PERIODIC SOLUTIONS OF A TWO-SPECIES RATIO-DEPENDENT PREDATOR-PREY SYSTEM WITH TIME DELAY IN A TWO-PATCH ENVIRONMENT

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Abstract

By using the continuation theorem of coincidence degree theory, a sufficient condition is obtained for the existence of a positive periodic solution of a predator-prey diffusion system.

1. Introduction

Xu and Chen [4] considered a two-species ratio-dependent predator-prey diffusion model with time delay given by

\begin{align*}
    x'_1(t) &= x_1(t) \left( a_1 - a_{11}x_1(t) - \frac{a_{13}x_3(t)}{mx_3(t) + x_1(t)} \right) + D_1(x_2(t) - x_1(t)), \\
    x'_2(t) &= x_2(t) \left( a_2 - a_{22}x_2(t) \right) + D_2(x_1(t) - x_2(t)), \\
    x'_3(t) &= x_3(t) \left( -a_3 + \frac{a_{31}x_1(t - \tau)}{mx_3(t - \tau) + x_1(t - \tau)} \right),
\end{align*}

(1.1)

where \( x_i(t) \) represents the prey population in the \( i \)th patch, \( i = 1, 2 \), and \( x_3(t) \) represents the predator population. Here \( \tau > 0 \) is a constant delay due to gestation, \( D_i \) is a positive constant denoting the dispersal rate, \( i = 1, 2 \), and \( a_i (i = 1, 2, 3) \), \( a_{11}, a_{13}, a_{22}, a_{31} \) and \( m \) are positive constants.

In Xu and Chen [4], the local and global asymptotical stability of the positive equilibrium of the system (1.1) were studied. For an ecological interpretation of system (1.1), we refer to [4] and references cited therein.

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Realistic models require the inclusion of the effect of change in the environment. This motivates us to consider the following two species predator-prey diffusion model with time delay:

$$\begin{align*}
x'_1(t) &= x_1(t) \left( a_1(t) - a_{11}(t)x_1(t) - \frac{a_{13}(t)x_3(t)}{m(t)x_3(t) + x_1(t)} \right) + D_1(t)(x_2(t) - x_1(t)), \\
x'_2(t) &= x_2(t)(a_2(t) - a_{22}(t)x_2(t)) + D_2(t)(x_1(t) - x_2(t)), \\
x'_3(t) &= x_3(t) \left( -a_3(t) + \frac{a_{31}(t)x_1(t - \tau)}{m(t)x_3(t - \tau) + x_1(t - \tau)} \right).
\end{align*}$$

(1.2)

In addition, the effects of a periodically changing environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Therefore the assumptions of periodicity of the parameters are a way of incorporating the periodicity of the environment (for example, seasonal changes, food supplies, mating habits, and so on), which leads us to assume that $D_i$ ($i = 1, 2$), $a_i$ ($i = 1, 2, 3$), $a_{11}$, $a_{13}$, $a_{22}$, $a_{31}$ and $m$ are strictly positive continuous $u\tau$-periodic functions.

As pointed out by Freedman and Wu [1] and Kuang [3], it is of interest to study the global existence of periodic solutions for systems representing predator-prey or competition systems. In this paper, our aim is to use the continuation theorem of coincidence degree theory which was proposed in [2] by Gaines and Mawhin to establish the existence of at least one positive $u\tau$-periodic solution with $w > 0$ of system (1.2).

Let $X, Z$ be real Banach spaces, $L : \text{dom} L \subset X \rightarrow Z$ a Fredholm mapping of index zero and $P : X \rightarrow X, Q : Z \rightarrow Z$ continuous projectors such that $\text{Im} P = \text{Ker} L, \text{Ker} Q = \text{Im} L, X = \text{Ker} L \oplus \text{Ker} P$ and $Z = \text{Im} L \oplus \text{Im} Q$. Denote by $K_p : \text{Im} L \rightarrow \text{Ker} P \cap \text{dom} L$ the generalised inverse (of $L$) and by $J : \text{Im} Q \rightarrow \text{Ker} L$ an isomorphism of $\text{Im} Q$ onto $\text{Ker} L$.

For convenience we introduce a continuation theorem [2, page 40] as follows.

**Lemma 1.1.** Let $\Omega \subset X$ be an open bounded set and $N : X \rightarrow Z$ be a continuous operator which is $L$-compact on $\overline{\Omega}$ (that is, $QN : \overline{\Omega} \rightarrow Z$ and $K_p(I - Q)N : \overline{\Omega} \rightarrow Y$ are compact). Assume

(a) for each $\lambda \in (0, 1), x \in \partial\Omega \cap \text{dom} L, Lx \neq \lambda Nx$;
(b) for each $x \in \partial\Omega \cap \text{Ker} L, QNx \neq 0$;
(c) $\deg(J QNx, \Omega \cap \text{Ker} L, 0) \neq 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega}$. 

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2. Main result

For the sake of convenience we will use the notation

\[ \bar{f} = \frac{1}{w} \int_0^w f(t) \, dt, \quad f^l = \min_{t \in [0, w]} f(t) \quad \text{and} \quad f^M = \max_{t \in [0, w]} f(t), \]

where \( f \) is a strictly positive continuous \( w \)-periodic function.

We now state our fundamental theorem about the existence of a positive \( w \)-periodic solution of system (1.2).

**Theorem 2.1.** Assume the following:

(i) \( (a_1 - D_1) > a_{13}^M / m^1 \),
(ii) \( a_{31}^l > \bar{a}_3 \),
(iii) \( (a_2 - D_2) > 0 \).

Then system (1.2) has at least one positive \( w \)-periodic solution.

**Proof.** Let

\[ F_1(t, s) = \frac{a_{13}(t)e^{y_3(s)}}{m(t)e^{y_3(s)} + e^{y_1(s)}} \quad \text{and} \quad F_2(t, s) = \frac{a_{31}(t)e^{y_1(t-s)}}{m(t)e^{y_3(s-t)} + e^{y_1(t-s)}}. \]

Consider the system

\[
\begin{align*}
    y_1'(t) &= a_1(t) - D_1(t) - a_{11}(t)e^{y_1(t)} - F_1(t, t) + D_1(t)e^{y_1(t)} - y_1(t), \\
    y_2'(t) &= a_2(t) - D_2(t) - a_{22}(t)e^{y_2(t)} + D_2(t)e^{y_2(t)} - y_2(t), \\
    y_3'(t) &= -a_3(t) + F_2(t, t),
\end{align*}
\]

(2.1)

where \( \tau, D_i (i = 1, 2), a_i (i = 1, 2, 3), a_{11}, a_{13}, a_{22}, a_{31} \) and \( m \) are the same as those in system (1.2). It is easy to see that if the system (2.1) has an \( w \)-periodic solution \((y_1^*(t), y_2^*(t), y_3^*(t))^T\), then \((e^{y_1^*(t)}e^{y_2^*(t)}e^{y_3^*(t)})^T\) is a positive \( w \)-periodic solution of system (1.2). Therefore for (1.2) to have at least one positive \( w \)-periodic solution it is sufficient that (2.1) has at least one \( w \)-periodic solution. In order to apply Lemma 1.1 to system (2.1), we take

\[ X = \{(y_1(t), y_2(t), y_3(t))^T \in C^1(R, R^3) : y_i(t + w) = y_i(t), \text{ for } i = 1, 2, 3 \}, \]

\[ Z = \{(z_1(t), z_2(t), z_3(t))^T \in C(R, R^3) : z_i(t + w) = z_i(t), \text{ for } i = 1, 2, 3 \} \]

and

\[ \| (y_1(t), y_2(t), y_3(t))^T \| = \max_{t \in [0, w]} |y_1(t)| + \max_{t \in [0, w]} |y_2(t)| + \max_{t \in [0, w]} |y_3(t)|. \]
With this norm, $X$ and $Z$ are Banach spaces. Let

$$N \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} a_1(t) - D_1(t) - a_{11}(t)e^{y_1(t)} - F_1(t, t) + D_1(t)e^{y_1(t)-y_1(t)} \\ a_2(t) - D_2(t) - a_{22}(t)e^{y_2(t)} + D_2(t)e^{y_1(t)-y_2(t)} \\ -a_3(t) + F_2(t, t) \end{bmatrix},$$

$$L \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix}, \quad P \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} (1/w) \int_0^w y_1(t) \, dt \\ (1/w) \int_0^w y_2(t) \, dt \\ (1/w) \int_0^w y_3(t) \, dt \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in X,$$

$$Q \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} (1/w) \int_0^w z_1(t) \, dt \\ (1/w) \int_0^w z_2(t) \, dt \\ (1/w) \int_0^w z_3(t) \, dt \end{bmatrix}, \quad \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in Z.$$

We note that $\text{Ker } L = R^3$,

$$\text{Im } L = \left\{ \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} | \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in Z, \int_0^w z_i(t) \, dt = 0, \text{ for } i = 1, 2, 3 \right\}$$

is closed in $Z$ and $\text{dim Ker } L = \text{codim Im } L = 3$. Hence $L$ is a Fredholm mapping of index 0. Furthermore, the generalised inverse (of $L$) $K_p : \text{Im } L \to \text{Ker } P \cap \text{dom } L$ has the form

$$K_p(z) = \int_0^t z(s) \, ds - \frac{1}{w} \int_0^w \int_0^t z(s) \, ds \, dt, \quad \text{for } z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in Z.$$

Thus $QN : X \to Z$,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \mapsto \begin{bmatrix} \frac{1}{w} \int_0^w [a_1(t) - D_1(t) - a_{11}(t)e^{y_1(t)} - F_1(t, t) + D_1(t)e^{y_1(t)-y_1(t)}] \, dt \\ \frac{1}{w} \int_0^w [a_2(t) - D_2(t) - a_{22}(t)e^{y_2(t)} + D_2(t)e^{y_1(t)-y_2(t)}] \, dt \\ \frac{1}{w} \int_0^w [-a_3(t) + F_2(t, t)] \, dt \end{bmatrix}.$$

$$K_p(I - Q)N : X \to X$$

and

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \mapsto \begin{bmatrix} \int_0^t [a_1(s) - D_1(s) - a_{11}(s)e^{y_1(s)} - F_1(s, s) + D_1(s)e^{y_1(s)-y_1(s)}] \, ds \\ \int_0^t [a_2(s) - D_2(s) - a_{22}(s)e^{y_2(s)} + D_2(s)e^{y_1(s)-y_2(s)}] \, ds \\ \int_0^t [-a_3(s) + F_2(s, s)] \, ds \end{bmatrix} - \frac{1}{w} \int_0^w \int_0^t [a_1(s) - D_1(s) - a_{11}(s)e^{y_1(s)} - F_1(s, s) + D_1(s)e^{y_1(s)-y_1(s)}] \, ds \, dt 

- \frac{1}{w} \int_0^w \int_0^t [a_2(s) - D_2(s) - a_{22}(s)e^{y_2(s)} + D_2(s)e^{y_1(s)-y_2(s)}] \, ds \, dt 

- \frac{1}{w} \int_0^w \int_0^t [-a_3(s) + F_2(s, s)] \, ds \, dt.$$
Clearly \( QN \) and \( K_p (I - Q)N \) are continuous by the Lebesgue theorem and moreover \( QN (\Omega) \) and \( K_p (I - Q)N (\Omega) \) are relatively compact for any open bounded set \( \Omega \subset X \). Hence \( N \) is \( L - \)compact on \( \Omega \) for any open bounded set \( \Omega \subset X \).

Corresponding to the operator equation \( Lx = \lambda Nx, \lambda \in (0, 1) \), we have

\[
\begin{align*}
\dot{y}_1(t) &= \lambda \left[ a_1(t) - D_1(t) - a_{11}(t)e^{y_1(t)} - F_1(t, t) + D_1(t)e^{y_2(t)-y_1(t)} \right], \\
\dot{y}_2(t) &= \lambda \left[ a_2(t) - D_2(t) - a_{22}(t)e^{y_2(t)} + D_2(t)e^{y_1(t)-y_2(t)} \right], \\
\dot{y}_3(t) &= \lambda \left[ -a_3(t) + F_2(t, t) \right].
\end{align*}
\]

Suppose that \((y_1(t), y_2(t), y_3(t))^T \in X \) is a solution of system (2.2) for a certain \( \lambda \in (0, 1) \). By integrating (2.2) over the interval \([0, w]\), we obtain

\[
\begin{align*}
\int_0^w \left[ a_1(t) - D_1(t) - a_{11}(t)e^{y_1(t)} - F_1(t, t) + D_1(t)e^{y_2(t)-y_1(t)} \right] dt &= 0, \\
\int_0^w \left[ a_2(t) - D_2(t) - a_{22}(t)e^{y_2(t)} + D_2(t)e^{y_1(t)-y_2(t)} \right] dt &= 0
\end{align*}
\]

and

\[
\int_0^w \left[ -a_3(t) + F_2(t, t) \right] dt = 0.
\]

Thus

\[
\begin{align*}
\int_0^w \left[ a_{11}(t)e^{y_1(t)} + F_1(t, t) \right] dt &= (a_1 - D_1)w + \int_0^w D_1(t)e^{y_1(t)-y_1(t)} dt, \\
\int_0^w a_{22}(t)e^{y_2(t)} dt &= (a_2 - D_2)w + \int_0^w D_2(t)e^{y_1(t)-y_2(t)} dt, \\
\int_0^w F_2(t, t) dt &= \overline{a_3}w.
\end{align*}
\]

From (2.2)–(2.5), it follows that

\[
\begin{align*}
\int_0^w |\dot{y}_1'(t)| dt &\leq \lambda \int_0^w \left| a_1(t) - D_1(t) - a_{11}(t)e^{y_1(t)} - F_1(t, t) + D_1(t)e^{y_2(t)-y_1(t)} \right| dt \\
&< (a_1 - D_1)w + \int_0^w \left| a_{11}(t)e^{y_1(t)} + F_1(t, t) \right| dt \\
&\quad + \int_0^w D_1(t)e^{y_2(t)-y_1(t)} dt \\
&\quad = 2(a_1 - D_1)w + \int_0^w D_1(t)e^{y_2(t)-y_1(t)} dt, \\
\int_0^w |\dot{y}_2'(t)| dt &\leq \lambda \int_0^w \left| a_2(t) - D_2(t) - a_{22}(t)e^{y_2(t)} + D_2(t)e^{y_1(t)-y_2(t)} \right| dt
\end{align*}
\]
\[
\int_0^w y_3'(t) \, dt \leq \lambda \int_0^w \left| y_2(t) + F_2(t, t) \right| \, dt < \alpha_3 w + \int_0^w F_2(t, t) \, dt = 2\alpha_3 w. \tag{2.8}
\]

Multiplying the first equation and the second equation of system (2.2) by \( e^{y_1(t)} \) and \( e^{y_2(t)} \), respectively, and integrating both over \([0, w]\), we obtain
\[
\int_0^w e^{y_1(t)} y_1'(t) \, dt = \int_0^w \left[ (a_1(t) - D_1(t))e^{y_1(t)} - a_1(t) e^{2y_1(t)} - F_1(t, t)e^{y_1(t)} + D_1(t)e^{y_2(t)} \right] \, dt
\]
and
\[
\int_0^w e^{y_2(t)} y_2'(t) \, dt = \int_0^w \left[ (a_2(t) - D_2(t))e^{y_2(t)} - a_2(t) e^{2y_2(t)} + D_2(t)e^{y_1(t)} \right] \, dt.
\]
That is,
\[
\int_0^w a_1(t)e^{2y_1(t)} \, dt + \int_0^w F_1(t, t)e^{y_1(t)} \, dt = \int_0^w (a_1(t) - D_1(t))e^{y_1(t)} \, dt + \int_0^w D_1(t)e^{y_2(t)} \, dt \tag{2.9}
\]
and
\[
\int_0^w a_2(t)e^{2y_2(t)} \, dt = \int_0^w (a_2(t) - D_2(t))e^{y_2(t)} \, dt + \int_0^w D_2(t)e^{y_1(t)} \, dt. \tag{2.10}
\]
Equation (2.9) implies that
\[
a_1 \int_0^w e^{y_1(t)} \, dt < (a_1 - D_1)^M \int_0^w e^{y_1(t)} \, dt + D_1^M \int_0^w e^{y_2(t)} \, dt,
\]
from which, using the inequality \((\int_0^w e^{y_1(t)} \, dt)^2 \leq w \int_0^w e^{2y_1(t)} \, dt\), we obtain
\[
\frac{a_1}{w} \left( \int_0^w e^{y_1(t)} \, dt \right)^2 < (a_1 - D_1)^M \int_0^w e^{y_1(t)} \, dt + D_1^M \int_0^w e^{y_2(t)} \, dt.
\]
Thus
\[
2\frac{a_1}{w} \int_0^w e^{y_1(t)} \, dt < \left[ (a_1 - D_1)^M + [(a_1 - D_1)^M]^2 + 4\frac{a_1D_1}{w} \int_0^w e^{y_2(t)} \, dt \right]^{1/2},
\]
from which, using the inequality
\[
(a + b)^{1/2} < a^{1/2} + b^{1/2}, \quad \text{for } a > 0 \text{ and } b > 0, \tag{2.11}
\]
it follows that
\[
\frac{a_{11}}{w} \int_0^w e^{y_1(t)} \, dt < (a_1 - D_1)^M + \sqrt{\frac{a_{11}^2 D_1^M}{w}} \left( \int_0^w e^{y_1(t)} \, dt \right)^{1/2}.
\] (2.12)

A similar argument to (2.12) implies from (2.10) that
\[
\frac{a_{22}}{w} \int_0^w e^{y_2(t)} \, dt < (a_2 - D_2)^M + \sqrt{\frac{a_{22}^2 D_2^M}{w}} \left( \int_0^w e^{y_1(t)} \, dt \right)^{1/2}.
\] (2.13)

Substituting (2.13) into (2.12), we obtain
\[
\frac{a_{11}}{w} \int_0^w e^{y_1(t)} \, dt < (a_1 - D_1)^M
\]
\[
+ \sqrt{\frac{a_{11}^2 D_1^M}{w}} \left[ \frac{(a_2 - D_2)^M w}{a_{22}^2} + \frac{a_{22}^2 D_2^M w}{a_{22}^2} \left( \int_0^w e^{y_1(t)} \, dt \right)^{1/2} \right]^{1/2},
\]
from which, using (2.11), it follows that
\[
\frac{a_{11}}{w} \int_0^w e^{y_1(t)} \, dt < (a_1 - D_1)^M
\]
\[
+ \sqrt{\frac{a_{11}^2 D_1^M}{a_{22}^2}} \left[ (a_2 - D_2)^M \right]^{1/2} + \sqrt{\frac{a_{22}^2 D_2^M}{w}} \left( \int_0^w e^{y_1(t)} \, dt \right)^{1/4}.
\]

Therefore there exists a positive constant \( \rho_1 \) such that
\[
\int_0^w e^{y_1(t)} \, dt < \rho_1.
\] (2.14)

Substituting (2.14) into (2.13) implies that there exists a positive constant \( \rho_2 \) such that
\[
\int_0^w e^{y_2(t)} \, dt < \rho_2.
\] (2.15)

Choose \( t_i \in [0, w], i = 1, 2, \) such that \( y_i(t_i) = \min_{t \in [0, w]} y_i(t), i = 1, 2. \) Then it is clear that \( y_i'(t_i) = 0, i = 1, 2. \) In view of this and system (2.2), we obtain
\[
(a_1(t_1) - D_1(t_1) - a_{11}(t_1)e^{y_1(t_1)} - F_1(t_1, t_1) + D_1(t_1)e^{y_2(t_1) - y_1(t_1)} = 0
\] (2.16)

and
\[
a_2(t_2) - D_2(t_2) - a_{22}(t_2)e^{y_2(t_2)} + D_2(t_2)e^{y_1(t_2) - y_2(t_2)} = 0.
\] (2.17)
Thus
\[ a_{11}^{M} e^{y_{1}(t_{1})} > a_{11}(t_{1}) e^{y_{1}(t_{1})} = a_{1}(t_{1}) - D_{1}(t_{1}) - F_{1}(t_{1}, t_{1}) + D_{1}(t_{1}) e^{y_{1}(t_{1})-y_{1}(t_{1})} > (a_{1} - D_{1})^l - a_{13}^{M} / m^l \]
and
\[ a_{22}^{M} e^{y_{2}(t_{2})} > a_{22}(t_{2}) e^{y_{2}(t_{2})} = a_{2}(t_{2}) - D_{2}(t_{2}) + D_{2}(t_{2}) e^{y_{2}(t_{2})-y_{2}(t_{2})} > (a_{2} - D_{2})^l. \quad (2.18) \]

Therefore
\[ y_{1}(t_{1}) > \ln \frac{(a_{1} - D_{1})^l - a_{13}^{M} / m^l}{a_{11}^{M}}, \quad y_{2}(t_{2}) > \ln \frac{(a_{2} - D_{2})^l}{a_{22}^{M}}. \quad (2.19) \]

Substituting (2.14), (2.15) and (2.19) into (2.6) and (2.7), we obtain
\[ \int_{0}^{w} |y'_{1}(t)| \, dt < 2(a_{1} - D_{1})w + \frac{2D_{1}^{M} \rho_{1} a_{11}^{M}}{(a_{1} - D_{1})^l - a_{13}^{M} / m^l} \triangleq d_{1} \quad (2.20) \]
and
\[ \int_{0}^{w} |y'_{2}(t)| \, dt < 2(a_{2} - D_{2})w + \frac{2D_{2}^{M} \rho_{2} a_{22}^{M}}{(a_{2} - D_{2})^l} \triangleq d_{2}. \quad (2.21) \]

Equations (2.14) and (2.15) imply that there exist two points \( \xi, \eta \in (0, w) \) such that
\[ y_{1}(\xi) < \ln(\rho_{1}/w), \quad y_{2}(\eta) < \ln(\rho_{2}/w). \quad (2.22) \]

In view of this and (2.19), we have
\[ |y_{1}(\xi)| \leq \max \left\{ |\ln \frac{\rho_{1}}{w}|, \left| \ln \frac{(a_{1} - D_{1})^l - a_{13}^{M} / m^l}{a_{11}^{M}} \right| \right\} \quad (2.23) \]
and
\[ |y_{2}(\eta)| \leq \max \left\{ |\ln \frac{\rho_{2}}{w}|, \left| \ln \frac{(a_{2} - D_{2})^l}{a_{22}^{M}} \right| \right\}. \quad (2.24) \]

Since \( \forall t \in R \)
\[ |y_{1}(t)| \leq |y_{1}(\xi)| + \int_{0}^{w} |y'_{1}(s)| \, ds \quad \text{and} \quad |y_{2}(t)| \leq |y_{2}(\eta)| + \int_{0}^{w} |y'_{2}(s)| \, ds, \]
from (2.20), (2.21) and (2.23), we obtain
\[ |y_{1}(t)| < \max \left\{ |\ln \frac{\rho_{1}}{w}|, \left| \ln \frac{(a_{1} - D_{1})^l - a_{13}^{M} / m^l}{a_{11}^{M}} \right| \right\} + d_{1} \triangleq R_{1} \]
and
\[ |y_{2}(t)| < \max \left\{ |\ln \frac{\rho_{2}}{w}|, \left| \ln \frac{(a_{2} - D_{2})^l}{a_{22}^{M}} \right| \right\} + d_{2} \triangleq R_{2}. \]
Equation (2.5) implies that there exists a point \( t^*_3 \in (0, w) \) such that

\[
F_2(t^*_3 + \tau, t^*_3 + \tau) = \alpha_3.
\]

That is, \( \alpha_3 m(t^*_3 + \tau) e^{\eta(t^*_3)} = (a_{31}(t^*_3 + \tau) - \alpha_3) e^{\eta(t^*_3)} \). Hence

\[
|y_3(t^*_3)| = \left| \ln \frac{a_{31}(t^*_3 + \tau) - \alpha_3}{m(t^*_3 + \tau) \alpha_3} \right| + |y_1(t^*_3)| < \max_{t \in [0, w]} \left| \ln \frac{a_{31}(t) - \alpha_3}{m(t) \alpha_3} \right| + R_1. \tag{2.25}
\]

Since \( \forall t \in R, |y_3(t)| \leq |y_3(t^*_3)| + \int_0^w |y'_3(s)| ds \), from this and (2.8), we obtain

\[
|y_3(t)| < \max_{t \in [0, w]} \left| \ln \frac{a_{31}(t) - \alpha_3}{m(t) \alpha_3} \right| + R_1 + 2a_3w \triangleq R_3.
\]

Clearly \( R_i \ (i = 1, 2, 3) \) are independent of \( \lambda \). Denote \( M = R_1 + R_2 + R_3 + R_0 \); here \( R_0 \) is taken sufficiently large such that

\[
2 \max \left\{ \left| \ln \delta_1 \right|, \left| \ln \frac{(a_i - D_i) - (a_{13}/m)}{a_{11}} \right|, \left| \ln \frac{(a_2 - D_2) + \sqrt{a_{22}D_2 \delta_1}}{a_{22}} \right|, \left| \ln \frac{(a_2 - D_2) - \sqrt{a_{22}D_2 \delta_1}}{a_{22}} \right| \right\} < M. \tag{2.26}
\]

Here \( \sqrt{\delta_1} \) is the only real root of the equation

\[
\sqrt{a_{22}} a_{11} x^4 = \sqrt{a_{22}} (a_i - D_i) + \sqrt{a_{11}D_1 (a_2 - D_2)} + \sqrt{a_{11}D_1 \sqrt{a_{22}D_2}} x.
\]

We now take \( \Omega = \{(y_1(t), y_2(t), y_3(t))^T \in X : \|(y_1, y_2, y_3)\| < M \} \). This satisfies condition (a) of Lemma 1.1. When \( (y_1, y_2, y_3)^T \in \partial \Omega \cap \text{Ker } L = \partial \Omega \cap R^3 \), \( (y_1, y_2, y_3)^T \) is a constant vector in \( R^3 \) with \( |y_1| + |y_2| + |y_3| = M \). We will prove that when \( (y_1, y_2, y_3)^T \in \partial \Omega \cap \text{Ker } L = \partial \Omega \cap R^3 \),

\[
QN \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} (a_1 - D_1) - a_{11} e^{y_1} - \frac{1}{w} \int_{-i}^w \frac{a_{13}(t)}{m(t)e^{y_2} + e^{y_1}} dte^{y_2} + \bar{D}_1 e^{y_2-y_1} \\ (a_2 - D_2) - a_{22} e^{y_2} + \bar{D}_1 e^{y_2-y_1} \\ -\alpha_3 + \frac{1}{w} \int_{-i}^w \frac{a_{21}(t)}{m(t)e^{y_2} + e^{y_1}} dte^{y_1} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]

If the conclusion is not true, that is, \( QN(y_1, y_2, y_3)^T = (0, 0, 0)^T \) with \( |y_1| + |y_2| + |y_3| = M \). Since

\[
(a_1 - D_1) - a_{11} e^{y_1} - \frac{1}{w} \int_{-i}^w \frac{a_{13}(t) dt}{m(t)e^{y_2} + e^{y_1}} e^{y_2} + \bar{D}_1 e^{y_2-y_1} = 0, \tag{2.27}
\]

we have \( a_{11} e^{2y_1} < (a_1 - D_1) e^{y_1} + \bar{D}_1 e^{y_2} < (a_1 - D_1) e^{y_1} + \bar{D}_1 e^{y_2} \). Thus

\[
2a_{11} e^{y_1} < (a_1 - D_1) + \sqrt{(a_1 - D_1)^2 + 4a_{11}D_1 e^{y_2}} < 2(a_1 - D_1) + 2\sqrt{a_{11}D_1} e^{y_2/2}.
\]
That is,
\[ a_{11}e^{y_1} < (a_1 - D_1) + \sqrt{a_{11}D_1} e^{y_1/2}. \]  
(2.28)

Since
\[ \frac{(a_2 - D_2) - a_{22}e^{y_2} + D_2 e^{y_2 - y_2}}{a_{22}e^{y_2}} = 0, \]  
(2.29)
we obtain \( a_{22}e^{y_2} < (a_2 - D_2)e^{y_2} + D_2 e^{y_2} \). Thus
\[ a_{22}e^{y_2} < (a_2 - D_2) + \sqrt{a_{22}D_2} e^{y_2/2}. \]  
(2.30)

From (2.28) and (2.30), it follows that
\[ e^{y_1} < \delta_1, \quad e^{y_2} < \frac{(a_2 - D_2) + \sqrt{a_{22}D_2} \delta_1}{a_{22}}. \]  
(2.31)

From (2.27) and (2.29), we obtain
\[ e^{y_1} > \frac{(a_1 - D_1) - (a_{13}/m)}{a_{11}} \quad \text{and} \quad e^{y_2} > \frac{(a_2 - D_2)}{a_{22}}. \]  
(2.32)

Hence
\[ |y_1| < \max \left\{ \ln \delta_1, \left| \ln \frac{(a_1 - D_1) - (a_{13}/m)}{a_{11}} \right| \right\} \quad \text{and} \quad |y_2| < \max \left\{ \ln \frac{(a_2 - D_2) + \sqrt{a_{22}D_2} \delta_1}{a_{22}}, \left| \ln \frac{(a_2 - D_2)}{a_{22}} \right| \right\}. \]

Since \(-a_3 + (1/w) \int_0^w (a_3(t)/(m(t)e^{y_2} + e^{y_2})) dt e^{y_3} = 0\), the same argument as that used for (2.25) gives
\[ |y_3| \leq \left| \ln \frac{a_{31} - a_3}{m^t a_3} \right| + \max \left\{ \ln \delta_1, \left| \ln \frac{(a_1 - D_1) - (a_{13}/m)}{a_{11}} \right| \right\}. \]

Therefore
\[ \sum_{i=1}^3 |y_i| \leq 2 \max \left\{ |\ln \delta_1|, \left| \ln \frac{(a_1 - D_1) - (a_{13}/m)}{a_{11}} \right| \right\} + \max \left\{ |\ln (a_2 - D_2) + \sqrt{a_{22}D_2} \delta_1|, \left| \ln \frac{(a_2 - D_2)}{a_{22}} \right| \right\} + |\ln \frac{a_{31} - a_3}{m^t a_3}| < M, \]
which contradicts the fact that $|y_1| + |y_2| + |y_3| = M$. So when $(y_1, y_2, y_3)^T \in \partial \Omega \cap \ker L = \partial \Omega \cap R^3$, $QN(y_1, y_2, y_3)^T \neq (0, 0, 0)^T$.

Finally we will prove that condition (c) of Lemma 1.1 is satisfied.

Define $\phi : \text{Dom} L \times [0, 1] \to X$ by

$$
\phi(y_1, y_2, y_3, \mu) = \begin{bmatrix}
\frac{(a_1 - D_1) - \overline{a}_{11} e^{y_1}}{(a_2 - D_2) - \overline{a}_{22} e^{y_2}} \\
- \overline{a}_3 + (1/w) \int_0^w \frac{\overline{a}_{31}(t)}{m(t)e^{y_1} + e^{y_1}} dt e^{y_1}
\end{bmatrix}
+ \mu
\begin{bmatrix}
0 \\
- (1/w) \int_0^w \frac{\overline{a}_{31}(t)}{m(t)e^{y_1} + e^{y_1}} dt e^{y_1} + \overline{D}_1 e^{y_1 - y_2}
\end{bmatrix}.
$$

When $(y_1, y_2, y_3)^T \in \partial \Omega \cap \ker L = \partial \Omega \cap R^3$, $(y_1, y_2, y_3)^T$ is a constant vector in $R^3$ with $|y_1| + |y_2| + |y_3| = M$. Using a similar argument to that for $QN(y_1, y_2, y_3)^T \neq 0$, when $(y_1, y_2, y_3)^T \in \partial \Omega \cap \ker L$, we can show that when $(y_1, y_2, y_3)^T \in \partial \Omega \cap \ker L$, $\phi(y_1, y_2, y_3, \mu) \neq (0, 0, 0)^T$. As a result, we have

$$
deg(J QN(y_1, y_2, y_3)^T, \Omega \cap \ker L, (0, 0, 0)^T)
= \deg \left( \begin{bmatrix}
(a_1 - D_1) - \overline{a}_{11} e^{y_1}, (a_2 - D_2) - \overline{a}_{22} e^{y_2}, \\
- \overline{a}_3 + \frac{1}{w} \int_0^w \frac{\overline{a}_{31}(t)}{m(t)e^{y_1} + e^{y_1}} dt e^{y_1}
\end{bmatrix}^T, \Omega \cap \ker L, (0, 0, 0)^T \right)
= \deg \left( \begin{bmatrix}
(a_1 - D_1) - \overline{a}_{11} e^{y_1}, (a_2 - D_2) - \overline{a}_{22} e^{y_2}, \\
- \overline{a}_3 + \frac{\overline{a}_{31} e^{y_1}}{m(t^*) e^{y_1} + e^{y_1}}
\end{bmatrix}^T, \Omega \cap \ker L, (0, 0, 0)^T \right),
$$

where $t^* \in [0, w]$ is a constant.

Since the system of algebraic equations

$$
\begin{align*}
(a_1 - D_1) - \overline{a}_{11} x &= 0, \\
(a_2 - D_2) - \overline{a}_{22} y &= 0, \\
- \overline{a}_3 + \overline{a}_{31} x / (m(t^*) z + x) &= 0,
\end{align*}
$$

has a unique solution $(x^*, y^*, z^*)$ which satisfies $x^* > 0$, $y^* > 0$ and $z^* > 0$, thus

$$
deg \left( \begin{bmatrix}
(a_1 - D_1) - \overline{a}_{11} e^{y_1}, (a_2 - D_2) - \overline{a}_{22} e^{y_2}, \\
- \overline{a}_3 + \frac{\overline{a}_{31} e^{y_1}}{m(t^*) e^{y_1} + e^{y_1}}
\end{bmatrix}^T, \Omega \cap \ker L, (0, 0, 0)^T \right)
$$
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\[
\text{Consequently } \deg \left( J \Omega N(y_1, y_2, y_3)^T, \Omega \cap \text{Ker } L, (0, 0, 0)^T \right) \neq 0. \text{ This completes the proof of condition (c) of Lemma 1.1.}
\]

By now we know that $\Omega$ verifies all the requirements of Lemma 1.1 and that system (2.1) has at least one $w$-periodic solution. Therefore system (2.1) has at least one positive $w$-periodic solution. This completes the proof of Theorem 2.1.

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References