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FRACTIONAL MOMENTS OF SOLUTIONS TO STOCHASTIC RECURRENCE EQUATIONS

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Abstract

In this paper we study the fractional moments of the stationary solution to the stochastic recurrence equation $X_t = A_t X_{t-1} + B_t,\ t \in \mathbb{Z}$, where $((A_t, B_t))_{t\in\mathbb{Z}}$ is an independent and identically distributed bivariate sequence. We derive recursive formulae for the fractional moments $\mathbb{E}|X_0|^p,\ p \in \mathbb{R}$. Special attention is given to the case when B_t has an Erlang distribution. We provide various approximations to the moments $\mathbb{E}|X_0|^p$ and show their performance in a small numerical study.

Keywords: Moment; stochastic recurrence equation; GARCH; Erlang distribution; numerical approximation

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1. Introduction

We consider the stochastic recurrence equation

$$X_t = A_t X_{t-1} + B_t, \qquad t \in \mathbb{Z}, \tag{1.1}$$

for an independent and identically distributed (i.i.d.) sequence $((A_t, B_t))_{t\in\mathbb{Z}}$ of pairs (A_t, B_t) with values in $[0, \infty) \times \mathbb{R}$. We will write $A, B, C, \dots$ for a generic variable of the stationary sequences (A_t), (B_t), (C_t), $\dots$, respectively. A unique causal stationary solution to (1.1) exists if $\mathbb{E}\log A < 0$ and $\mathbb{E}\log B^+ < \infty$ (see [25]), and the solution can be written in the form

$$X_t = B_t + \sum_{i=-\infty}^{t-1} A_{i+1} \cdots A_t B_i, \qquad t \in \mathbb{Z}. \tag{1.2}$$

In what follows, we always assume that the stationary solution (1.2) exists.

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The stochastic recurrence equation (1.1) and its solution (1.2) have attracted significant attention in the literature; see, e.g. [12], [21], [25], [33], and the references therein. This interest is due to the numerous applications of the model (1.1). Among the most popular applications are the ARCH(1) and GARCH(1, 1) processes in financial time series analysis introduced by Engle [18] and Bollerslev [6], respectively. Another recent application is the modeling of the TCP in telecommunications networks; see, e.g. [17], [23], and [28]. Boxma *et al.* [7] considered (1.1) in the context of growth–collapse processes with renewal collapse epochs. The stochastic recurrence equation (1.1) has also been used in the context of insurance risk models; see [11] and [20]. Moreover, this equation is closely related to exponential functionals of Lévy processes; see, e.g. [5], [9], [10], [24], and [29].

The distributional properties of the stationary solution to (1.1) are rather sophisticated. This fact is highlighted by a famous result of Kesten [25] concerning the tails of X; see also [21]. Write

$$f(\kappa) = \mathbb{E}A^{\kappa} \quad \text{and} \quad \phi(\kappa) = 1 - f(\kappa), \qquad \kappa \in \mathbb{R}.$$

Under the assumptions that there exists a positive α such that $f(\alpha) = 1$, $\mathbb{E}A^{\alpha} \log A$ and $\mathbb{E}|B|^{\alpha}$ are both finite, the law of $\log A$ is nonarithmetic, and, for every x, $\mathbb{P}(A_1 x + B_1 = x) < 1$, there exist constants $c_+, c_- \geq 0$ such that $c_+ + c_- > 0$, and

$$\mathbb{P}(X > x) \sim c_+ x^{-\alpha} \quad \text{and} \quad \mathbb{P}(X \leq -x) \sim c_- x^{-\alpha} \quad \text{as } x \to \infty. \tag{1.3}$$

Because of the convexity of f and since $f(0) = 1$, we necessarily have $f(\kappa) < 1$ for $\kappa \in (0, \alpha)$. Goldie [21] gave an alternative proof of (1.3) and determined the explicit form of the constants c_+ and c_-. In particular, for $A, B \geq 0$, he proved that

$$c_+ = \frac{\mathbb{E}[(A_1 X_0 + B_1)^{\alpha} - (A_1 X_0)^{\alpha}]}{\alpha \mathbb{E}A^{\alpha} \log A}. \tag{1.4}$$

Note that, due to the tail behavior (1.3), $\mathbb{E}X^{\alpha} = \infty$; hence, both $\mathbb{E}(A_1 X_0 + B_1)^{\alpha}$ and $\mathbb{E}(A_1 X_0)^{\alpha}$ are infinite while the nominator in the previous formula is finite.

If $A \leq 1$ almost surely (a.s.), $\mathbb{P}(0 < A < 1) > 0$, and $\mathbb{E}\mathrm{e}^{r|B|} < \infty$ for some $r > 0$, Goldie and Grübel [22] showed that (1.3) does not remain true. In this case, the tails of X decay exponentially fast, thus ensuring the existence of all moments of X. If we formally set $\alpha = \infty$, we have $f(\kappa) = \mathbb{E}A^{\kappa} < 1$ for $\kappa < \alpha$, just as in the Kesten–Goldie case. Under mild conditions on A and B, Alsmeyer *et al.* [1] showed that $A \leq 1$ a.s. and the existence of exponential moments of B are conditions which are necessary to ensure that exponential moments of X exist. In the same paper, the authors quote a technical report of Kellerer (1992), who proved for $A, B \geq 0$ a.s. that $\mathbb{E}\mathrm{e}^{rX} < \infty$ for some $r > 0$ if and only if $A \leq 1$ a.s. and $\mathbb{E}\mathrm{e}^{rB} < \infty$.

In this paper we are concerned with the calculation of the moments of the solution $(X_t)_{t \in \mathbb{Z}}$ to the stochastic recurrence equation (1.1). The positive integer moments of X can be calculated by using the recursive argument given in [33]. First, observe that, from (1.1) for integer $n \geq 1$,

$$X^n \overset{\mathrm{D}}{=} (A_1 X_0 + B_1)^n,$$

where X_0 and (A_1, B_1) are independent. Then, assuming that $\mathbb{E}|X|^p$ is finite and $f(p) < 1$ for some $p \geq 1$, an application of the binomial formula yields a recursive relation for the moments $\mathbb{E}X^l$, $1 \leq l \leq n = [p]$, given by

$$\mathbb{E}X^l = (\phi(l))^{-1} \sum_{k=0}^{l-1} \binom{l}{k} \mathbb{E}(A_1^k B_1^{l-k}) \mathbb{E}X^k. \tag{1.5}$$

Note that $f(l) < 1$, $l \le n$, if $p < \alpha$ for the value $\alpha \le \infty$ introduced above. It follows from [1, Theorem 1.4] that $\mathbb{E}|X|^p < \infty$ if and only if $\mathbb{E}A^p < 1$ and $\mathbb{E}|B|^p < \infty$ for any $p > 0$, provided that the mild conditions $A \ne 0$ a.s., $\mathbb{P}(B = 0) < 1$, and $\mathbb{P}(A_1 x + B_1 = x) < 1$, $x \in \mathbb{R}$, hold.

In some special cases the distribution of X can be calculated explicitly; see, e.g. [1], [7], [13], [14], [15], [16], [21], [30], and the references therein. Then, in principle, one could also calculate all moments of X, both the integer and the fractional moments.

In this paper we focus on the derivation of explicit formulae for the fractional moments of X. Such moments were derived for models of telecommunication networks in [10] and [23], and for exponential functionals of Lévy processes in [10], [23], and [29]. We also mention that, for exponential B, the calculation of Goldie's constant c_+ in (1.4) reduces to calculating the fractional moment $\mathbb{E}X^{\alpha-1}$; see the comments at the end of Example 2.1. Moments are highly relevant in statistical and econometric applications. In financial time series analysis, the ARCH and GARCH processes introduced by Engle [18] and Bollerslev [6], respectively, constitute a major class which is closely related to multivariate stochastic recurrence equations of the type (1.1); see, e.g. [3] and [4]. The GARCH(1, 1) process fits into the one-dimensional stochastic recurrence equation framework: it is a stationary process $Y_t = \sigma_t Z_t$, $t \in \mathbb{Z}$, where (Z_t) is a mean zero, unit variance i.i.d. sequence and (σ_t^2) solves the stochastic recurrence equation

$$\sigma_t^2 = \alpha_0 + (\alpha_1 Z_{t-1}^2 + \beta_1)\sigma_{t-1}^2, \qquad t \in \mathbb{Z}, \tag{1.6}$$

for positive coefficients α_i and β_1. In applications Z is often assumed standard normal or student distributed with $\beta > 2$ degrees of freedom. Then one is in the Kesten–Goldie framework, i.e. there exist positive $c_0, \alpha > 0$ with $\mathbb{E}(\alpha_1 Z^2 + \beta_1)^\alpha = 1$ such that $\mathbb{P}(\sigma^2 > x) \sim c_0 x^{-\alpha}$, and a result of Breiman [8] ensures that the distribution of Y inherits the power law tail of σ. In applications we are interested in the moments $\mathbb{E}|Y|^p = \mathbb{E}\sigma^p \mathbb{E}|Z|^p$, $p > 0$, as well as in the correlations of the sequences $(|Y_t|^p)$ for positive p, most often for $p = 1, 2$.

The paper is organized as follows. In Section 2 we consider some recursive formulae for the moments of X. In Section 3 we consider some special cases. Our main focus is on cases which are related to exponential random variables B_t. We use the results and techniques proved in [10] and [23]. We also consider the case when $0 \le A < 1$ and B is bounded, and derive explicit formulae for $\mathbb{E}X^p$.

2. Preliminaries

2.1. A simple recursive formula for moments

For the calculation of the fractional moments, the following observation is useful. A similar formula was applied for calculating the moments of exponential functionals of Lévy processes in [10], [23], and [29]. In what follows, we write F_Z for the distribution function of any random variable Z and $\overline{F}_Z = 1 - F_Z$ for its right tail.

Lemma 2.1. *Let $p \ne 0$ be any real number. Assume that $A, B \ge 0$ a.s. are independent. Then*

$$\mathbb{E}[(A_1X_0 + B_1)^p - (A_1X_0)^p] = p\int_0^\infty \mathbb{E}(A_1X_0 + u)^{p-1}\overline{F}_B(u)\,\mathrm{d}u, \tag{2.1}$$

where both sides are finite or infinite at the same time. If $0 < \mathbb{E}B < \infty$, (2.1) can be written in the form

$$\mathbb{E}[(A_1X_0 + B_1)^p - (A_1X_0)^p] = p\mathbb{E}B\mathbb{E}(A_1X_0 + B^*)^{p-1}$$

for a random variable B^* which is independent of A_1, X_0 and has the integrated tail distribution of B given by

$$F_B^*(b) = \frac{\int_0^b \overline{F}_B(u)\,\mathrm{d}u}{\mathbb{E}B}, \qquad b > 0. \tag{2.2}$$

Proof. We observe that, for any $p \in \mathbb{R}$,

$$(A_1X_0 + B_1)^p - (A_1X_0)^p = p\int_0^{B_1}(A_1X_0 + u)^{p-1}\,\mathrm{d}u.$$

Hence, by the independence of A_1X_0 and B_1,

$$\begin{aligned}\mathbb{E}[(A_1X_0 + B_1)^p - (A_1X_0)^p] &= p\mathbb{E}\bigg[\int_0^{B_1}(A_1X_0 + u)^{p-1}\,\mathrm{d}u\bigg]\\ &= p\int_0^\infty\bigg[\int_0^b \mathbb{E}(A_1X_0 + u)^{p-1}\,\mathrm{d}u\bigg]F_B(\mathrm{d}b)\\ &= p\int_0^\infty \mathbb{E}(A_1X_0 + u)^{p-1}\overline{F}_B(u)\,\mathrm{d}u.\end{aligned}$$

Then the statement of the lemma follows.

Remark 2.1. If $\mathbb{E}X^p < \infty$, $\mathbb{E}B < \infty$, and $f(p) \neq 1$, then the lemma yields

$$\mathbb{E}X^p = \frac{p\mathbb{E}B}{\phi(p)}\mathbb{E}(A_1X_0 + B^*)^{p-1}. \tag{2.3}$$

Following the argument after (1.5), we conclude that $f(p) < 1$ is a necessary condition for $\mathbb{E}X^p < \infty$ provided $p > 0$ and some mild conditions on A, B are satisfied. Since f is convex and $f(0) = 1$, $f(p) > 1$ for $p < 0$. Hence, $f(p) \neq 1$ is satisfied for all $p \neq 0$ such that $f(p)$ is finite.

The idea of the proof of Lemma 2.1 can be applied iteratively. We explain the approach via an example. Assume the conditions of the lemma are satisfied. Write F_B^{n*} for the distribution function of a random variable B^{n*}, which is obtained by applying the integrated tail operation (2.2) n times and assuming that B^{n*} is independent of (A_1, X_0). Then, assuming that all moments involved are finite and $p \neq 0$,

$$\begin{aligned}\mathbb{E}X^p &= \frac{p\mathbb{E}B}{\phi(p)}[\mathbb{E}(A_1X_0 + B^*)^{p-1} - \mathbb{E}(A_1X_0)^{p-1} + \mathbb{E}(A_1X_0)^{p-1}]\\ &= \frac{p\mathbb{E}B}{\phi(p)}[(p-1)\mathbb{E}B^*\mathbb{E}(A_1X_0 + B^{2*})^{p-2} + \mathbb{E}(A_1X_0)^{p-1}]\\ &= \frac{p(p-1)\mathbb{E}B}{\phi(p)}\bigg[\mathbb{E}B^*\mathbb{E}(A_1X_0 + B^{2*})^{p-2} + \frac{f(p-1)\mathbb{E}B}{\phi(p-1)}\mathbb{E}(A_1X_0 + B^*)^{p-2}\bigg].\end{aligned}$$

To illustrate the use of Lemma 2.1, we include the following benchmark example.

Example 2.1. Assume that B has a standard exponential distribution, i.e. $\overline{F}_B(x) = \mathrm{e}^{-x}$, $x > 0$. Then $B^* \stackrel{\mathrm{D}}{=} B$ and $A_1X_0 + B^* \stackrel{\mathrm{D}}{=} X$. Multiple use of (2.3) yields

$$\mathbb{E}X^p = \frac{p\cdots(p-n+1)}{\phi(p)\cdots\phi(p-n+1)}\mathbb{E}X^{p-n}, \qquad n \geq 1. \tag{2.4}$$

Relation (2.4) can be found in the literature on exponential functionals of Lévy processes and related topics; see, e.g. [5], [9], [10], [24], and [29]. An explicit solution to (2.4) for $A_t = \beta^{Y_t}$, positive Y_t, $\beta \in (0, 1)$, was given in [23]:

$$\mathbb{E}X^p = \Gamma(p+1) \prod_{k=1}^{\infty} \frac{\phi(p+k)}{\phi(k)}, \tag{2.5}$$

provided $p > 0$ or $p < 0$, $-p \notin \mathbb{N}$, $f(p+1) < \infty$, and $\mathbb{E}[(1-A)^{-1}] < \infty$.

Under the assumptions of the Kesten–Goldie theory [21], [25] (see Section 1), the constant c_+ in (1.4) satisfies

$$\begin{aligned} c_+ \mathbb{E}A^\alpha \log A &= \alpha^{-1}\mathbb{E}[(A_1X_0 + B_1)^\alpha - (A_1X_0)^\alpha] \\ &= \mathbb{E}(A_1X_0 + B_1)^{\alpha-1} \\ &= \mathbb{E}X^{\alpha-1} \\ &< \infty, \end{aligned}$$

although $\mathbb{E}X^\alpha = \infty$. Then, if c_+ and $\mathbb{E}A^\alpha \log A$ are known, we can calculate the moments $\mathbb{E}X^{\alpha-k}$, $k = 1, 2, \ldots$, using (2.4). Alternatively, if we know $\mathbb{E}X^{\alpha-1}$ and $\mathbb{E}A^\alpha \log A$, we could calculate c_+. Unfortunately, (2.5) is not available in the Kesten–Goldie setting.

2.2. A result about the convergence of moments

The following result will be useful.

Lemma 2.2. *Consider three i.i.d. sequences (A_t), (B_t), and $(B_t^{(n)})$ of nonnegative random variables defined on the same probability space. Assume that (A_t) and (B_t) are independent, and that (A_t) and $(B_t^{(n)})$ are also independent. Let (X_t) be the solution to (1.1), and let $(X_t^{(n)})$ be the corresponding solution with (B_t) replaced by $(B_t^{(n)})$. If $\mathbb{E}A^p < 1$ and $\mathbb{E}|B_0 - B_0^{(n)}|^p \to 0$ as $n \to \infty$ for some $p > 0$, then*

$$X^{(n)} \xrightarrow{\text{D}} X \quad \textit{and} \quad \mathbb{E}|X^{(n)}|^p \to \mathbb{E}|X|^p \quad \textit{as } n \to \infty.$$

Proof. Recall the Mallows metric $d_p(R, S) = \inf(\mathbb{E}|R-S|^p)^{\min(1,1/p)}$, $p > 0$, where the infimum is taken over all joint distributions of the bivariate vectors (R, S) with fixed marginals and pth finite moments. It is well known (see, e.g. [31]) that d_p metrizes convergence in distribution and L^p convergence. Under the assumptions of the lemma, for $p \le 1$,

$$\begin{aligned} d_p(X_0^{(n)}, X_0) &\le \mathbb{E}|X_0^{(n)} - X_0|^p \\ &= \mathbb{E}\left|\sum_{i=-\infty}^{t} A_{i+1}\cdots A_t[B_i^{(n)} - B_i]\right|^p \\ &\le \sum_{i=-\infty}^{t} (\mathbb{E}A^p)^{t-i}\mathbb{E}|B^{(n)} - B|^p. \end{aligned} \tag{2.6}$$

But $\sum_{i=-\infty}^{t}(\mathbb{E}A^p)^{t-i} < \infty$ since $\mathbb{E}A^p < 1$ and $\mathbb{E}|B_0^{(n)} - B_0|^p \to 0$. Then (3.6) follows for $p \le 1$. For $p > 1$, the same idea of proof applies if we use the Minkowski inequality of order p in (2.6).

The following example illustrates the use of Lemma 2.2.

Example 2.2. Assume (A_t) and (B_t) are independent i.i.d. sequences of nonnegative random variables, and that (X_t) solves (1.1). We also assume that $\mathbb{E}A^p < 1$ and $\mathbb{E}B^p < \infty$ for some $p > 0$. Let $B^{(a)}$ be a random variable with distribution given by

$$\begin{aligned} &\mathbb{P}(B^{(a)} = 0) = 1 - a \quad \text{for some } a \in (0, 1] \\ \text{and} \quad &\mathbb{P}(B^{(a)} > x) = a(1 - \mathbb{P}(B \le x)), \qquad x > 0. \end{aligned}$$

Write F_a for the distribution of $B^{(a)}$. Denote by $(X_t^{(a)})$ the solution to (1.1) for independent i.i.d. sequences (A_t) and $(B_t^{(a)})$, where $B_t^{(a)} \overset{\mathrm{D}}{=} B^{(a)}$. Consider an i.i.d. uniform sequence (U_i) on $(0, 1)$ independent of (A_t). Then $F_a^{\leftarrow}(U_t)$ has the same distribution as $B^{(a)}$, where $F_a^{\leftarrow}$ denotes the quantile function of F_a. Obviously, $\mathbb{E}|B_0^{(a)} - B_0^{(1)}|^p \to 0$ as $a \to 1$. An application of Lemma 2.2 now yields

$$X_0^{(a)} \overset{\mathrm{D}}{\to} X \quad \text{and} \quad \mathbb{E}|X_0^{(a)}|^p \overset{\mathrm{D}}{\to} \mathbb{E}|X|^p.$$

3. Special cases

In this section we consider several special choices for the distributions of A and B, and give recursive or explicit expressions for the fractional moments of X.

3.1. Cases related to exponential *B*

The examples of this section are closely related to Example 2.1, i.e. to the case of exponentially distributed B. We will show how Lemma 2.1 and the results of Example 2.1 can be applied to classes of distributions of B beyond the exponential distribution.

3.1.1. *Erlang distributed B.* In Example 2.1 we explained how Lemma 2.1 can be applied to exponentially distributed B. A natural extension is to assume that B has an Erlang distribution.

Lemma 3.1. *Assume that (E_i) is an i.i.d. standard exponential, independent of (A_i), and that $B \overset{\mathrm{D}}{=} \Gamma_n = E_1 + \cdots + E_n$ for some $n \ge 1$. Also, assume that $A \ge 0$ a.s. and A, B are independent. Then, for $p \in \mathbb{R}$,*

$$\phi(p)\mathbb{E}X^p = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \mathbb{E}X^{p-k} p(p-1)\cdots(p-k+1), \tag{3.1}$$

where we assume that all moments in this formula are finite.

Proof. For $n = 1$, this is just (2.4). Now assume that $n \ge 2$. We have, with $\Gamma_0 = 0$,

$$\begin{aligned} \phi(p)\mathbb{E}X^p &= \mathbb{E}(A_1X_0 + \Gamma_n)^p - \mathbb{E}(A_1X_0)^p \\ &= \sum_{l=1}^{n} [\mathbb{E}(A_1X_0 + \Gamma_l)^p - \mathbb{E}(A_1X_0 + \Gamma_{l-1})^p]. \end{aligned}$$

As in the proof of Lemma 2.1, we obtain

$$\mathbb{E}(A_1X_0 + \Gamma_l)^p - \mathbb{E}(A_1X_0 + \Gamma_{l-1})^p = p\mathbb{E}(A_1X_0 + \Gamma_l)^{p-1}. \tag{3.2}$$

For $l = n$, $A_1X_0 + \Gamma_n \overset{\mathrm{D}}{=} A_1X_0 + B_1 \overset{\mathrm{D}}{=} X$. Therefore,

$$\mathbb{E}(A_1X_0 + \Gamma_{n-1})^p = \mathbb{E}X^p - p\mathbb{E}X^{p-1}. \tag{3.3}$$

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Using (3.2) and (3.3), we obtain

$$\begin{aligned}\mathbb{E}(AX+\Gamma_{n-1})^p-\mathbb{E}(AX+\Gamma_{n-2})^p &= p\mathbb{E}(AX+\Gamma_{n-1})^{p-1}\\ &= p\mathbb{E}X^{p-1}-p(p-1)\mathbb{E}X^{p-2}.\end{aligned}$$

Induction yields, for $l=1,\dots,n$,

$$\begin{aligned}&\mathbb{E}(A_1X_0+\Gamma_l)^p-\mathbb{E}(A_1X_0+\Gamma_{l-1})^p\\ &\quad=\sum_{j=0}^{n-l}(-1)^j\binom{n-l}{j}p(p-1)\cdots(p-j)\mathbb{E}X^{p-j-1}.\end{aligned}$$

This concludes the proof of (3.1).

Relation (3.1) yields a recursive relation for $\mathbb{E}X^p$ in terms of the lower moments $\mathbb{E}X^{p-1},\dots,\mathbb{E}X^{p-n}$. In contrast to the case $n=1$ (exponential B), we could not solve the recursion (3.1) explicitly. There is evidence that (3.1) yields good approximations to higher moments if one can approximate $\mathbb{E}X^p$ for small initial values of p (e.g. by Monte Carlo simulation); this has already been reported in [12].

In the Kesten–Goldie setting, the right-hand side of (3.1) remains valid for the tail index $p=\alpha$ if the left-hand side is replaced by $\mathbb{E}[(A_1X_0+\Gamma_n)^\alpha-(A_1X_0)^\alpha]$. The resulting formula yields an expression for Goldie's constant c_+ in (1.4) in terms of the moments $\mathbb{E}X^{\alpha-1},\dots,\mathbb{E}X^{\alpha-n}$.

3.1.2. *The distribution of B is a multiplicative mixture of an exponential distribution.* A particular case of (1.1) has attracted some attention:

$$X_t=A_t(X_{t-1}+C_t),\qquad t\in\mathbb{R}, \tag{3.4}$$

for independent sequences (A_t) and (C_t) of i.i.d. nonnegative random variables. In this case, $B_t=A_tC_t$, and A_t and B_t are dependent for every t with the exception of constant A. The marginal distribution of the solution to (3.4) is known in some particular cases when A and C have gamma- or beta-like distributions; see [13], [14], [15], [16], and [7], [10].

Lemma 3.2. *Consider model (3.4), and assume that C is standard exponential and $(A_t)=(\beta^{Y_t})$ for some $\beta\in(0,1)$ and an i.i.d. sequence (Y_t) with $Y>0$ a.s. Then*

$$\mathbb{E}X^p=f(p)\mathbb{E}(X_0+C_1)^p=f(p)\Gamma(p+1)\prod_{k=1}^{\infty}\frac{\phi(p+k)}{\phi(k)}, \tag{3.5}$$

provided $p>0$ or $p<0$, $-p\notin\mathbb{N}$, $f(p+1)<\infty$, and $\mathbb{E}(1-A)^{-1}<\infty$, and both sides in (3.5) are finite or infinite at the same time.

Proof. Assume that $\mathbb{E}X^p<\infty$ for some $p>0$. Then, by Lemma 2.1,

$$\begin{aligned}\mathbb{E}X^p&=f(p)\mathbb{E}(X_0+C_1)^p\\ &=f(p)[\mathbb{E}(X_0+C_1)^p-\mathbb{E}X^p+\mathbb{E}X^p]\\ &=f(p)[p\mathbb{E}(X_0+C_1)^{p-1}+\mathbb{E}X^p].\end{aligned}$$

Hence,

$$\mathbb{E}(X_0+C_1)^p=\frac{p}{\phi(p)}\mathbb{E}(X_0+C_1)^{p-1}.$$

Then one is in the situation of Example 2.1. Replacing in (2.4) the moments $\mathbb{E}X^{p-k}$, $k = 0, 1, \dots,$ by the corresponding moments of $X_0 + C_1$, the proof of Proposition 7 of [23] applies to conclude that

$$\mathbb{E}(X_0 + C_1)^p = \Gamma(p+1) \prod_{k=1}^{\infty} \frac{\phi(p+k)}{\phi(k)}.$$

3.1.3. *The distribution of B is exponential with an atom at 0.* In this subsection we assume that

$$\mathbb{P}(B = 0) = 1 - a \quad \text{for some } a \in (0, 1) \quad \text{and} \quad \mathbb{P}(B > x) = a\mathrm{e}^{-x} \quad \text{for } x > 0. \tag{3.6}$$

Then B^* has a standard exponential distribution and it is plausible that the results for exponential B are applicable. This is the content of the next result.

Lemma 3.3. *Assume that B has the distribution (3.6), that $(A_t) = (\beta^{Y_t})$ for some $\beta \in (0, 1)$ and an i.i.d. sequence (Y_t) of positive random variables, and that (A_t) and (B_t) are independent. Then, for $p > 0$, the relation*

$$\mathbb{E}X^p = \Gamma(p+1) \prod_{k=1}^{\infty} \frac{\phi(p+k)}{\phi(k)} \frac{1 - (1-a)\mathbb{E}A^{k-1}}{1 - (1-a)\mathbb{E}A^{p+k-1}} \tag{3.7}$$

holds, provided $\mathbb{E}[(1 - A)^{-1}] < \infty$ and

$$(1 - a)\mathbb{E}A^{p-1} < 1 \quad \textit{for } p \in (0, 1). \tag{3.8}$$

Proof. An application of Lemma 2.1 and (2.3) yields, for real p and standard exponential B,

$$\mathbb{E}X^p = \frac{pa}{\phi(p)} \mathbb{E}(A_1 X_0 + B^*)^{p-1}, \tag{3.9}$$

where both sides are finite or infinite at the same time. The moment on the right-hand side is finite for $p > 0$. For $p \geq 1$, this is elementary, and, for $p \in (0, 1)$, we have

$$\mathbb{E}(A_1 X_0 + B^*)^{p-1} \leq \mathbb{E}(B^*)^{p-1} = \int_0^{\infty} x^{-(1-p)} \mathrm{e}^{-x} \, \mathrm{d}x < \infty.$$

Under the assumption (3.8), we also have $\mathbb{E}X^{p-1} < \infty$ for $p \in (0, 1)$, as we will show next. We have the representation $B_i = E_i r_i$, $i \in \mathbb{Z}$, where (E_i) is an i.i.d. standard exponential sequence independent of the i.i.d. sequence (r_i), where $\mathbb{P}(r_i = 1) = 1 - \mathbb{P}(r_i = 0) = a$. Now consider $T_0 = 0$ and $T_1 = \min\{i \geq 1 \colon r_i = 1\}$. This random variable is geometrically distributed with success probability a: $\mathbb{P}(T_1 = k) = a(1-a)^{k-1}$, $k \geq 1$. By induction, define $T_{i+1} = \min\{k > T_i \colon r_k = 1\}$, $i \geq 1$. The sequence $(T_{i+1} - T_i)$ is i.i.d. Using the representation $X \stackrel{\mathrm{D}}{=} \sum_{i=0}^{\infty} B_{i+1} \Pi_i$, where $\Pi_i = \prod_{k=1}^{i} A_k$ with the convention that $\Pi_0 = 1$, we also have $X \stackrel{\mathrm{D}}{=} \sum_{k=1}^{\infty} E_{T_k} \Pi_{T_k - 1}$. Hence,

$$\begin{aligned}
\mathbb{E}X^{p-1} &\le \mathbb{E}(E_{T_1} \Pi_{T_1})^{p-1} \\
&= \sum_{k=1}^{\infty} a(1-a)^{k-1} \mathbb{E}(E_k \Pi_k)^{p-1} \\
&= \sum_{k=1}^{\infty} a(1-a)^{k-1} \mathbb{E}(B^*)^{p-1} (\mathbb{E}A^{p-1})^k \\
&= a\mathbb{E}A^{p-1} \mathbb{E}(B^*)^{p-1} (1-(1-a)\mathbb{E}A^{p-1})^{-1} \\
&< \infty,
\end{aligned}$$

where we used (3.8) in the last step. Relation (3.9) can be written in the form

$$\begin{aligned}
\mathbb{E}X^p &= \frac{p}{\phi(p)} [a\mathbb{E}(A_1 X_0 + B^*)^{p-1} + (1-a)\mathbb{E}(A_1 X_0)^{p-1}] - (1-a)\mathbb{E}(A_1 X_0)^{p-1} \\
&= \frac{p}{\phi(p)} [1-(1-a)\mathbb{E}A^{p-1}] \mathbb{E}X^{p-1} \\
&= \frac{p}{g(p)} \mathbb{E}X^{p-1}, \qquad p > 0.
\end{aligned} \tag{3.10}$$

We will verify that, for $p > 0$, provided $\mathbb{E}[(1-A)^{-1}] < \infty$, (3.7) coincides with the following limit:

$$\begin{aligned}
\mathbb{E}X^p &= \Gamma(p+1) \lim_{n\to\infty} \prod_{k=1}^{n} \frac{g(p+k)}{g(k)} \\
&= \Gamma(p+1) \prod_{k=1}^{\infty} \frac{\phi(p+k)}{\phi(k)} \frac{1-(1-a)\mathbb{E}A^{k-1}}{1-(1-a)\mathbb{E}A^{p+k-1}}.
\end{aligned}$$

For the convergence of $\prod_{k=1}^{\infty} g(p+k),\ p \ge 0$, to a finite positive limit, we verify that

$$\sum_{k=1}^{\infty} |g(p+k) - 1| < \infty. \tag{3.11}$$

However, for some constant $c > 0$,

$$\begin{aligned}
\sum_{k=1}^{\infty} |g(p+k) - 1| &= \sum_{k=1}^{\infty} \frac{|-f(p+k) + (1-a)f(p+k-1)|}{1-(1-a)f(p+k-1)} \\
&\le c \sum_{k=0}^{\infty} f(p+k) \\
&\le c\mathbb{E}[(1-A)^{-1}] \\
&< \infty.
\end{aligned}$$

Therefore, (3.11) is satisfied and the infinite products in (3.7) have finite positive limits. Now we proceed as in the proof of Proposition 7 of [23]. Consider the function

$$\psi(p) = \mathbb{E}(X^{p-1}) \prod_{k=1}^{\infty} \frac{g(k)}{g(p+k-1)}.$$

In view of (3.10) it satisfies the relations $\psi(p+1) = p\psi(p)$ for $p > 0$ and also $\psi(1) = 1$. We will show that ψ is the gamma function. As in [23], we will use the Bohr–Mollerup theorem (see [2]), according to which it remains to verify that $\log\psi$ is convex on $(0, \infty)$. We have

$$\log\psi(p) = \log\bigg(\prod_{k=1}^{\infty}\frac{g(k)}{g(p+k-1)}\bigg) + \log\mathbb{E}X^{p-1}.$$

Following [23], the second derivative of $\log\mathbb{E}X^{p-1}$ is nonnegative. Direct calculation shows that the second derivative of $\sum_{k=1}^{\infty}\log(g(k)/g(p+k-1))$ is nonnegative as well.

Now, in order to indicate that (B_t) and (X_t) depend on a, we write $(B_t^{(a)})$ and $(X_t^{(a)})$. Of course, $B^{(a)} \xrightarrow{\mathrm{D}} B^{(1)}$ as $a \uparrow 1$, and the limiting random variable has a standard exponential distribution. In our context, it is natural to ask whether the moments $\mathbb{E}(X^{(a)})^p$ converge to $\mathbb{E}(X^{(1)})^p$ as $a \uparrow 1$. The answer is indeed positive as follows from the discussion in Example 2.2. Under the assumptions of Lemma 3.3, we have, for $p > 0$, as $a \uparrow 1$,

$$X^{(a)} \xrightarrow{\mathrm{D}} X^{(1)} \quad \text{and} \quad \mathbb{E}(X^{(a)})^p \to \mathbb{E}(X^{(1)})^p.$$

3.1.4. *Geometric α-stable B.* In the previous examples we considered light-tailed distributions of B. The present example shows that B can have a very heavy-tailed distribution, while the exponential benchmark (Example 2.1) is still useful for determining the moments of X.

We assume that

$$B_n = E_n^{1/\alpha}C_n, \qquad n \in \mathbb{Z}, \tag{3.12}$$

where (E_n) is an i.i.d. standard exponential sequence, (C_n) is an i.i.d. sequence of strictly α-stable random variables for some $0 < \alpha \le 2$ (cf. [32]), $A > 0$ a.s., and (A_n), (E_n), and (C_n) are independent. Then (3.12) defines a strictly geometric α-stable random variable; see [26] and [27].

Let Z be an α-stable Lévy motion on $[0, \infty)$ such that $Z_1 \stackrel{\mathrm{D}}{=} C_1$ and $N_t = \#\{i \ge 1 \colon \Gamma_n \le t\}$, $t \ge 0$, $\Gamma_0 = 0$, $\Gamma_n = E_1 + \cdots + E_n$, $n \ge 1$, be the Poisson process generated by (E_n), $\xi_t = \sum_{i=1}^{N_t}\log A_i$, $t \ge 0$. Also, assume that Z, (E_n), and (A_n) are independent. The stationary solution to $X_n = A_nX_{n-1} + B_n$, $n \in \mathbb{Z}$, has the representation in law

$$X \stackrel{\mathrm{D}}{=} \int_0^{\infty} \mathrm{e}^{\xi_t}\,\mathrm{d}Z_t.$$

Indeed, recalling that $Z_{E_1} \stackrel{\mathrm{D}}{=} E_1^{1/\alpha}C_1$, we have

$$\begin{aligned}\int_0^{\infty} \mathrm{e}^{\xi_t}\,\mathrm{d}Z_t &= \sum_{k=1}^{\infty}\exp\bigg(\sum_{t=1}^{k}\log A_i\bigg)(Z_{\Gamma_k} - Z_{\Gamma_{k-1}}) \\ &\stackrel{\mathrm{D}}{=} \sum_{k=1}^{\infty} A_1\cdots A_kE_k^{1/\alpha}C_k.\end{aligned}$$

Write $Y_0 = \int_0^{\infty}\mathrm{e}^{\alpha\xi_t}\,\mathrm{d}t$, and assume that Y_0 and C_1 are independent. By the strict α-stability of Z,

$$X \stackrel{\mathrm{D}}{=} C_1Y_0^{1/\alpha},$$

and then, for $p < \alpha$,

$$\mathbb{E}|X|^p = \mathbb{E}|C|^p \mathbb{E} Y_0^{p/\alpha},$$

where $\mathbb{E}|C|^p$ is a known constant (see [34, Chapter 2]). It is not difficult to see that

$$Y_0 \stackrel{\mathrm{D}}{=} A_1^\alpha Y_0 + E_1,$$

where A_1, E_1, and Y_0 are independent. Now, the moments of Y_0 can be determined by using (2.5) and, in a similar way, one can determine the moments of the positive and negative parts of X.

3.2. The case when B is bounded

In Section 1 we mentioned a popular class of models from financial time series analysis, the ARCH-GARCH family, and the fact that it is common to analyze the fractional moments of such processes. The GARCH(1, 1) squared volatility process (σ_t^2) is described by the stochastic recurrence equation (1.6). In this case, $B = \alpha_0$ is a constant.

The goal of this section is to study the fractional moments of X for bounded B. Using the same idea as above, we can obtain similar formulae for recursion (1.1) with bounded A and B.

Proposition 3.1. *Assume that A and B are independent, and that* $0 \le A \le a < 1$, $0 \le B \le b$ *a.s. for some positive constants* a, b. *Write* $f(p) = \mathbb{E}A^p$ *and* $g(p) = \mathbb{E}B^p$, $p > 0$. *Then the following relation holds for* $p > 0$*:*

$$\begin{aligned}\mathbb{E}X^p = \sum_{n=0}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!} \frac{f(n)g(p-n)}{\phi(n)} \\ \times \sum_{k_1=0}^{n-1} \frac{f(k_1)g(n-k_1)}{\phi(k_1)} \sum_{k_2=0}^{k_1-1} \frac{f(k_2)g(k_1-k_2)}{\phi(k_2)} \cdots \sum_{k_n=0}^{k_{n-1}-1} \frac{f(k_n)g(k_{n-1}-k_n)}{\phi(k_n)} \\ \times \binom{n}{k_1-k_2 k_2-k_3 \cdots k_{n-1}-k_n k_n}.\end{aligned} \tag{3.13}$$

Proof. We start by observing that $A_1 X_0 \le ab(1-a)^{-1}$. Thus, if $a < (b+1)^{-1}$, a Taylor expansion of $(B_1 + x)^p$ for $0 \le x \le ab(1-a)^{-1} < 1$ yields

$$\mathbb{E}(B_1 + A_1 X_0)^p = \sum_{n=0}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!} f(n) \mathbb{E}B^{p-n} \mathbb{E}X^n.$$

Then multiple use of (1.5) yields (3.13).

For any $c > 0$, write $B_t(c) = B_t/c$ and $X_t(c) = X_t/c$. The stochastic recurrence equation (1.1) is equivalent to

$$X_t(c) = A_t X_{t-1}(c) + B_t(c), \qquad t \in \mathbb{Z}, \tag{3.14}$$

and $\mathbb{E}X^p = c^p \mathbb{E}(X(c))^p$. Since $B(c) \le b/c$ a.s. and $X(c) \le a(b/c)(1-a)^{-1}$, the above calculations are valid for the new equation (3.14) under the condition $a < (b/c + 1)^{-1}$. Thus, if c is sufficiently large, the value a may be arbitrarily close to 1 and all moment calculations can be extended for arbitrary $a < 1$ and $b > 0$.

Remark 3.1. The condition $a < 1$ is crucial for the above calculations. If $\mathbb{P}(A = 1) > 0$ then all of the moments $\mathbb{E}X^p$, $p > 0$, can still be finite. For example, assume that we have

$q = \mathbb{P}(A = 0) = 1 - \mathbb{P}(A = 1) \in (0, 1)$ and $B \equiv 1$ a.s. Then X has a geometric distribution with parameter q and the moments are given by

$$\mathbb{E}X^p = (1 - q)\sum_{n=1}^{\infty} q^{n-1} n^p, \qquad p > 0.$$

The latter example extends to the case when $q = \mathbb{P}(A = 0) = 1 - \mathbb{P}(A = a) \in (0, 1)$ for some $a > 0$. Then $X = \sum_{n=1}^{N} a^n$, where N is geometric with success probability q and the moments are given by

$$\mathbb{E}X^p = (1 - q) \sum_{n=1}^{\infty} q^{n-1} \Big(\sum_{k=1}^{n} a^k\Big)^p.$$

For $a \neq 1$, this turns into

$$\mathbb{E}X^p = (1 - q) \frac{a^p}{|1 - a|^p} \sum_{n=1}^{\infty} q^{n-1} |1 - a^n|^p,$$

which is finite for $qa^p < 1$.

In the remainder of this section, we assume that $B \equiv 1$ a.s. This case can be understood as a limiting case when B is Erlang distributed and the parameter of the Erlang distribution tends to ∞. Indeed, consider the sequence of stochastic recurrence equations $X_t^{(n)} = A_t X_{t-1}^{(n)} + B_t^{(n)}$, $t \in \mathbb{Z}$, where $(B_t^{(n)})$ is an i.i.d. sequence with $B^{(n)} \stackrel{\mathrm{D}}{=} n^{-1}(E_1 + \cdots + E_n)$ for an i.i.d. sequence (E_t) with standard exponential marginal distribution. We also assume that (A_t) and $(B_t^{(n)})$ are independent. By the strong law of large numbers, $B_t^{(n)} \to 1$ a.s. as $n \to \infty$ for every t and we also have $\mathbb{E}|B^{(n)} - 1|^p \to 0$ for every $p > 0$. Lemma 2.2 now yields $\mathbb{E}|X^{(n)}|^p \to \mathbb{E}|X|^p$.

A change of the indices in (3.13) and $g \equiv 1$ yield the alternative expression

$$\mathbb{E}X^p = \sum_{j=1}^{\infty} \sum_{n_1=1}^{\infty} \cdots \sum_{n_j=1}^{\infty} \frac{p(p-1)\cdots(p-(n_1+\cdots+n_j)+1)}{\prod_{d=1}^{j} n_d!} \times \frac{f(n_1)}{\phi(n_1)} \frac{f(n_1+n_2)}{\phi(n_1+n_2)} \cdots \frac{f(n_1+\cdots+n_j)}{\phi(n_1+\cdots+n_j)}.$$

This formula can be easily evaluated and converges very fast. We illustrate this aspect for A with a uniform distribution on $(0, a)$, $a < 1$. Then

$$\sum_{j=1}^{\infty} \sum_{n_1=1}^{\infty} \cdots \sum_{n_j=1}^{\infty} \frac{p(p-1)\cdots(p-(n_1+\cdots+n_j)+1)}{\prod_{d=1}^{j} n_d!} \frac{a^{n_1}}{(n_1+1) - a^{n_1}} \times \frac{a^{n_1+n_2}}{(n_1+n_2+1) - a^{n_1+n_2}} \cdots \frac{a^{n_1+\cdots+n_j}}{(n_1+\cdots+n_j+1) - a^{n_1+\cdots+n_j}}. \qquad (3.15)$$

In Figure 1 we approximate the latter expression for $p = 0.6$ and values $a \in (0, 1)$ by replacing all infinite sums in (3.15) by the sums truncated at some integer $M \geq 1$. The value M is indicated on the x-axis. The corresponding absolute change of the approximated moment from M to $M + 1$ is tabulated in Table 1.

We mention for the sake of completeness that the calculation of Goldie's constant c_+ in (1.4) in the case $B \equiv 1$ a.s. has been addressed in [19]. However, since we assume that $A < 1$ a.s., the results of this section are beyond the Kesten–Goldie setting.

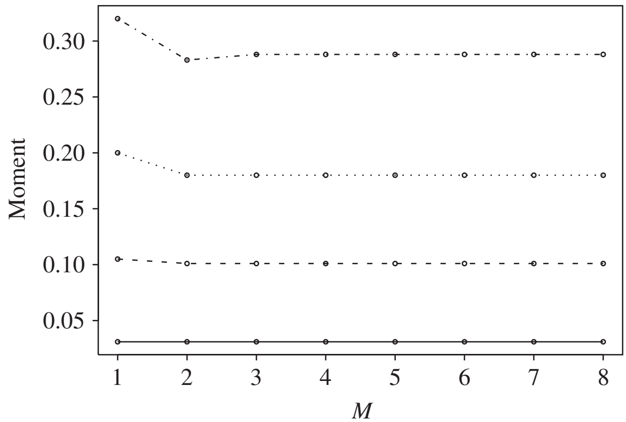

FIGURE 1: Approximation of $\mathbb{E}X^{0.6}$ for $B = 1$ and A uniform on $(0, a)$, taking into account only the first M (indicated on the x-axis) summands in each of the infinite series in (3.15). The curves from bottom to top correspond to $a = 0.1, 0.3, 0.5, 0.7$.

TABLE 1: Absolute values of the differences of the approximated moments for $p = 0.6$. The symbol $\Delta_{M,M+1}$ describes the absolute value of the change of the approximation from M to $M + 1$.

a	Δ_{12}	Δ_{23}	Δ_{34}	Δ_{45}	Δ_{56}	Δ_{67}	Δ_{78}
0.1	4×10^{-4}	1×10^{-5}	6×10^{-7}	4×10^{-8}	2×10^{-9}	1×10^{-10}	1×10^{-11}
0.3	5×10^{-3}	3×10^{-4}	5×10^{-5}	8×10^{-6}	1×10^{-6}	3×10^{-7}	6×10^{-8}
0.5	2×10^{-2}	2×10^{-3}	3×10^{-4}	7×10^{-5}	2×10^{-5}	6×10^{-6}	2×10^{-6}
0.7	4×10^{-2}	4×10^{-3}	9×10^{-4}	3×10^{-4}	9×10^{-5}	4×10^{-5}	3×10^{-5}

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