X \tag{20}$$

be the projectivisation of E; here Proj denotes the relative Proj over X. Its fibre in a point $x \in X$ consists of one-dimensional quotients of E_x. The universal line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a quotient of p^*E.

Let F be another vector bundle on X, and consider the projectivisation

$$q\colon \mathbb{P}(F) \to X. \tag{21}$$

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Define Y as the following fibre product.

$$\begin{array}{ccc} Y & \xrightarrow{q'} & \mathbb{P}(E) \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{P}(F) & \xrightarrow[q]{} & X. \end{array} \tag{22}$$

Then

$$Y \cong \mathbb{P}_{\mathbb{P}(E)}(p^*F) \cong \mathbb{P}_{\mathbb{P}(F)}(q^*E). \tag{23}$$

Let $m, n \in \mathbb{Z}$. We define

$$\mathcal{O}_Y(m,n) := q'^*\mathcal{O}_{\mathbb{P}(E)}(m) \otimes p'^*\mathcal{O}_{\mathbb{P}(F)}(n). \tag{24}$$

Applying the projection formula and flat base change, together with [Sta23, Lemma 01XX], we have that

$$\mathrm{H}^i(Y, \mathcal{O}_Y(1,1)) \cong \mathrm{H}^i(X, E \otimes F). \tag{25}$$

Double framing construction. Let Q be a quiver, $\mathbf{d}$ a dimension vector, and θ a stability parameter, satisfying Assumption 2.4.

Let $X := \mathrm{M}^{\theta\text{-st}}(Q, \mathbf{d})$ denote the moduli space of θ-(semi)stable representations of dimension vector $\mathbf{d}$, and let $\mathcal{U}$ be the universal representation on X, which depends on the choice of a character $\mathbf{a}$ of weight 1 as in §2.

Fix vertices $i, j \in Q_0$; we allow $i = j$. Consider the fibre product

$$Y = \mathbb{P}_X(\mathcal{U}_i^\vee) \times_X \mathbb{P}_X(\mathcal{U}_j). \tag{26}$$

Our goal is to describe Y as a quiver moduli space. Let $\overline{Q}$ be the doubly-framed quiver defined by

$$\begin{cases} \overline{Q}_0 := Q_0 \sqcup \{0, \infty\}, \\ \overline{Q}_1 := Q_1 \sqcup \{0 \to i, j \to \infty\}, \end{cases} \tag{27}$$

where 0 and ∞ are two new symbols. This quiver depends on the choice of i and j, but we will suppress this in the notation. The notation $\overline{Q}$ is not to be confused with the doubled quiver, as for instance in the definition of the preprojective algebra.

We define a doubly-framed dimension vector $\overline{\mathbf{d}}$ by

$$\overline{d}_k := \begin{cases} d_k & k \in Q_0 \\ 1 & k \in \{0, \infty\}. \end{cases} \tag{28}$$

We will also denote this dimension vector by $\overline{\mathbf{d}} = (1, \mathbf{d}, 1)$.

A representation of $\overline{Q}$ of dimension vector $\overline{\mathbf{d}}$ is a triple (v, M, ϕ) which consists of:

- a representation M of Q with dimension vector $\mathbf{d}$;
- a vector $v \in M_i$;
- a linear form $\phi \in M_j^\vee$.

The representation variety $\mathrm{R}(\overline{Q}, \overline{\mathbf{d}})$ can therefore be written as

$$\mathrm{R}(\overline{Q}, \overline{\mathbf{d}}) = \mathbb{A}^{d_i} \times \mathrm{R}(Q, \mathbf{d}) \times \mathbb{A}^{d_j}, \tag{29}$$

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where the factor $\mathbb{A}^{d_i}$ encodes the choice of $v \in M_i$, and the factor $\mathbb{A}^{d_j}$ encodes the choice of $\phi \in M_j^\vee$.

Next, we fix a natural number N, which we will choose sufficiently big later on. We define a stability parameter $\overline{\theta} = \overline{\theta}_N$ as $(1, N\theta, -1)$ using the identification $\mathbb{Z}^{\overline{Q}_0} = \mathbb{Z} \oplus \mathbb{Z}^{Q_0} \oplus \mathbb{Z}$ where the first (respectively, third) summand corresponds to $k = 0$ (respectively, $k = \infty$). The following proposition describes the stable and semistable locus with respect to $\overline{\theta}$.

Proposition 3.1. *Let (v, M, ϕ) be a representation of $\overline{Q}$ of dimension vector $\overline{\mathbf{d}}$. The following assertions are equivalent:*

(i) *(v, M, ϕ) is $\overline{\theta}$-stable.*
(ii) *(v, M, ϕ) is $\overline{\theta}$-semistable.*
(iii) *M is θ-(semi)stable, $v \neq 0$, and $\phi \neq 0$.*

We first prove a lemma. For a stability parameter θ for Q such that $\theta(\mathbf{d}) = 0$ we define the following sets of dimension vectors:

$$B_+(\theta) = \{\mathbf{e} \in \mathbb{N}_0^{Q_0} \mid 0 \leq \mathbf{e} \leq \mathbf{d} \text{ and } \theta(\mathbf{e}) > 0\}, \tag{30}$$

$$B_-(\theta) = \{\mathbf{e} \in \mathbb{N}_0^{Q_0} \mid 0 \leq \mathbf{e} \leq \mathbf{d} \text{ and } \theta(\mathbf{e}) < 0\}, \tag{31}$$

$$B_0(\theta) = \{\mathbf{e} \in \mathbb{N}_0^{Q_0} \mid 0 \leq \mathbf{e} \leq \mathbf{d} \text{ and } \theta(\mathbf{e}) = 0\}. \tag{32}$$

They depend on $\mathbf{d}$ but as the dimension vector will be clear from the context, we choose to neglect the dependency on $\mathbf{d}$ in the notation. We determine the analogous sets for $\overline{\theta}$ (with respect to $\overline{\mathbf{d}} = (1, \mathbf{d}, 1)$ as in (28)).

Lemma 3.2. *For N sufficiently large we have*

$$\begin{aligned} B_+(\overline{\theta}) &= \{(1,\mathbf{e},0) \mid \mathbf{e} \in B_0(\theta)\} \cup \{(0,\mathbf{e},0),(1,\mathbf{e},0),(0,\mathbf{e},1),(1,\mathbf{e},1) \mid \mathbf{e} \in B_+(\theta)\}, \\ B_-(\overline{\theta}) &= \{(0,\mathbf{e},1) \mid \mathbf{e} \in B_0(\theta)\} \cup \{(0,\mathbf{e},0),(1,\mathbf{e},0),(0,\mathbf{e},1),(1,\mathbf{e},1) \mid \mathbf{e} \in B_-(\theta)\}, \\ B_0(\overline{\theta}) &= \{(0,\mathbf{e},0),(1,\mathbf{e},1) \mid \mathbf{e} \in B_0(\theta)\}. \end{aligned} \tag{33}$$

Proof. Let $\mathbf{e}$ be a dimension vector such that $0 \leq \mathbf{e} \leq \mathbf{d}$. For any dimension vector of the form $(a, \mathbf{e}, b)$ with $a, b \in \{0, 1\}$ we have

$$\overline{\theta}(a, \mathbf{e}, b) = a + N\theta(\mathbf{e}) - b. \tag{34}$$

The equalities in (33) are then immediate, using that $\overline{\theta}(a, \mathbf{e}, b) > 0$ for $N \gg 0$ if $\mathbf{e} \in B_+(\theta)$, $\overline{\theta}(a, \mathbf{e}, b) < 0$ for $N \gg 0$ if $\mathbf{e} \in B_-(\theta)$, and $\overline{\theta}(a, \mathbf{e}, b) = a - b$ if $\mathbf{e} \in B_0(\theta)$. □

Proof of Proposition 3.1. Item (i) obviously implies item (ii).

Now we prove that Item (ii) implies Item (iii). Let (v, M, ϕ) be a $\overline{\theta}$-semistable representation. Let $\mathbf{e} \in B_+(\theta)$. As $(0, \mathbf{e}, 1)$ lies in $B_+(\overline{\theta})$, we see that M cannot have a subrepresentation M' of dimension vector $\mathbf{e}$, for otherwise $(0, M', \phi|_{M'_j})$ would be a subrepresentation of (v, M, ϕ) of dimension vector $(0, \mathbf{e}, 1)$, contradicting semistability of (v, M, ϕ). Also, $v \neq 0$ because $(1, \mathbf{0}, 0) \in B_+(\overline{\theta})$ and $\phi \neq 0$ because $(1, \mathbf{d}, 0) \in B_+(\overline{\theta})$.

Finally, we show that Item (iii) implies Item (i). Let $(a, \mathbf{e}, b) \in B_+(\overline{\theta}) \cup B_0(\overline{\theta})$. By the description of the sets $B_+(\overline{\theta})$ and $B_0(\overline{\theta})$ in Lemma 3.2, we see that $\theta(\mathbf{e}) \geq 0$. Assume that (v, M, ϕ) had a subrepresentation of dimension vector $(a, \mathbf{e}, b)$. In particular, M would have a subrepresentation of dimension vector $\mathbf{e}$ which, by semistability, implies that $\theta(\mathbf{e}) \leq 0$. So $\theta(\mathbf{e}) = 0$ must hold. As

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M is θ-stable, this implies that $\mathbf{e} \in \{\mathbf{0}, \mathbf{d}\}$. This shows that $(a, \mathbf{e}, b)$ is one of the following six dimension vectors:

$$(0, \mathbf{0}, 0),\ (1, \mathbf{0}, 0),\ (1, \mathbf{0}, 1),\ (0, \mathbf{d}, 0),\ (1, \mathbf{d}, 0),\ (1, \mathbf{d}, 1). \tag{35}$$

There can be no subrepresentations of (v, M, ϕ) of dimension vector $(1, \mathbf{0}, 0)$ or $(1, \mathbf{0}, 1)$ because $v \neq 0$. There can also be no subrepresentations of dimension vectors $(0, \mathbf{d}, 0)$ or $(1, \mathbf{d}, 0)$, because $\phi \neq 0$. The remaining two possibilities are the zero dimension vector and $\overline{\mathbf{d}}$. We have established that (v, M, ϕ) is $\overline{\theta}$-stable. □

Remark 3.3. The double framing construction resembles the framing construction in [ER09, Definition 3.1] for the construction of smooth models. Given a triple $(Q, \mathbf{d}, \theta)$ consisting of a quiver, a dimension vector, and a stability parameter, loc. cit. yields, when choosing $\mathbf{n} = \mathbf{e}_i$ another such triple $(\hat{Q}, \hat{\mathbf{d}}, \hat{\theta})$ for which a $\hat{\theta}$-(semi)stable representation is a pair (v, M) such that M is a θ-semistable representation of Q and $v \in M_i \setminus \{0\}$.

Dualising the construction of loc. cit., we obtain for a triple $(Q, \mathbf{d}, \theta)$ as above a triple $(\tilde{Q}, \tilde{\mathbf{d}}, \tilde{\theta})$ for which a $\tilde{\theta}$-(semi)stable representation is a pair (M, ϕ) such that M is a θ-semistable representation of Q and $\phi\colon M_j \to \mathbf{k}$ is a non-zero linear form.

Applying both constructions yields, independently of the order, the doubly-framed quiver $\overline{Q}$ and the dimension vector $\overline{\mathbf{d}}$. For the stability parameter, though, the order matters. The two resulting stability parameters do not agree, that is,

$$\tilde{\hat{\theta}} \neq \hat{\tilde{\theta}}, \tag{36}$$

and they are both different from $\overline{\theta}$. One can a posteriori use Proposition 3.1 to conclude that they are all GIT-equivalent, meaning that the (semi)stable loci with respect to these stability parameters agree.

Remark 3.4. Note also that the present double framing construction is essentially different from the one in [ABHR22], used for modelling neural network architectures using quiver moduli. Namely, the double framing in op. cit. adds a single vertex to the quiver, as well as arrows to and from it, in contrast to the two different extension vertices used here.

Proposition 3.1 implies the following result.

Corollary 3.5. *The doubly-framed $\overline{Q}$, $\overline{\mathbf{d}}$, and $\overline{\theta}$ satisfy (i), (ii), and (iii) of Assumption 2.4.*

Therefore $\mathrm{M}^{\overline{\theta}\text{-st}}(\overline{Q}, \overline{\mathbf{d}})$ is smooth and projective, and it possesses a universal representation $\overline{\mathcal{U}} = \overline{\mathcal{U}}(\overline{\mathbf{a}})$ dependent on the choice of an $\overline{\mathbf{a}}$ such that $\overline{\mathbf{a}} \cdot \overline{\mathbf{d}} = 1$. The summands $\overline{\mathcal{U}}_0$ and $\overline{\mathcal{U}}_\infty$ are line bundles.

The following lemma explains why we might still have to modify $\overline{Q}$, so that we can guarantee (iv) of Assumption 2.4.

Lemma 3.6. *The dimension vector $\overline{\mathbf{d}}$ is amply stable for $\overline{\theta}$ if and only if $d_i > 1$ and $d_j > 1$.*

Proof. By Proposition 3.1, the $\overline{\theta}$-unstable locus is the union

$$\mathrm{R}(\overline{Q}, \overline{\mathbf{d}}) \setminus \mathrm{R}^{\overline{\theta}\text{-sst}}(\overline{Q}, \overline{\mathbf{d}}) = \{(v, M, \phi) \mid M \text{ is } \overline{\theta} - \text{unstable}\} \cup \{(v, M, \phi) \mid v = 0\} \cup \{(v, M, \phi) \mid \phi = 0\}. \tag{37}$$

The first set has codimension at least 2 by θ-ample stability of $\mathbf{d}$. The second set has codimension d_i and the third has codimension d_j. This proves the claimed equivalence. □

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We have the forgetful morphism

$$u\colon \mathrm{M}^{\overline{\theta}\text{-st}}(\overline{Q},\overline{\mathbf{d}}) \to \mathrm{M}^{\theta\text{-st}}(Q,\mathbf{d}) : [v,M,\phi] \mapsto [M], \tag{38}$$

which forgets the framing data. We obtain $u^*\mathcal{U}_k \cong \overline{\mathcal{U}}_k$ for $k \in Q_0$, by choosing the character $\overline{\mathbf{a}}$ as $(0,\mathbf{a},0)$.

Using Proposition 3.1, we will now show that the two spaces

$$\overline{X} := \mathrm{M}^{\overline{\theta}-\mathrm{st}}(\overline{Q},\overline{\mathbf{d}}), \tag{39}$$

$$Y := \mathbb{P}(\mathcal{U}_i^\vee) \times_X \mathbb{P}(\mathcal{U}_j) \tag{40}$$

are isomorphic. To prove this, let $f\colon Y \to X$ be the diagonal morphism in diagram (22).

Proposition 3.7. *There exists an isomorphism $Y \cong \overline{X}$ over X such that:*

- *$\overline{\mathcal{U}}_k$ is identified with $f^*\mathcal{U}_k$ for all $k \in Q_0$ and $\overline{\mathcal{U}}_a$ is identified with $f^*\mathcal{U}_a$ for all $a \in Q_1$;*
- *$\overline{\mathcal{U}}_0$ is identified with $\mathcal{O}_Y(-1,0)$ and $\overline{\mathcal{U}}_{0\to i}$ is identified with $\mathcal{O}_Y(-1,0) \to f^*\mathcal{U}_i$;*
- *$\overline{\mathcal{U}}_\infty$ is identified with $\mathcal{O}_Y(0,1)$ and $\overline{\mathcal{U}}_{j\to\infty}$ is identified with $f^*\mathcal{U}_j \to \mathcal{O}_Y(0,1)$.*

Proof. We obtain morphisms in both directions from the universal properties of Y and $\overline{X}$ as follows.

Let (v,M,ϕ) be a $\overline{\theta}$-stable doubly-framed representation. Then $v \neq 0$ and $\phi \neq 0$ by Proposition 3.1. This implies that $\overline{\mathcal{U}}_0$ is a line subbundle of $\overline{\mathcal{U}}_i = u^*\mathcal{U}_i$ and $\overline{\mathcal{U}}_\infty$ is a line bundle quotient of $\overline{\mathcal{U}}_j = u^*\mathcal{U}_j$. The universal property of Y thus yields a morphism $\overline{X} \to Y$ over X under which $f^*\mathcal{U}_i^\vee \to \mathcal{O}_Y(1,0)$ pulls back to $\overline{\mathcal{U}}_{0\to i}^\vee\colon \overline{\mathcal{U}}_i^\vee \to \overline{\mathcal{U}}_0^\vee$ and $f^*\mathcal{U}_j \to \mathcal{O}_Y(0,1)$ pulls back to $\overline{\mathcal{U}}_{j\to\infty}\colon \overline{\mathcal{U}}_j \to \overline{\mathcal{U}}_\infty$.

Conversely, consider the pullback $f^*\mathcal{U}$ of the universal representation on $X = \mathrm{M}^{\theta-\mathrm{st}}(Q,\mathbf{d})$. Using the morphisms $\mathcal{O}_Y(0,1)^\vee \to f^*\mathcal{U}_i$ and $\mathcal{O}_Y(1,0)$ and $\mathcal{U}_j \to \mathcal{O}_Y(1,0)$, we obtain a representation of $\overline{Q}$ over Y of rank vector $\overline{\mathbf{d}}$ such that the fibre over every point in Y is $\overline{\theta}$-stable; the last statement follows again from Proposition 3.1. The universal property of $\overline{X}$ yields a morphism $Y \to \overline{X}$ under which the universal representation $\overline{\mathcal{U}}$ pulls back to the representation over Y described above.

The two morphisms can easily be seen to be inverse bijections on closed points, which suffices to show that $\overline{X}$ and Y are isomorphic. The identification of the universal bundles follows from the universal properties used in the construction. This proves the proposition. □

The previous proposition implies, using (25), the following result.

Corollary 3.8. *The diagram*

$$\begin{array}{ccccc} p & e_j\mathbf{k}Qe_i & \xrightarrow{h^{\mathcal{U}}_{i,j}} & \mathrm{H}^0(X,\mathcal{U}_i^\vee\otimes\mathcal{U}_j) & s \\ \downarrow & \downarrow & & \downarrow & \downarrow \\ (j\to\infty)p(0\to i) & e_\infty\mathbf{k}\overline{Q}e_0 & \xrightarrow{h^{\overline{\mathcal{U}}}_{0,\infty}} & \mathrm{H}^0(\overline{X}\ \overline{\mathcal{U}}_0^\vee\otimes\overline{\mathcal{U}}_\infty) & \overline{\mathcal{U}}_{j\to\infty}\circ u^*s\circ\overline{\mathcal{U}}_{0\to i} \end{array} \tag{41}$$

is commutative and the vertical maps are isomorphisms.

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Next, let us give a description of $\overline{X}$ from (39) as a quiver moduli space, which in addition satisfies condition (iv) of Assumption 2.4.

Proposition 3.9. *Let Q, $\mathbf{d}$ and θ be as in* Assumption 2.4, *and let $\overline{Q}$, $\overline{\mathbf{d}}$ and $\overline{\theta}$ be as constructed above. There exists a full subquiver Q' of $\overline{Q}$ with $Q_0 \subseteq Q'_0$ and two vertices $i', j' \in Q'_0$ such that the following assertions hold.*

(i) *There exist paths $q_0 \colon 0 \to i'$ and $q_\infty \colon j' \to \infty$ of lengths at most 1 such that*

$$e_{j'}\mathbf{k}Q'e_{i'} \to e_\infty \mathbf{k}\overline{Q}e_0, \;\; p \mapsto q_\infty p q_0 \tag{42}$$

is an isomorphism.

(ii) *Let $\mathbf{d}' = \overline{\mathbf{d}}|_{Q'_0}$ and consider the forgetful map*

$$\mathrm{R}(\overline{Q}, \overline{d}) \to \mathrm{R}(Q', \mathbf{d}'), \;\; N \mapsto N|_{Q'}. \tag{43}$$

There exists a stability parameter θ' with $\theta'(\mathbf{d}') = 0$ such that for every representation N of $\overline{Q}$ of dimension vector $\overline{\mathbf{d}}$ which is $\overline{\theta}$-(semi)stable the representation $N|_{Q'}$ is θ'-(semi)stable.

(iii) *For*

$$X' := \mathrm{M}^{\theta'\text{-st}}(Q', \mathbf{d}'), \tag{44}$$

with universal bundle $\mathcal{U}'$ the map $g \colon \overline{X} \to X'$ induced by $N \mapsto N|_{Q'}$ is an isomorphism such that $\overline{\mathcal{U}}_{q_0} \colon \overline{\mathcal{U}}_0 \to \overline{\mathcal{U}}_{i'} = g^*\mathcal{U}'_{i'}$ and $\overline{\mathcal{U}}_{q_\infty} \colon g^*\mathcal{U}'_{j'} = \overline{\mathcal{U}} \to \overline{\mathcal{U}}_\infty$ are isomorphisms.

(iv) *The data Q', $\mathbf{d}'$, and θ' fulfil* Assumption 2.4, *and $d'_{i'} = d'_{j'} = 1$.*

Proof. We need to distinguish between different scenarios.

(a) If $d_i > 1$ and $d_j > 1$, then we set $Q' := \overline{Q}$, $i' := 0$, and $j' := \infty$. Item (i) is fulfilled with $q_0 = e_0$ and $q_\infty = e_\infty$. We get $\mathbf{d}' = \overline{\mathbf{d}}$ and take $\theta' := \overline{\theta}$. The map g is the identity. Lemma 3.6 tells us that $\mathbf{d}'$ is amply stable for θ' in this case. Together with Corollary 3.5, we now deduce that Assumption 2.4 is satisfied.

(b) If $d_i > 1$ and $d_j = 1$, then let Q' be the full subquiver with $Q'_0 = \overline{Q}_0 \setminus \{\infty\} = Q_0 \sqcup \{0\}$, which is acyclic. Let $i' = 0$ and $j' = j$. Then with $q_0 = e_0$ and $q_\infty = (j \to \infty)$, Item (i) is satisfied. The dimension vector $\mathbf{d}' = (1, \mathbf{d})$ is indivisible, so it meets Assumption 2.4(ii). Let θ' be given by $(|\mathbf{d}|, (|\mathbf{d}|+1)N\theta - 1)$ for $N \gg 0$. Here, $|\mathbf{d}| = \sum_{i \in Q_0} d_i$. A representation of Q' of dimension vector $\mathbf{d}'$ is a pair (v, M) consisting of a representation M of Q of dimension vector $\mathbf{d}$ and a vector $v \in M_i$. In [ER09, Proposition 3.3] it is shown that the following are equivalent:

- (v, M) is θ'-semistable;
- (v, M) is θ'-stable;
- M is θ-semistable and $v \neq 0$.

This shows that Assumption 2.4(iii) holds, and also, with the same proof as Lemma 3.6, that Assumption 2.4(iv) is satisfied. The forgetful map is

$$\mathrm{R}(\overline{Q}, \overline{d}) \to \mathrm{R}(Q', \mathbf{d}'), \quad (v, M, \phi) \mapsto (v, M). \tag{45}$$

A representation (v, M, ϕ) of dimension vector $\overline{\mathbf{d}}$ is $\overline{\theta}$-semistable if and only if ϕ is an isomorphism and (v, M) is θ'-stable. As we can use the $\mathrm{G}_{\overline{\mathbf{d}}}$-action to transform ϕ to the

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identity, we obtain an isomorphism

$$g\colon \mathrm{M}^{\overline{\theta}\text{-st}}(\overline{Q},\overline{\mathbf{d}}) \to \mathrm{M}^{\theta'\text{-st}}(Q',\mathbf{d}'), \tag{46}$$

which pulls $\mathcal{U}'_{i'}$ back to $\overline{\mathcal{U}}_0$ and $\mathcal{U}'_{j'}$ back to $\overline{\mathcal{U}}_j \cong \overline{\mathcal{U}}_\infty$.

(c) If $d_i = 1$ and $d_j > 1$, then we define Q' as the full subquiver with $Q'_0 = \overline{Q}_0 \setminus \{0\} = Q_0 \sqcup \{\infty\}$. Let $i' = i$, $j' = \infty$, and θ' be defined by $((|\mathbf{d}|+1)N\theta + 1, -|\mathbf{d}|)$ for $N \gg 0$. We may argue dually to the proof of Item (ii).

(d) If $d_i = d_j = 1$, then we let $Q' = Q$, $i' = i$, and $j' = j$. With $\theta' = \theta$ the claim is then obviously true.

□

Corollary 3.10. *The diagram*

$$\begin{array}{ccccc} p & e_j\,\mathbf{k}Q'e_i & \xrightarrow{h^{\mathcal{U}'}_{i',j'}} & \mathrm{H}^0(X,\mathcal{U}_{i'}'^{\vee}\otimes\mathcal{U}'_{j'}) & s \\ \big\downarrow & \big\downarrow & & \big\downarrow & \big\downarrow \\ q_\infty p q_0 & e_\infty\mathbf{k}\overline{Q}e_0 & \xrightarrow{h^{\overline{\mathcal{U}}}_{0,\infty}} & \mathrm{H}^0(\overline{X},\overline{\mathcal{U}}_0^{\vee}\otimes\overline{\mathcal{U}}_\infty) & \overline{\mathcal{U}}_{q_\infty}\circ g^*s\circ\overline{\mathcal{U}}_{q_0} \end{array} \tag{47}$$

is commutative and the vertical maps are isomorphisms.

In the next section we will compute the global sections of the *line bundle* $\mathcal{U}_{i'}^\vee \otimes \mathcal{U}_{j'}$.

Remark 3.11. The double framing construction performed here, in order to obtain the reduction of the cohomology of vector bundles to that of line bundles as in Corollary 3.8, is parallel to what is done in [LM23, Proof of Theorem 7.1] for computing the cohomology of $\mathcal{U}_{x_1}^\vee \otimes \mathcal{U}_{x_2}$ for two distinct closed points $x_1, x_2 \in C$ on the moduli space $\mathrm{M}_C(r,\mathcal{L})$ of vector bundles on a curve.

Namely, denote $\mathcal{U}_x := \mathcal{U}|_{\{x\}\times\mathrm{M}_C(r,\mathcal{L})}$ where $\mathcal{U}$ is the universal vector bundle on $C \times \mathrm{M}_C(r,\mathcal{L})$. In op. cit. the double framing is performed by considering the moduli space $\mathrm{M}_C(r,\mathcal{L},\mathbf{e})$ of parabolic bundles, where the parabolic structure is considered in the two points x_1 and x_2, leading to the isomorphism

$$\mathrm{H}^i(\mathrm{M}_C(r,\mathcal{L}),\mathcal{U}_{x_1}^\vee\otimes\mathcal{U}_{x_2}) \cong \mathrm{H}^i(\mathrm{M}_C(r,\mathcal{L},\mathbf{e}),\mathcal{O}(1,1)). \tag{48}$$

One difference is that in op. cit. wall-crossing methods are used to show vanishing of all cohomology, whereas that is impossible in our set-up, as we can (and will) have global sections.

4. Global sections

4.1 Global sections of the endomorphism bundle

The universal representation $\mathcal{U}$ depends on the choice of the normalisation $\mathbf{a}$. But this dependence disappears when considering $\mathcal{H}om(\mathcal{U},\mathcal{U})$. We will thus compute the global sections of its summands $\mathcal{U}_i^\vee \otimes \mathcal{U}_j$. The higher cohomology will be dealt with using Theorem 4.3, which is taken from [BBF⁺23].

The proof of Theorem A uses the reduction techniques from section 3 and a lemma which is a consequence of the celebrated result of Le Bruyn and Procesi [BP90, Theorem 1], whose statement we first recall.

THEOREM 4.1 (Le Bruyn and Procesi). *Let Q be a (not necessarily acyclic) quiver and let $\mathbf{d}$ be a dimension vector. The ring $\mathbf{k}[\mathrm{R}(Q,\mathbf{d})]^{\mathrm{G}_\mathbf{d}}$ of invariant functions is generated, as a $\mathbf{k}$-algebra, by the functions*

$$\varphi_c\colon \mathrm{R}(Q,\mathbf{d}) \to \mathbf{k}, \quad M \mapsto \mathrm{tr}(M_c), \tag{49}$$

where c ranges over all oriented cycles of Q.

Here, and elsewhere, we use that an oriented cycle c has a distinguished starting vertex $\mathrm{s}(c)$, which is also the end vertex $\mathrm{t}(c)$. Thus, M_c is an endomorphism of $M_{\mathrm{s}(c)} = M_{\mathrm{t}(c)}$. It is possible to cyclically permute the vertices in the cycle c, and thus change the distinguished starting and end vertices. However, this does not change the value of $\mathrm{tr}(M_c)$, by the cyclic invariance of the trace.

Let $p = a_\ell \cdots a_1$ be an oriented path in the quiver Q. Let $\mathbf{d}$ be a dimension vector for which we have $d_{\mathrm{s}(p)} = d_{\mathrm{t}(p)} = 1$. As every $N \in \mathrm{R}(Q,\mathbf{d})$ comes equipped with a basis for each of the vector spaces N_i, we obtain $\mathrm{Hom}_\mathbf{k}(N_{\mathrm{s}(p)}, N_{\mathrm{t}(p)}) \cong \mathbf{k}$. We define a regular function

$$f_p\colon \mathrm{R}(Q,\mathbf{d}) \to \mathrm{Hom}_\mathbf{k}(N_{\mathrm{s}(p)}, N_{\mathrm{t}(p)}) \cong \mathbf{k}, \quad N \mapsto N_p, \tag{50}$$

where we identify the linear map $N_p = N_{a_\ell} \circ \ldots \circ N_{a_1}$ with the factor of the scalar multiplication which it performs on the basis vectors. As

$$f_p(g \cdot N) = g_{\mathrm{t}(p)} g_{\mathrm{s}(p)}^{-1} N_p, \tag{51}$$

for every $N \in \mathrm{R}(Q,\mathbf{d})$ and every $g \in \mathrm{G}_\mathbf{d}$, we see that f_p is a semi-invariant function of weight $\delta_{\mathrm{t}(p)} - \delta_{\mathrm{s}(p)}$.

This allows us to prove the following lemma.

LEMMA 4.2. *Let Q be an acyclic quiver. Let $0, \infty \in Q_0$ be two vertices, and let $\mathbf{d}$ be a dimension vector such that $d_0 = d_\infty = 1$. Then the morphism*

$$e_\infty \mathbf{k} Q e_0 \to \mathbf{k}[\mathrm{R}(Q,\mathbf{d})]^{\mathrm{G}_\mathbf{d}, \delta_\infty - \delta_0} : p \mapsto f_p \tag{52}$$

is an isomorphism of $\mathbf{k}$-vector spaces.

Proof. If $0 = \infty$ then the theorem of Le Bruyn and Procesi directly applies – note that the quiver is assumed to be acyclic. We may therefore assume that $0 \neq \infty$.

We consider the subgroup

$$H = \{g = (g_i)_{i \in Q_0} \in \mathrm{G}_\mathbf{d} \mid g_0 = g_\infty\} \tag{53}$$

of $\mathrm{G}_\mathbf{d}$. There is an isomorphism $\chi\colon H \times \mathbb{G}_\mathrm{m} \to \mathrm{G}_\mathbf{d}$ of algebraic groups defined by $\chi(h,t) = g$, where

$$g_i = \begin{cases} h_i & \text{if } i \neq \infty \\ t h_\infty & \text{if } i = \infty. \end{cases} \tag{54}$$

Now consider the algebra $\mathbf{k}[\mathrm{R}(Q,\mathbf{d})]^H$ of H-invariant polynomial functions. It is a $\mathbb{Z}$-graded algebra by the action of $\mathbb{G}_\mathrm{m}$. The space of semi-invariant functions with respect to $\mathrm{G}_\mathbf{d}$ of weight $\delta_\infty - \delta_0$ identifies, via χ, with the subspace of $\mathbf{k}[\mathrm{R}(Q,\mathbf{d})]^H$ on which $\mathbb{G}_\mathrm{m}$ acts linearly, that is,

$$\mathbf{k}[\mathrm{R}(Q,\mathbf{d})]^{\mathrm{G}_\mathbf{d}, \delta_\infty - \delta_0} = \left(\mathbf{k}[\mathrm{R}(Q,\mathbf{d})]^H\right)_1. \tag{55}$$

We will identify the latter with the invariant polynomial functions on a representation variety of a different quiver. Namely, let $Q^\flat$ be the quiver which arises from Q by identifying the vertices 0 and ∞; denote the 'merged' vertex by 0∞. Let $\mathbf{d}^\flat$ be the dimension vector with $d_i^\flat = d_i$ for all

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$i \in Q_0^\flat \setminus \{0\infty\}$ and $d^\flat_{0\infty} = 1$. There exist isomorphisms

$$\mathrm{R}(Q^\flat, \mathbf{d}^\flat) \cong \mathrm{R}(Q, \mathbf{d}) \tag{56}$$

and

$$\mathrm{G}_{\mathbf{d}^\flat} \cong H. \tag{57}$$

Under these identifications, the action of $\mathrm{G}_{\mathbf{d}^\flat}$ agrees with the action of H. Now we may apply Theorem 4.1. This tells us that $\mathbf{k}[\mathrm{R}(Q^\flat, \mathbf{d}^\flat)]^{\mathrm{G}_{\mathbf{d}^\flat}} \cong \mathbf{k}[\mathrm{R}(Q, \mathbf{d})]^H$ is generated by traces along oriented cycles in $Q^\flat$. As Q is acyclic, all oriented cycles in $Q^\flat$ must involve 0∞. Moreover, by cyclic invariance of the trace, the starting point of the cycle is irrelevant, whence $\mathbf{k}[\mathrm{R}(Q^\flat, \mathbf{d}^\flat)]^{\mathrm{G}_{\mathbf{d}^\flat}}$ is generated by cycles which start at the vertex 0∞.

For $i, j \in Q_0$ we will denote by $Q_*(i,j)$ the set of paths from i to j. We have a bijection between oriented cycles in $Q^\flat$ starting at 0∞ and words $p_n \cdots p_1$ in paths $p_\nu \in Q_*(0,\infty) \cup Q_*(\infty, 0)$. Let $p_n \cdots p_1$ be such a word, and let $c \in Q^\flat_*(0\infty, 0\infty)$ be the associated cycle. Then for any representation $N \in \mathrm{R}(Q^\flat, \mathbf{d}^\flat)$ we have

$$\varphi_c(N) = \mathrm{tr}(N_{p_n} \cdots N_{p_1}) = N_{p_n} \cdots N_{p_1} = f_{p_n}(N) \cdots f_{p_1}(N). \tag{58}$$

The group $\mathbb{G}_{\mathrm{m}}$ acts on φ_c with weight

$$w = \#\{r \in \{1, \ldots, n\} \mid p_r \in Q_*(0,\infty)\} - \#\{s \in \{1, \ldots, n\} \mid p_s \in Q_*(\infty, 0)\}. \tag{59}$$

As Q is acyclic, at least one of the sets $Q_*(0,\infty)$ and $Q_*(\infty,0)$ is empty. So either $w = n$ if $Q_*(\infty, 0) = \emptyset$, or $w = -n$ if $Q_*(0,\infty) = \emptyset$.

For φ_c to be a semi-invariant function of weight $\delta_\infty - \delta_0$, we must have $w = 1$. So this can only occur when there are no oriented paths from ∞ to 0 and $n = 1$. A semi-invariant function of weight $\delta_\infty - \delta_0$ is therefore a linear combination of the functions f_p with $p \in Q_*(0,\infty)$, as claimed. □

We will use this in the following proof.

Proof of Theorem A. Let $\overline{X} = \mathrm{M}^{\overline{\theta}\text{-st}}(\overline{Q}, \overline{\mathbf{d}})$ be the doubly-framed moduli space at i and j. Using the construction from Proposition 3.9 we let $X' = \mathrm{M}^{\theta'-\mathrm{st}}(Q', \mathbf{d}')$. We know from Corollary 3.8 and Corollary 3.10 that

$$\begin{array}{ccccccc}
p & & e_j\mathbf{k}Qe_i & \xrightarrow{h^{\mathcal{U}}_{i,j}} & \mathrm{H}^0(X, \mathcal{U}_i^\vee \otimes \mathcal{U}_j) & s & \\
\downarrow & & \downarrow & & \downarrow & \downarrow & \\
(j \to \infty)p(0 \to i) & q_\infty p' q_0 & {}_\infty\mathbf{k}\overline{Q}e_0 & \xrightarrow{h^{\overline{\mathcal{U}}}_{0,\infty}} & \mathrm{H}^0(\overline{X}, \overline{\mathcal{U}}_0^\vee \otimes \overline{\mathcal{U}}_\infty) & \overline{\mathcal{U}}_{j\to\infty} \circ u^*s \circ \overline{\mathcal{U}}_{0\to i} & \overline{\mathcal{U}}_{q_\infty} \circ {}^*s' \circ \overline{\mathcal{U}}_{q_0} \\
& \uparrow & \uparrow & & \uparrow & & \uparrow \\
& p' & e_{j'}\mathbf{k}Q'e_{i'} & \xrightarrow{h^{\mathcal{U}'}_{i',j'}} & \mathrm{H}^0(X, \mathcal{U}'^\vee_{i'} \otimes \mathcal{U}'_{j'}) & & s'
\end{array} \tag{60}$$

is commutative and all vertical maps are isomorphisms. To show that the horizontal maps are isomorphisms, it is therefore enough to show that one of them is. We will show that $h^{\mathcal{U}'}_{i',j'}$ is an isomorphism in what follows.

The line bundle $\mathcal{U}'^\vee_{i'} \otimes \mathcal{U}'_{j'}$ is the line bundle $\mathcal{L}(\delta_{j'} - \delta_{i'})$. The data Q', $\mathbf{d}'$, and θ' satisfy Assumption 2.4 by Proposition 3.9. We may therefore apply [FRS21, Lemma 3.3], and obtain the identification

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$$\mathbf{k}[\mathrm{R}(Q',\mathbf{d}')]^{\mathrm{G}_{\mathbf{d}'},\delta_{j'}-\delta_{i'}} = \mathrm{H}^0(X',\mathcal{L}(\delta_{j'}-\delta_{i'})). \tag{61}$$

Note that, in loc. cit., it is required that $\mathbf{d}'$ is θ'-coprime. This assumption may be relaxed: it is enough to assume that the stable and semi-stable locus for $\mathbf{d}'$ agree, which is the case by Proposition 3.1.

From Lemma 4.2 we have the explicit isomorphism

$$e_{j'}\mathbf{k}Q'e_{i'} \xrightarrow{\cong} \mathbf{k}[\mathrm{R}(Q',\mathbf{d}')]^{\mathrm{G}_{\mathbf{d}'},\delta_{j'}-\delta_{i'}}. \tag{62}$$

The composition of (61) and (62) is the map $h^{\mathcal{U}'}_{i',j'}$. Thus the horizontal maps in (60) are all isomorphisms, and through the identifications in the diagram we conclude that $h^{\mathcal{U}}_{i,j}$ is an isomorphism, which concludes the proof of the first part of the theorem.

The isomorphism (3) follows because the algebra structure on $\mathbf{k}Q$ is given in terms of the composition of paths, and on $\mathrm{End}_X(\mathcal{U})$ in terms of the composition of morphisms. □

4.2 Global sections of the tangent bundle

For smooth projective quiver moduli there is, for example, by [BF24, Lemma 4.2], the four-term exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{\Phi} \bigoplus_{i\in Q_0} \mathcal{U}_i^\vee \otimes \mathcal{U}_i \xrightarrow{\Psi} \bigoplus_{a\in Q_1} \mathcal{U}_{\mathrm{s}(a)}^\vee \otimes \mathcal{U}_{\mathrm{t}(a)} \to \mathrm{T}_X \to 0. \tag{63}$$

The morphism Φ is the standard inclusion, and the morphism $\Psi = \sigma_{\mathcal{U},\mathcal{U}}$ is defined in Section 3.1 of op. cit. Over an open subset $V \subseteq X$, the map Ψ sends a section $f = (f_i)_{i\in Q_0}$ of $\mathcal{U}_{\mathrm{s}(a)}^\vee \otimes \mathcal{U}_{\mathrm{t}(a)} \cong \mathcal{H}om(\mathcal{U}_{\mathrm{s}(a)},\mathcal{U}_{\mathrm{t}(a)})$ over V to

$$(f_{\mathrm{t}(a)} \circ \mathcal{U}_a|_V - \mathcal{U}_a|_V \circ f_{\mathrm{s}(a)})_{a\in Q_1}. \tag{64}$$

For the description of the vector fields we recall the main result of [BBF+23], which is also used to prove Theorem D.

Theorem 4.3 (Cohomology vanishing). *Let $Q, \mathbf{d}$ and θ be as in* Assumption 2.4, *and assume in addition that $\mathbf{d}$ is θ-strongly amply stable. Consider $X = \mathrm{M}^{\theta\text{-st}}(Q,\mathbf{d})$. For all $i,j \in Q_0$ we have*

$$\mathrm{H}^{\geq 1}(X, \mathcal{U}_i^\vee \otimes \mathcal{U}_j) = 0. \tag{65}$$

In op. cit. it is used to establish the following (infinitesimal) rigidity result.

Corollary 4.4 (Rigidity). *Let $Q, \mathbf{d}$ and θ be as in* Assumption 2.4, *and assume in addition that $\mathbf{d}$ is θ-strongly amply stable. Consider $X = \mathrm{M}^{\theta\text{-st}}(Q,\mathbf{d})$. Then*

$$\mathrm{H}^{\geq 1}(X, \mathrm{T}_X) = 0. \tag{66}$$

We can now give the description (4) of the vector fields.

Proof of Theorem B. By Theorem A, a global section of $\bigoplus_{i\in Q_0} \mathcal{U}_i^\vee \otimes \mathcal{U}_i$ is of the form $(z_i \,\mathrm{id}_{\mathcal{U}_i})_{i\in Q_0}$, which by (64) is then mapped to $((z_{\mathrm{t}(a)} - z_{\mathrm{s}(a)})\mathcal{U}_a)_{a\in Q_1}$. This shows that the induced map of Ψ on global sections corresponds to ψ from (6) under the identifications

$$\begin{aligned} \bigoplus_{i\in Q_0} \mathcal{U}_i^\vee \otimes \mathcal{U}_i &\cong \bigoplus_{i\in Q_0} e_i\mathbf{k}Qe_ie_i\mathbf{k}Qe_i, \\ \bigoplus_{a\in Q_1} \mathcal{U}_{\mathrm{s}(a)}^\vee \otimes \mathcal{U}_{\mathrm{t}(a)} &\cong \bigoplus_{a\in Q_1} e_{\mathrm{t}(a)}\mathbf{k}Qe_{\mathrm{s}(a)}. \end{aligned} \tag{67}$$

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It is immediate that the map induced by Φ from (63) on global sections corresponds to the morphism ϕ defined in (5). This provides us with an exact sequence

$$0 \to \mathbf{k} \xrightarrow{\phi} \bigoplus_{i\in Q_0} e_i\mathbf{k}Qe_i \xrightarrow{\psi} \bigoplus_{a\in Q_1} e_{\mathrm{t}(a)}\mathbf{k}Qe_{\mathrm{s}(a)} \to \mathrm{H}^0(X,\mathrm{T}_X) \to \bigoplus_{i\in Q_0} \mathrm{H}^1(X,\mathcal{U}_i^\vee\otimes\mathcal{U}_i). \tag{68}$$

The rightmost term vanishes by Theorem 4.3. This proves Theorem B. □

We want to point out a similarity between Theorem B and how the Hochschild cohomology of the path algebra $\mathbf{k}Q$ is computed in [Hap89, §1.6]. The following statement is obtained by inspecting the proof in loc. cit.

Theorem 4.5 (Happel). *Let Q be an acyclic quiver. Then there exists a four-term exact sequence*

$$0 \to \mathbf{k} \to \bigoplus_{i\in Q_0} \mathbf{k} \to \bigoplus_{\alpha\in Q_1} e_{\mathrm{t}(\alpha)}\mathbf{k}Qe_{\mathrm{s}(\alpha)} \to \mathrm{HH}^1(\mathbf{k}Q) \to 0. \tag{69}$$

This allows us to explain the origin of Conjecture C. Recall that the outer automorphism group $\mathrm{Out}(\mathbf{k}Q)$ is the affine algebraic group $\mathrm{Aut}(\mathbf{k}Q)/\mathrm{Inn}(\mathbf{k}Q)$, where $\mathrm{Inn}(\mathbf{k}Q)$ are the inner automorphisms. By [Str02, Proposition 1.1] we have an isomorphism of Lie algebras

$$\mathrm{HH}^1(\mathbf{k}Q) \cong \mathrm{Lie}\,\mathrm{Out}(\mathbf{k}Q), \tag{70}$$

where the Lie algebra structure on the left is given by the Gerstenhaber bracket. On the other hand we have, for any smooth projective variety X, and thus in particular for the quiver moduli we are interested in, an isomorphism of Lie algebras

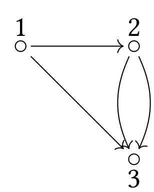
$$\mathrm{H}^0(X,\mathrm{T}_X) \cong \mathrm{Lie}\,\mathrm{Aut}(X), \tag{71}$$

where the Lie algebra structure on the left is given by the Schouten–Nijenhuis bracket of vector fields. This explains the origin of Conjecture C.

The following example shows how Theorem A (and thus Theorem B) fails without ample stability.

Example 4.6. Consider the three-vertex quiver

Q:



(72)

for the (so-called thin) dimension vector $\mathbf{d} = \mathbf{1} = (1,1,1)$. As discussed at the end of [FRS21], there exists an identification

$$\mathrm{M}^{\theta-\mathrm{st}}(Q,\mathbf{1}) \cong \mathrm{Bl}_p\,\mathbb{P}^2, \tag{73}$$

where $\theta = \theta_{\mathrm{can}} = (2,1,-3)$ is the canonical stability condition.

One can compute that there is precisely one other stability chamber where the associated moduli space is not empty. Let θ' be a stability parameter in this chamber, for example, $\theta' = (2,-1,-1)$. Then

$$\mathrm{M}^{\theta'\text{-st}}(Q,\mathbf{1}) \cong \mathbb{P}^2, \tag{74}$$

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because the moduli space is a smooth projective rational surface of Picard rank 1, which can be determined by computing the Betti numbers using [Rei03, Corollary 6.9], as implemented in [Bel].

Every condition except the ample stability in Assumption 2.4 holds, and ample stability fails because the Picard rank drops. The summands of the universal representation are line bundles on $\mathbb{P}^2$. But it is impossible to have an isomorphism

$$\mathrm{H}^0(\mathrm{M}^{\theta'-\mathrm{st}}(Q,\mathbf{1}),\mathcal{U}_2^\vee\otimes\mathcal{U}_3)\cong e_3\mathbf{k}Qe_2 \tag{75}$$

because the right-hand side is two-dimensional, which is impossible for the global sections of the line bundle$\mathcal{U}_2^\vee\otimes\mathcal{U}_3$ on $\mathbb{P}^2$, because those dimensions are necessarily of the form $\binom{n+2}{n}$, so $0,1,3,6,\ldots$.

We can also observe the failure of Theorem B. Through its identification with $\mathbb{P}^2$ we obtain

$$\mathrm{H}^0(\mathrm{M}^{\theta'\text{-st}}(Q,\mathbf{1}),\mathrm{T}_{\mathrm{M}^{\theta'\text{-st}}(Q,\mathbf{1})})\cong\mathbf{k}^8, \tag{76}$$

whereas

$$\mathrm{HH}^1(\mathbf{k}Q)\cong\mathbf{k}^6 \tag{77}$$

by (69).

5. Functors associated to the universal representation

Given a smooth projective variety S, an acyclic quiver Q, and a locally free left $\mathcal{O}_SQ$-module $\mathcal{M}$, we can consider *four* associated Fourier–Mukai-like functors. The conditions on S, Q, and $\mathcal{M}$ can be relaxed, at the cost of not working with the bounded derived category, but we will not do this.

Usually Fourier–Mukai functors are considered in the context of smooth projective varieties (without a sheaf of algebras) where these variations can be ignored, but because we work with noncommutative algebras we want to point out how they can be compared.

Using the sheaf Hom we have a covariant and a contravariant version

$$\begin{aligned}\Phi_{\mathcal{M}}&:=\mathbf{R}\mathcal{H}om_{\mathcal{O}_SQ}(\mathcal{M},-\otimes\mathcal{O}_S):\mathbf{D}^{\mathrm{b}}(\mathbf{k}Q)\to\mathbf{D}^{\mathrm{b}}(S),\\ \Psi_{\mathcal{M}}&:=\mathbf{R}\mathcal{H}om_{\mathcal{O}_SQ}(-\otimes\mathcal{O}_S,\mathcal{M}):\mathbf{D}^{\mathrm{b}}(\mathbf{k}Q)^{\mathrm{op}}\to\mathbf{D}^{\mathrm{b}}(S),\end{aligned} \tag{78}$$

and using the tensor product we have two covariant versions, but the first considers *right* $\mathbf{k}Q$-modules, leading to

$$\begin{aligned}\mathrm{X}_{\mathcal{M}}&:=-\otimes^{\mathbf{L}}_{\mathbf{k}Q}\mathcal{M}:\mathbf{D}^{\mathrm{b}}(\mathbf{k}Q^{\mathrm{op}})\to\mathbf{D}^{\mathrm{b}}(S),\\ \Omega_{\mathcal{M}}&:=(\mathrm{D}\mathcal{M})\otimes^{\mathbf{L}}_{\mathbf{k}Q}-:\mathbf{D}^{\mathrm{b}}(\mathbf{k}Q)\to\mathbf{D}^{\mathrm{b}}(S),\end{aligned} \tag{79}$$

where $\mathrm{D}\mathcal{M}=\mathcal{M}^\vee=\mathcal{H}om_{\mathcal{O}_S}(\mathcal{M},\mathcal{O}_S)$, equipped with the natural structure of a *right* $\mathcal{O}_SQ$-module. We will use similar notation for turning a left $\mathbf{k}Q$-module into a right $\mathbf{k}Q$-module, which can also be considered as a left $\mathbf{k}Q^{\mathrm{op}}$-module.

Their relationship is explained by the following lemmas.

LEMMA 5.1. *Let N be an object in $\mathbf{D}^{\mathrm{b}}(\mathbf{k}Q^{\mathrm{op}})$. Then*

$$\mathrm{X}_{\mathcal{M}}(N)^\vee\cong\Psi_{\mathrm{D}\mathcal{M}}(N)\cong\Phi_{\mathcal{M}}(\mathrm{D}N) \tag{80}$$

in $\mathbf{D}^{\mathrm{b}}(S)$.

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Proof. By definition we have

$$\mathrm{X}_{\mathcal{M}}(N)^{\vee} = \mathbf{R}\mathcal{H}om_{\mathcal{O}_S}(N \otimes^{\mathbf{L}}_{\mathbf{k}Q} \mathcal{M}, \mathcal{O}_S). \tag{81}$$

Using the isomorphism $N \otimes^{\mathbf{L}}_{\mathbf{k}Q} \mathcal{M} \cong N \otimes^{\mathbf{L}}_{\mathbf{k}} \mathcal{O}_S \otimes^{\mathbf{L}}_{\mathcal{O}_S Q} \mathcal{M}$ and the tensor-Hom adjunction, we can rewrite the right-hand side as

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_S Q^{\mathrm{op}}}(N \otimes^{\mathbf{L}}_{\mathbf{k}} \mathcal{O}_S, \mathrm{D}\mathcal{M}), \tag{82}$$

which is $\Psi_{\mathrm{D}\mathcal{M}}(N)$, or to

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_S Q}(\mathcal{M}, \mathrm{D}M \otimes^{\mathbf{L}}_{\mathbf{k}} \mathcal{O}_S), \tag{83}$$

which is $\Psi_{\mathcal{M}}(\mathrm{D}M)$. □

Similarly, we have the following lemma.

Lemma 5.2. *Let M be an object in $\mathbf{D}^{\mathrm{b}}(\mathbf{k}Q)$. Then*

$$\Omega_{\mathcal{M}}(M)^{\vee} \cong \Psi_{\mathcal{M}}(M) \cong \Phi_{\mathrm{D}\mathcal{M}}(\mathrm{D}M) \tag{84}$$

in $\mathbf{D}^{\mathrm{b}}(S)$.

5.1 Admissible embeddings

Following the sheaf-theoretic examples of admissible embeddings cited in the introduction, we will now prove Theorem D. In fact, the admissible embeddings cited in the introduction are preceded by an admissible embedding in a restricted setting of quiver moduli, obtained by Altmann–Hille [AH99, Theorem 1.3], which we will recall to illustrate the methods. In a different restricted setting, that of quiver flag varieties, it was obtained by Craw, Ito, and Karmazyn in [CIK18, Example 2.9].

Their condition that the canonical stability parameter θ_{can} does not lie on any $(1,0)$- or (t,t)-walls in the terminology of [AH99, §2.2] is implied by Assumption 2.4, where $\mathbf{d} = \mathbf{1}$.

Theorem 5.3 (Altmann–Hille). *Let $Q, \mathbf{d} = \mathbf{1}$ and θ_{can} satisfy* Assumption 2.4. *Consider $X = \mathrm{M}^{\theta_{\mathrm{can}}\text{-st}}(Q, \mathbf{1})$. Then X is a smooth projective toric Fano variety, with $\mathrm{rk}\,\mathrm{Pic}\,X = \#Q_0 - 1$, and*

$$\mathrm{X}_{\mathcal{U}} \colon \mathbf{D}^{\mathrm{b}}(\mathbf{k}Q^{\mathrm{op}}) \to \mathbf{D}^{\mathrm{b}}(X) \tag{85}$$

is fully faithful.

In the thin case, where $d_i = 1$ for all $i \in Q_0$, the proof reduces to:

- higher cohomology vanishing for the tensor products $\mathcal{U}_i^{\vee} \otimes \mathcal{U}_j$ [AH99, Theorem 3.6];
- an identification of the global sections $\mathrm{H}^0(\mathcal{U}_i^{\vee} \otimes \mathcal{U}_j) \cong e_j \mathbf{k}Q e_i$ [AH99, Theorem 4.3].

The vanishing is an application of the Kodaira vanishing theorem, for which it is important that $\mathcal{U}_i^{\vee} \otimes \mathcal{U}_j$ is a line bundle and that X is a Fano variety, and hence why Theorem 5.3 is stated only for θ_{can}. The identification of the global sections uses the toric description in [AH99, Proposition 3.1]. A minor but important detail which is omitted in the proof of [AH99, Theorem 4.3] is the fact that the isomorphism needs to be induced by the functor (85). We will address this in our more general setting in the proof of Theorem D.

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Admissible embedding in the general case. To check the full faithfulness in Theorem D we will apply the following full faithfulness criterion [Huy06, Proposition 1.49].

Proposition 5.4. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between triangulated categories, which admits a left and a right adjoint. Let $\mathcal{S}$ be a spanning class for $\mathcal{C}$ such that for all $C, C' \in \mathcal{S}$ and all $i \in \mathbb{Z}$ the natural morphism*

$$F_{C,C'}\colon \mathrm{Hom}_{\mathcal{C}}(C, C'[i]) \to \mathrm{Hom}_{\mathcal{D}}(F(C), F(C'[i])) \tag{86}$$

is an isomorphism. Then F is fully faithful.

We will apply Proposition 5.4 using the first spanning class in the following standard lemma.

Lemma 5.5. *Let Q be an acyclic quiver. Then the following are spanning classes of $\mathbf{D}^{\mathrm{b}}(kQ)$:*

- *the set $\{P_i \mid i \in Q_0\}$ of indecomposable projectives;*
- *the set $\{I_i \mid i \in Q_0\}$ of indecomposable injectives.*

We will apply Proposition 5.4 using the spanning class of indecomposable projectives from Lemma 5.5.

Proof of Theorem D. Let us first show that $\Psi_{\mathcal{U}}$ is fully faithful. As $P_i \otimes \mathcal{O}_X$ is a projective left $\mathcal{O}_X Q$-module, we have

$$\Psi_{\mathcal{U}}(P_i) = \mathbf{R}\mathcal{H}om_{\mathcal{O}_X Q}(P_i \otimes \mathcal{O}_X, \mathcal{U}) \cong \mathcal{H}om_{\mathcal{O}_X Q}(P_i \otimes \mathcal{O}_X, \mathcal{U}) \cong \mathcal{U}_i. \tag{87}$$

The isomorphism $\mathcal{H}om_{\mathcal{O}_X Q}(P_i \otimes \mathcal{O}_X, \mathcal{U}) \cong \mathcal{U}_i$ can be checked affine locally. Let $U = \operatorname{Spec} A$ and let M be the left AQ-module belonging to $\mathcal{U}|_U$. Over U we have the obvious isomorphism $\mathrm{Hom}_{AQ}(P_i \otimes A, M) \cong M_i$. They glue to a global isomorphism over X.

We have:

- $\mathrm{Hom}_{\mathbf{k}Q}(P_j, P_i) \cong e_j \mathbf{k}Q e_i$;
- $\mathrm{Ext}^n_{\mathbf{k}Q}(P_j, P_i) = 0$ for all $n \geq 1$, because P_j is projective.

So we need to show that:

- the natural map
$$\Psi_{\mathcal{U},P_i,P_j}\colon \mathrm{Hom}_{\mathbf{k}Q}(P_j, P_i) \to \mathrm{Hom}_X(\mathcal{U}_i, \mathcal{U}_j) \cong \mathrm{H}^0(X, \mathcal{U}_i^\vee \otimes \mathcal{U}_j) \tag{88}$$
is an isomorphism;
- $\mathrm{Ext}^n_X(\mathcal{U}_i, \mathcal{U}_j) = 0$ for all $n \geq 1$.

The second point follows from the cohomology vanishing in [BBF+23], recalled in Theorem 4.3.

For the first point, let us consider the basis of $e_j \mathbf{k}Q e_i$ of paths from i to j. Let q be such a path. The associated homomorphism $\psi_q\colon P_j \to P_i$ is given on basis elements by mapping a path p starting at j to $\psi_q(p) = pq$. Using this, we see that the diagram

$$\begin{array}{ccc} \mathcal{H}\mathrm{om}_{\mathcal{O}_X Q}(P_i \otimes \mathcal{O}_X, \mathcal{U}) & \xrightarrow{\cong} & \mathcal{U}_i \\ \downarrow{\scriptstyle \mathcal{H}\mathrm{om}(\psi_q \otimes \mathrm{id}, -)} & & \downarrow{\scriptstyle \mathcal{U}_q} \\ \mathcal{H}\mathrm{om}_{\mathcal{O}_X Q}(P_j \otimes \mathcal{O}_X, \mathcal{U}) & \xrightarrow{\cong} & \mathcal{U}_j \end{array} \tag{89}$$

commutes; this can again be checked affine locally. The image of ψ_q under $\Psi_{\mathcal{U},P_i,P_j}$ is therefore the morphism $\mathcal{U}_q\colon \mathcal{U}_i \to \mathcal{U}_j$. This shows that the composition

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$$\begin{array}{ccccc} e_j\mathbf{k}Qe_i & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbf{k}Q}(P_j,P_i) & \xrightarrow{\Psi_{\mathcal{U},P_i,P_j}} & \mathrm{H}^0(X,\mathcal{U}_i^\vee\otimes\mathcal{U}_j) \\ q & \longmapsto & \psi_q & & \end{array} \tag{90}$$

agrees with the isomorphism from Theorem A. Therefore $\Psi_{\mathcal{U},P_i,P_j}$ must be an isomorphism as well.

The proof of the full faithfulness of $\Phi_{\mathcal{U}}$ is similar. Here we use that $\Phi_{\mathcal{U}}(I_i)\cong\mathcal{U}_i^\vee$. The comparison in Lemmas 5.1 and 5.2 gives the full faithfulness of $\mathrm{X}_{\mathcal{U}}$ and $\Omega_{\mathcal{U}}$. □

5.2 Rigidity and vector fields from the admissible embedding

In [BFR19, §4] the fully faithful functor $\Phi_{\mathcal{J}}$ from (14) (provided $\mathcal{O}_S$ is exceptional) is used to relate the Hochschild cohomology of S to the deformation theory of $\mathrm{Hilb}^n S$. We will now explain how a similar reasoning allows us to describe $\mathrm{H}^i(X,\mathrm{T}_X)$, where X still denotes $\mathrm{M}^{\theta\text{-st}}(Q,\mathbf{d})$. Because the proof of Theorem D is heavily dependent on Theorem 4.3 and Theorem A this is not an independent description of $\mathrm{H}^i(X,\mathrm{T}_X)$ (i.e., the combination of Theorem A and Corollary 4.4), but it highlights an important parallel between the behaviour of different moduli problems.

Recall from [BF24, Equation (24)] the local-to-global spectral sequence

$$\mathrm{E}_2^{p,q}=\mathrm{H}^p(X,\mathcal{E}xt^p_{\mathcal{O}_XQ}(\mathcal{U},\mathcal{U}))\Rightarrow\mathrm{Ext}^{p+q}_{\mathcal{O}_XQ}(\mathcal{U},\mathcal{U}). \tag{91}$$

The following lemma identifies the abutment of (91) with the Hochschild cohomology of $\mathbf{k}Q$. It is the analogue of [BFR19, Lemmas 16 and 17]. We have opted to phrase it using a variation of the functor $\mathrm{X}_{\mathcal{U}}$, but it can also be phrased using any of the other three functors.

Lemma 5.6. *The functor*

$$-\otimes^{\mathbf{L}}_{\mathbf{k}Q}\mathcal{U}\colon\mathbf{D}^{\mathrm{b}}(\mathbf{k}Q\otimes\mathbf{k}Q^{\mathrm{op}})\to\mathbf{D}^{\mathrm{b}}(\mathcal{O}_XQ) \tag{92}$$

is fully faithful, and sends the diagonal $\mathbf{k}Q$-bimodule $\mathbf{k}Q$ to $\mathcal{U}$. In particular, there exists an isomorphism of vector spaces

$$\mathrm{HH}^\bullet(\mathbf{k}Q)\cong\mathrm{Ext}^\bullet_{\mathcal{O}_XQ}(\mathcal{U},\mathcal{U}). \tag{93}$$

Proof. The right adjoint to the functor (92) is $\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{U},-)$, and to check full faithfulness of (92) we will check that the unit of the adjunction is a natural equivalence. Let M be a complex of finitely generated $\mathbf{k}Q$-$\mathbf{k}Q$-bimodules. We want to show that natural morphism in $\mathbf{D}^{\mathrm{b}}(\mathbf{k}Q\otimes\mathbf{k}Q^{\mathrm{op}})$,

$$M\to\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{U},M\otimes^{\mathbf{L}}_{\mathbf{k}Q}\mathcal{U}), \tag{94}$$

is an isomorphism. We have established in Theorem D (or rather, in its proof) that $\mathrm{X}_{\mathcal{U}}$ is fully faithful. Its right adjoint is $\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{U},-)\colon\mathbf{D}^{\mathrm{b}}(X)\to\mathbf{D}^{\mathrm{b}}(\mathbf{k}Q^{\mathrm{op}})$. So we know that (94) is an isomorphism after applying the forgetful functor to $\mathbf{D}^{\mathrm{b}}(\mathbf{k}Q^{\mathrm{op}})$. As this functor reflects isomorphisms, it was an isomorphism already in $\mathbf{D}^{\mathrm{b}}(\mathbf{k}Q\otimes_{\mathbf{k}}\mathbf{k}Q^{\mathrm{op}})$, and thus the unit of the adjunction is an isomorphism.

The natural isomorphism $\mathbf{k}Q\otimes^{\mathbf{L}}_{\mathbf{k}Q}\mathcal{U}\cong\mathcal{U}$ identifies the image of the diagonal bimodule with the universal representation $\mathcal{U}$. Denoting the functor in (92) by F, the isomorphism of vector spaces is given by composing the isomorphism

$$F_{\mathbf{k}Q,\mathbf{k}Q}\colon\mathrm{Ext}^\bullet_{\mathbf{k}Q\otimes\mathbf{k}Q^{\mathrm{op}}}(\mathbf{k}Q,\mathbf{k}Q)\to\mathrm{Ext}^\bullet_{\mathcal{O}_XQ}(\mathcal{U},\mathcal{U}), \tag{95}$$

with the standard isomorphism $\mathrm{Ext}^\bullet_{\mathbf{k}Q\otimes\mathbf{k}Q^{\mathrm{op}}}(\mathbf{k}Q,\mathbf{k}Q)\cong\mathrm{HH}^\bullet(\mathbf{k}Q)$. □

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Now we turn out attention to the objects on the E_2-page. We have the following description.

Lemma 5.7. *We have*

$$\mathcal{E}xt^i_{\mathcal{O}_X Q}(\mathcal{U},\mathcal{U}) \cong \begin{cases} \mathcal{O}_X & i=0 \\ \mathrm{T}_X & i=1 \\ 0 & i \geq 2. \end{cases} \tag{96}$$

Proof. The first isomorphism is given by the natural morphism $\mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X Q}(\mathcal{U},\mathcal{U})$, which can be checked to be an isomorphism fibrewise, because stable representations are simple. The second isomorphism is [BF24, Proposition 3.7(2)]. The vanishing for $i \geq 2$ can be checked fibrewise, using that $\mathrm{Ext}^{\geq 2}$ vanishes for any representation of Q over $\mathbf{k}$. □

The following proposition proves how admissibility gives the promised identification between the Hochschild cohomology of $\mathbf{k}Q$ and deformation theory of X. We do not need to require strong ample stability, it suffices that $\mathcal{U}$ exists and gives a fully faithful functor.

Proposition 5.8. *Let Q, $\mathbf{d}$ and θ be as in* Assumption 2.4*. Let $X = \mathrm{M}^{\theta\text{-st}}(Q,\mathbf{d})$. Assume that $\Phi_{\mathcal{U}}$ is fully faithful. Then*

$$\mathrm{H}^i(X,\mathrm{T}_X) \cong \begin{cases} \mathrm{HH}^1(\mathbf{k}Q) & i=0 \\ 0 & i \geq 1. \end{cases} \tag{97}$$

Proof. Because X is a smooth projective rational variety, as shown in [Sch01, Theorem 6.4], we have that $\mathrm{H}^{\geq 1}(X,\mathcal{O}_X)=0$ and $\mathrm{H}^0(X,\mathcal{O}_X) \cong \mathbf{k}$. This fact, together with Lemma 5.7 shows that the E_2-page of the spectral sequence (91) looks like

$$\begin{matrix} \vdots & \vdots & \vdots & \iddots \\ 0 & \mathrm{H}^2(X,\mathrm{T}_X) & 0 & \dots \\ 0 & \mathrm{H}^1(X,\mathrm{T}_X) & 0 & \dots \\ \mathbf{k} & \mathrm{H}^0(X,\mathrm{T}_X) & 0 & \dots \end{matrix} \tag{98}$$

From Lemma 5.6 we know the abutment of the spectral sequence, and we see that $\mathrm{H}^{\geq 1}(X,\mathrm{T}_X)$ needs to be cancelled in the spectral sequence, as $\mathrm{HH}^{\geq 2}(\mathbf{k}Q)=0$. But this is impossible, because the spectral sequence necessarily already degenerates on the E_2-page, thus they are zero to begin with. Similarly, we obtain an isomorphism

$$\mathrm{H}^0(X,\mathrm{T}_X) \cong \mathrm{HH}^1(\mathbf{k}Q). \tag{99}$$

□

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Conflicts of Interest

None

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