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mathematical hall of fame (here, with portraits), the same in number as the previous handbook with a few differences. Both handbooks agree on their women mathematicians (Hypatia, Maria Agnesi, Sonia Kovalevskaya and Emmy Noether) though this one adds Sophie Germain who worked on pure and applied mathematics and initially wrote under the pen-name Monsieur LeBlanc. Perhaps, because it contains less information, but also because of its attractive layout, I found this handbook less overpowering than the first. Again, it is excellent value.

The third and fourth handbooks cater mainly for specialists. The reduced sales potential accounts for the higher prices. The Handbook of function and generalized function transformations presents many transformations likely to be used by engineers, physicists and mathematicians. A prominent example is the Fourier transform. Mathematicians who have contributed to this area in recent times include Ruel V. Churchill, I. N. Sneddon, E. C. Titchmarsh, all of whom have written classic books on the subject. In recent years an impulse has been given to this subject by the popularity of wavelets, and in this connection the transformations due to V. Bargmann, I. M. Gel’fand/J. Zac and Dennis Gabor (of hologram fame) are explored. That was in the 1960s, while holograms are now being used in areas from Art to Medical Science. It is good to see that he has a transformation named after him. The handbook assumes a knowledge of real and complex analysis at graduate level. These are the ‘non-experts’ the author had in mind and the reason he included only the ‘basic facts’! This handbook is almost a text book since it gives definitions and theorems in a connected way, though proofs are omitted and examples substituted for explanatory purposes. The book will be an indispensable guide for the specialist and perhaps for the general reader. The author’s discussion of fractional differentiation, for example, is brief but lucid.

The last handbook for review, CRC handbook of tables for the use of order statistics in estimation is specialised to order statistics (for example, the smallest member of a sample drawn at random from some population) and functions of them. The first two hundred pages describe various order statistics from different populations (including rectangular, exponential, and normal populations). An extensive list of references is given at the end of each chapter. The final two-thirds of the book consists of statistical tables including variances and covariances of normal, exponential, Weibull and Gamma distributions.

Each one of these handbooks deserves a place in the central reference section of any university mathematics library. The first two could be usefully consulted by a keen A level student.

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This book is a collection of translations of papers by Paul Finsler on set theory, with some additional material prepared by the editors.

In the translated papers, Finsler expresses his views on the philosophy of set theory and the philosophy of mathematics generally. He discusses the paradoxes of self-reference from a philosophical standpoint. He also analyses the paradoxes of set theory from a more mathematical standpoint, with the aim of identifying the error of reasoning which led to the paradoxes.

Finsler proposes a set theory of his own, which he believes will avoid the paradoxes. The consensus of modern set theorists is that Finsler’s set theory is
incoherent (the word ‘incoherent’ is not used here in any technical sense, but in the literal English sense; it is not just that Finsler’s axioms are inconsistent, but that it is hard to determine what they mean!); the reviewer concurs, and thereby takes issue not only with Finsler himself but with the editors of the volume. We state Finsler’s axioms and discuss them:

‘We consider a system of things, which we call sets, and a relation, which we symbolise by \( \beta \). The exact and complete description is achieved by means of the following axioms.

I. Axiom of Relation: For arbitrary sets \( M \) and \( N \) it is always uniquely determined whether \( M \) possesses the relation \( \beta \) to \( N \) or not.

II. Axiom of Identity: Isomorphic sets are identical.

III. Axiom of Completeness: The sets form a system of things which, by strict adherence to the axioms I and II, is no longer capable of extension. That is, it is not possible to adjoin further things in such a way that the axioms I and II are satisfied.’

The relation \( \beta \) is the converse of the usual membership relation; it might be read ‘contains’. Axiom I asserts that the extensions of sets must be well-defined. Axiom II needs explanation (it is explained in Finsler’s paper some time after it is stated). The idea is that the identity of a set should be completely determined by the isomorphism class of the relation \( \beta \) restricted to the transitive closure of the set (i.e. the collection of its elements, the elements of its elements, and so forth). A consequence of this axiom is the usual axiom of extensionality (sets with the same elements are the same) but axiom II is stronger than extensionality: for example, it implies that two sets which are their own sole elements must be equal. It is the strong extensionality axiom associated with an anti-foundation axiom of the kind studied by Peter Aczel in [1]: Aczel includes Finsler’s axiom among those that he considers.

The problem with Finsler’s theory is his Axiom III. Literally, what it says is that the universe of sets is a maximal structure satisfying axioms I and II. The only way that a structure satisfying axioms I and II can be maximal is for its domain to be the universal class; otherwise, as Reinhold Baer, a contemporary German mathematician, observed, one could take a new object and assign to it the extension of the Russell class of the purported maximal structure, thus extending it.

Finsler reads his axiom as implying that all consistent set definitions will be satisfied in his theory. But this is not possible, as Ernst Specker pointed out: the sets \( \{ x \mid x = x \} \) (the universe) and \( \{ x \mid x \notin x \text{ and for some } y, x \in y \} \) (the class of all elements which are not elements of themselves) can each be satisfied in a model of axioms I and II, but they cannot both be present in a single model of axioms I and II. The second set cannot be an element, and so cannot coexist with the set of all sets, which must have every set as an element.

Both Finsler and the editors attempt objections to Specker’s refutation, but I find these objections unintelligible.

Further, the model theory of axioms I and II does not support Finsler’s assertions about the consequences of axioms I-III. Finsler claims that the existence of the universal set is a consequence of his axioms, but the reviewer has shown that any maximal model of axioms I-II with a universal set can be converted to a maximal model of axioms I-II without a universal set.

Finsler goes on to introduce a further concept, which he does not support using
reasoning based on axioms I-III. He argues convincingly that the problem with the set-theoretical paradoxes is that the definitions of the paradoxical sets are in some sense ‘circular’. The new concept he defines is that of a ‘circle-free’ set. He points out that a ‘circle-free’ set certainly will not appear in its own transitive closure (it will not itself be one of the components from which it is constructed). But this is not enough; the collection of all sets which do not appear in their own transitive closures is itself paradoxical and so ‘circular’ (think about whether it is an element of itself!) Finsler concluded that the correct definition of ‘circle-free’ is

Definition: A set is circle-free if it does not belong to its own transitive closure and its definition does not refer to the concept ‘circle-free’.

This is a ‘circular’ definition, of course. He proceeds to develop the consequences of this definition, very largely independently of axioms I-III. The fascinating thing is that the consequences that he develops are basically those of the set theory of Ackermann (for a full description of this theory the reader is referred to [2]). Ackermann's theory has sets and classes. The comprehension principle given for classes is that an arbitrary condition serves to define a class of sets (though there may, and indeed must, be classes which have non-set elements).

The comprehension principle for sets is that any class of sets which can be defined without reference to the concept of sethood (and without non-set parameters) is a set. The similarity to the definition of the circle-free sets should be clear. The similarity extends to detail; every axiom of Ackermann's theory is reflected by a conclusion of Finsler's about circle-free sets, and Finsler anticipated proofs of Ackermann in detail, including the (perhaps surprising) proof of the axiom of infinity. Ackermann’s set theory is consistent iff \( \text{ZFC} \) is consistent, so this much of Finsler's work can be put on a sound foundation.

Our conclusion about Finsler’s set theory is that, while the entire theory is untenable (and difficult to understand) the ideas behind axiom II and the notion of ‘circle-free’ sets prove to be sound. Finsler appears to have had good intuition. Space forbids discussion of Finsler's other contributions, made evident from the papers in this volume, except to say that we find his thoughts about paradoxes of self-reference to be interesting, his thoughts about the distinction between sets and classes to be important, and his defence of mathematical Platonism to be admirable. We think that making these papers available in translation is a service to scholarship.

We found the supporting materials prepared by the editors to be unsatisfactory in most cases, with the notable exception of calling attention to the relation to the set theory of Ackermann. This material called for the services of editors with a clearer understanding of the mathematical issues involved. The editors attempt to defend the coherence of Finsler's axiom III against the objections of Baer, Specker, and others; no such defence is possible. The paper by Ziegler on paradoxes of self-refernce presents a ‘solution’ to paradoxes of self-reference which makes it impossible for two letters in an algebraic expression to refer to the same object! (Finsler's approach is much more sensible.) The editors should have presented an explanation of the real relation of Finsler's derivation of a 'formally undecidable proposition' to the later work of Gödel; they do not. (Finsler does not actually succeed in presenting a formally undecidable proposition, though he is on the right track; his 'proposition' would need a truth predicate, forbidden by Tarski's theorem, to be actually expressible in his language).

The combination of the fascinating but ultimately untenable set-theoretical claims of Finsler and the unsatisfactory support provided to the reader by the editors led the reviewer to write an extended review of the book [3].
References

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How many lottery tickets must one buy in order to guarantee at least three correct numbers? For which values of \( n \) can a whist tournament for 4\( n \) players be designed, so that each player plays in each round, and partners everyone else once and opposes everyone else twice? How can Bob be sure that a message purporting to come from Alice has not been tampered with by Oscar?

These are the kinds of question which can be answered by the mathematical theory of Combinatorial Designs. The archetypal question was first raised in the first half of the nineteenth century. Can \( v \) objects be arranged in blocks of size \( k \), in such a way that each subset of size \( t \) is contained in exactly \( \lambda \) blocks? In 1853 Jakob Steiner asked about systems with \( k - 3, t = 2, \) and \( \lambda = 1 \), and such a system subsequently became known as a Steiner Triple System (STS). A simple counting argument leads to the conclusion that an STS is possible only when \( v \) is congruent to 1 or 3 modulo 6, but Steiner was unable to decide if this necessary condition is also sufficient. Remarkably, that very question had been settled six years earlier, by the Reverend Thomas Kirkman, who had shown how to construct an STS for every value of \( v \) satisfying the simple condition.

Kirkman's paper was little known at the time, although he did become famous as the author of a simple puzzle derived from it. Suppose that fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast. Kirkman's Schoolgirls Problem asks for an STS with \( v = 15 \), whose blocks are split into seven sets of five, each of which contains all 15 girls. The problem is fairly easy to solve, but for some reason it captured the imagination of mathematicians in the second half of the nineteenth century, and much was written about it. The problem for general values of \( v \) was also studied. Here too there is a simple necessary condition: an STS with \( v \) objects can be split up in the Kirkman manner only if \( v \) is congruent to 3 modulo 6. For many years the sufficiency of this condition was an open question until (it is reported) in the unlikely environment of China at the time of the Cultural Revolution, Li Juaxi settled the question in the affirmative. His work remained unpublished and unknown until 1990, and, in the meantime, an independent solution had been published by Ray-Chaudhuri and Wilson.

Kirkman's Problem, in its general form, typifies the difficulties of the theory of Combinatorial Designs. It is usually easy to obtain simple numerical conditions which must be satisfied if a particular kind of design exists, but it is much harder to construct such a design. The development of algebraic methods, especially those based upon finite fields, has helped to provide methods of construction which work...