Canad. J. Math. Vol. **59** (5), 2007 pp. 943–965

A Weighted L^2-Estimate of the Witten Spinor in Asymptotically Schwarzschild Manifolds

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Abstract. We derive a weighted L^2-estimate of the Witten spinor in a complete Riemannian spin manifold (M^n, g) of non-negative scalar curvature which is asymptotically Schwarzschild. The interior geometry of M enters this estimate only via the lowest eigenvalue of the square of the Dirac operator on a conformal compactification of M.

1 Introduction

Since Witten's proof of the positive mass theorem [3, 17, 20], spinors have been a valuable tool for the analysis of asymptotically flat manifolds; see for example [8, 10] or the integral estimates of Riemannian curvature [4–6]. These last estimates have the disadvantage that they involve the isoperimetric constant, which depends on the geometry in the interior (*i.e.,* away from the asymptotic end) and is therefore in most situations not known. In order to get curvature estimates which do not involve the isoperimetric constant, one needs better control of the Witten spinor. This was our motivation for looking at weighted integral norms of the Witten spinor. In order to concentrate on the role of the interior geometry, we chose the geometry in the asymptotic end as simple as possible, *i.e.,* the Schwarzschild metric. Thus the question under consideration is to what extent the unknown interior geometry can affect the behavior of the Witten spinor in the asymptotic end. In this paper, we quantify this effect by an integral inequality. We find that the effect of the interior geometry on a suitable weighted L^2-norm is described purely in terms of the lowest eigenvalue of the square of the Dirac operator on a conformal compactification of M.

To be more specific, we now describe the problem and our main results in the most familiar and physically most interesting case of dimension three. Thus let (M^3, g) be a complete Riemannian manifold of non-negative scalar curvature which is asymptotically Schwarzschild, *i.e.,* the metric in the asymptotic end is

$$g = \left(1 + \frac{m}{2r}\right)^4 g_0,$$

where g_0 is the Euclidean metric, and $r = |x|$ is the Euclidean norm of $x \in \mathbb{R}^3$. The Witten spinor is a solution of the massless Dirac equation which at infinity goes over

Received by the editors February 1, 2005.
AMS subject classification: Primary: 83C60; secondary: 35Q75, 35J45, 58J05.

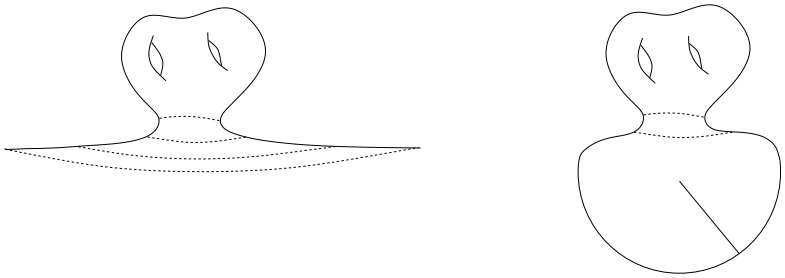

Figure 1: The asymptotically Schwarzschild manifold (M, g) and its conformal compactification $(\overline{M}, \tilde{g})$.

to a constant spinor ψ_0 with $\|\psi_0\| = 1$. More precisely, a Witten spinor ψ has the following asymptotic approximation at infinity,

$$\psi(x) = \Big(1 + \frac{m}{2r}\Big)^{-2} \psi_0 + \mathcal{O}\Big(\frac{1}{r^2}\Big). \tag{1.1}$$

This expansion is used in [17,20] for the proof of the positive mass theorem. In order to get an estimate of the error term, we point compactify the manifold with a conformal transformation of the form $\tilde{g} = \lambda^2 g$, in such a way that the geometry of K remains unchanged, the scalar curvature stays non-negative, and the compactification of the asymptotic end is isometric to a cap $C \subset S^n_\sigma$ of a sphere of radius σ (for an illustrating example see Figure 1). Then the compactification $(\overline{M}, \tilde{g})$ is a closed manifold of non-negative scalar curvature, and it is clear from the Lichnerowicz–Weitzenböck formula that the square of the Dirac operator on $\overline{M}$ is positive, $\tilde{\mathcal{D}}^2 \geq 0$. Since the scalar curvature is strictly positive in the spherical cap, we even know that the lowest eigenvalue is non-zero: $\inf \sigma(\tilde{\mathcal{D}}^2) > 0$. Moreover, using methods of spin geometry, it is possible under various geometric conditions (for example involving only scalar curvature) to bound the lowest eigenvalue of $\tilde{\mathcal{D}}^2$ from below (see [1, 2, 7, 11, 12, 14, 16]). Then the following inequality gives a detailed estimate of the Witten spinor.

Theorem 1.1 *There is a constant c independent of the geometry of K such that*

$$\int_K \|\psi\|^2 \, d\mu_M + \int_{M\setminus K} \Big\| \psi - \Big(1 + \frac{m}{2r}\Big)^{-2} \psi_0 \Big\|^2 \lambda \, d\mu_M \leq \frac{c}{\inf \sigma(\tilde{\mathcal{D}}^2)}.$$

In the course of proving this theorem, we derive an identity involving Witten spinors, which is of some interest in its own right, as we now explain. We choose an orthonormal basis $(\psi_{0,i})_{i=1,\ldots,4}$ of the spinors at infinity and consider the corresponding family ψ_i of Witten spinors,

$$\mathcal{D}\psi_i = 0, \quad \lim_{|x|\to\infty} \psi_i(x) = \psi_{0,i} \quad (i = 1, \ldots, 4). \tag{1.2}$$

https://doi.org/10.4153/CJM-2007-040-1 Published online by Cambridge University Press

We let G and $G_{S^3_\sigma}$ be the Green's functions of the square of the Dirac operator on $\overline{M}$ and the sphere S^3_σ, respectively, and denote their integral kernels by $G(x, y)$ and $G_{S^3_\sigma}(x, y)$. The next theorem expresses the weighted L^2-norms of the Witten spinors (1.2) in terms of the difference of these integral kernels, with an explicit error term.

Theorem 1.2 *For sufficiently large R, the Witten spinors satisfy the identity*

$$\begin{aligned}\int_K \sum_{i=1}^4 \|\psi_i\|^2 \, d\mu_M + \int_{M\setminus K} \sum_{i=1}^4 \Big\| \psi_i - \Big(1+\frac{m}{2r}\Big)^{-2} \psi_{0,i} \Big\|^2 \lambda \, d\mu_M \\ = 64\pi^2\sigma^4 \lim_{\mathfrak{n}\neq y\to\mathfrak{n}} \operatorname{Tr}(G(\mathfrak{n}, y) - G_{S^3_\sigma}(\mathfrak{n}, y)) \\ + 4\int_{B_R(0)} \frac{2\sigma^2}{\sigma^2+r^2}\, d^3x - 4\int_{B_R(0)\cap(M\setminus K)} \Big(1+\frac{m}{2r}\Big)^2 \lambda \, d^3x,\end{aligned}$$

where d^3x is the Lebesgue measure on $\mathbb{R}^3$. Here we assume that the asymptotic end $M\setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus B_\rho(0)$ and work in the corresponding chart.

This *equality* clearly gives finer information than the inequality of Theorem 1.1. The interesting point is that the interior geometry enters only via the Green's function G. We learn that the influence of the interior geometry on the weighted L^2-norm is described precisely by the behavior of $G(\mathfrak{n}, y)$ as $y \to \mathfrak{n}$.

In this paper, we prove the analog of Theorem 1.1 and Theorem 1.2 in general dimension. For the proof, we work as in [5] with the spinor operator, which is composed of a basis of Witten spinors (1.2). Our first step is to get a connection between the conformally transformed spinor operator and a quadratic expression in the Dirac Green's function on $\overline{M}$ (see Section 3). In Section 4 we find that after subtracting suitable counter terms, we can integrate this expression over $\overline{M}$ to obtain the Green's function G of the square of the Dirac operator minus suitable counter terms. Then our task becomes to analyze the behavior of G near the pole $\mathfrak{n}$ of the spherical cap. This is done in Section 5, where we estimate the difference of G and the corresponding Green's function on the sphere using Sobolev techniques. In Section 6, we compute the Green's functions on the sphere explicitly. Finally, in Section 7 we combine the results of Sections 3 and 4 to obtain an identity for an integral of the trace of the spinor operator (Theorem 7.1). Using a positivity argument together with the estimates of Sections 5 and 6, we then conclude our main results (Corollary 7.4, Theorem 7.5 and Corollary 7.6).

We finally remark that our methods also work similarly for harmonically flat manifolds (see [19] for the definition), except for Lemma 7.3, where an angular dependence of the function λ would lead to a "mixing" of the angular momentum modes, making the situation more complicated. Since we did not find a simple argument to overcome this problem, we here restrict our attention to a Schwarzschild end. A generalization to harmonically flat manifolds would be desirable in view of the fact that the metric of every asymptotically flat manifold can be made harmonically flat by an arbitrarily small perturbation [19].

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2 The Conformal Compactification

Let (M^n, g) be a complete Riemannian manifold of dimension $n \geq 3$ with non-negative scalar curvature. For simplicity, we assume that the manifold has one asymptotic end which is isometric to Schwarzschild. By rescaling, we can assume without loss of generality that the ADM mass is equal to two.

Definition 2.1 A complete Riemannian manifold (M^n, g) of dimension $n \geq 3$ is said to be *asymptotically Schwarzschild* if there is a parameter $\rho > 0$, a compact set $K \subset M$ and a diffeomorphism $\phi: M \setminus K \to \mathbb{R}^n \setminus \overline{B_\rho(0)}$ such that

$$\phi_* g = \left(1 + \frac{1}{|x|^{n-2}}\right)^{\frac{4}{n-2}} g_0. \tag{2.1}$$

Here $B_r(0)$ denotes an open Euclidean ball in $\mathbb{R}^n$, and g_0 is the Euclidean metric on $\mathbb{R}^n$.

In the asymptotic end it is most convenient to work in the chart $(\phi, M \setminus K)$; we use the notation

$$r(x) = |\phi(x)| \quad \text{for } x \in M \setminus K.$$

The metric (2.1) is obviously conformally equivalent to the Euclidean metric. Moreover, the Euclidean $\mathbb{R}^n$ is conformal to a sphere S^n_σ of radius σ with the north pole removed. This is is seen explicitly in the usual chart obtained by stereographic from the north pole, where

$$g_{S^n_\sigma} = \left(\frac{2\sigma^2}{\sigma^2 + r^2}\right)^2 g_0. \tag{2.2}$$

Therefore, we can arrange by a conformal transformation that the asymptotic end is isometric to the cap of a sphere of radius σ with the north pole removed. Furthermore, we want to keep the metric inside K unchanged, and we want to preserve the positivity of scalar curvature. In the next lemma we construct a function λ such that the conformal transformation

$$\tilde{g} = \lambda^2 g \tag{2.3}$$

gives us a metric with the desired properties.

Lemma 2.2 *There is a function $\lambda \in C^\infty(M)$ satisfying the following conditions:*

(i) $\lambda|_K \equiv 1$,

(ii) *For some radius $R > \rho$,*

$$\lambda(x) = \left(\frac{2\sigma^2}{\sigma^2 + r(x)^2}\right) \cdot \left(1 + \frac{1}{r(x)^{n-2}}\right)^{-\frac{2}{n-2}} \quad \text{on } \phi^{-1}(\mathbb{R}^n \setminus B_R(0)). \tag{2.4}$$

(iii) *The scalar curvature corresponding to the metric* (2.3) *is non-negative.*

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More specifically, we can arrange that

$$\rho \leq \sigma \leq R \leq c(n)(\rho + 1) \tag{2.5}$$

with a constant c that depends only on the dimension.

Proof It is more convenient to write the metric in the asymptotic end as $\tilde{g} = \mu^2 g_0$ with

$$\mu(x) = \Big(1 + \frac{1}{r(x)^{n-2}}\Big)^{\frac{2}{n-2}} \lambda(x).$$

Our first attempt is to define μ piecewise. To this end, we choose $R_* = \rho + C(n)$ with a suitable constant $C(n)$ and set

$$\mu = \mu(r(x)) = \begin{cases} \Big(1 + \dfrac{1}{r^{n-2}}\Big)^{-\frac{2}{n-2}} & \text{if } r < R_*, \\ \dfrac{2\sigma^2}{\sigma^2 + r^2} & \text{if } r \geq R_*. \end{cases}$$

In order to make this function continuous at $r = R_*$, we let

$$\sigma = R_* \Big[2\Big(1 + \frac{1}{R_*^{n-2}}\Big)^{-\frac{2}{n-2}} - 1\Big]^{-\frac{1}{2}}.$$

By choosing $C(n)$ sufficiently large, we can arrange that the square bracket is uniformly bounded from above and below. In the region $r < R_*$, the metric $\tilde{g}$ coincides with g, whereas in the region $r > R_*$, $\tilde{g}$ is the metric on S^n_σ. Obviously, in both of these regions the conformally transformed metric has all the required properties. Unfortunately, the function μ is not smooth at $r = R_*$. More precisely, a short calculation shows that the first derivative of μ makes a negative jump, *i.e.,*

$$\lim_{R_* < r \to R_*} \mu'(r) - \lim_{R_* > r \to R_*} \mu'(r) < 0.$$

As a consequence, the scalar curvature $\tilde{s}$ corresponding to $\tilde{g}$, given by the formula

$$\tilde{s} = 4\frac{n-1}{n-2} \mu^{-\frac{n+2}{2}} \Delta \mu^{\frac{n-2}{2}}$$

(where $\Delta = -\nabla^i \nabla_i$ is the Laplacian in $\mathbb{R}^n$) is positive at $r = R_*$ in the distributional sense. By mollifying $\mu(r)$ in a small neighborhood of $r = R_*$, we can thus arrange that $\tilde{s} \geq 0$. We finally set $R = \sigma + 1$. ∎

For clarity, we denote the manifold M with metric (2.3) by $(\tilde{M}, \tilde{g})$. Then by (i), the manifolds M and $\tilde{M}$ are isometric on K. By (ii), $\tilde{M}$ is isometric on $\phi^{-1}(\mathbb{R}^n \setminus B_R(0))$ to the cap of the sphere S^n_σ with the north pole $\mathfrak{n}$ removed. We denote the geodesic distance from the north pole by d,

$$d\colon S^n_\sigma \to \mathbb{R}_+, \tag{2.6}$$

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and let $B_s(\mathfrak{n})$ be the geodesic balls of radius s around the north pole. Then

$$\phi^{-1}(\mathbb{R}^n \setminus B_R(0)) \cong \big(B_\delta(\mathfrak{n}) \setminus \{\mathfrak{n}\}\big) \subset S^n_\sigma \quad \text{with} \quad \delta = 2\sigma \arctan\Big(\frac{\sigma}{R}\Big).$$

We now compactify $\tilde{M}$ by adding the north pole. The resulting manifold, denoted by $(\overline{M}, \tilde{g})$, is called the *conformal compactification* of (M, g). For $r \geq R$ we set

$$C_r = \overline{(\varphi^{-1}(\mathbb{R}^n \setminus B_r(0)), \tilde{g})} \subset \tilde{M}.$$

We also identify C_r via the isometry of the stereographic projection with a closed subset of S^n_σ. We refer to the set $C \equiv C_R$ as the *spherical cap* of $(\overline{M}, \tilde{g})$. We always identify it with the set $B_\delta(\mathfrak{n}) \subset S^n_\sigma$.

We remark that there are also conformal compactifications with the above properties (i)–(iii), for which σ is arbitarily large, thus violating (2.5). However, such conformal compactifications do not seem to give good estimates of the Witten spinor, and we shall not consider them here.

3 The Spinor Operator and the Dirac Green's Function

From now on we need to assume that $(\overline{M}, \tilde{g})$ is a *spin* manifold. This assumption is no restriction in dimension three, whereas in general it poses a constraint for the topology of $\overline{M}$. As a consequence, the manifold M is also spin. In fact, taking out the point $\mathfrak{n}$ and performing a conformal transformation, every spin structure on $\overline{M}$ induces a spin structure on M. We fix corresponding spin structures on M and $\overline{M}$ throughout.

We let Σ and $\tilde{\Sigma}$ be conformally equivalent spinor bundles over (M, g) and $(M, \tilde{g})$ and denote the corresponding Dirac operators by $\mathcal{D}$ and $\tilde{\mathcal{D}}$. According to [11, 13] there is a fiberwise isometry $\Sigma \to \tilde{\Sigma}, \psi \mapsto \tilde{\psi}$ such that

$$\tilde{\mathcal{D}}\tilde{\psi} = \lambda^{-\frac{n+1}{2}} \widetilde{(\mathcal{D}(\lambda^{\frac{n-1}{2}}\psi))}. \tag{3.1}$$

In the coordinates induced by the diffeomorphism ϕ of Definition 2.1, we choose a family of constant spinors $\psi_{0,i}$ such that $\psi_{0,1}, \ldots, \psi_{0,N}, N = 2^{[n/2]}$, is an orthonormal basis in the asymptotic end $M \setminus \varphi^{-1}(\mathbb{R}^n \setminus B_R(0))$, and we consider the boundary value problem

$$\mathcal{D}\psi_i = 0, \qquad \lim_{|x|\to\infty} \psi_i(x) = \psi_{0,i}. \tag{3.2}$$

This boundary value problem was first considered in [20]; its solutions are called *Witten spinors*. The existence and uniqueness of the Witten spinors was proven in [3, 17]. They decay at infinity as

$$\psi_i = \psi_{0,i} + \mathcal{O}(r^{2-n}), \quad \partial_j \psi_i = \mathcal{O}(r^{1-n}), \quad \partial_k \partial_l \psi_i = \mathcal{O}(r^{-n}).$$

In [5] the *spinor operator* Π_x was introduced, which we now slightly generalize, using the following.

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Notation 3.1 Let $E_1 \to X$, $E_2 \to X$ be vector bundles over the manifold X. By

$$E_1 \boxtimes E_2 \to X \times X$$

we denote the vector bundle $\pi_1^* E_1 \otimes \pi_2^* E_2$ with the projections $\pi_i \colon X \times X \to X$ to the i-th factor.

Definition 3.2 Let $\psi_1, \ldots, \psi_N$ be a family of solutions of (3.2). Then the *spinor operator* $\Pi \in \Gamma(\Sigma \boxtimes \Sigma^*)$ is defined by

$$\Pi(x, y) = \sum_{i=1}^{N} \langle \psi_i(y), \cdot \rangle \psi_i(x).$$

This definition reduces to the spinor operator as used in [5] if x and y are equal,

$$\Pi(x) := \Pi(x, x) \colon \Sigma_x \to \Sigma_x.$$

From the boundary values (3.2) it is obvious that $\lim_{|x| \to \infty} \Pi(x) = \mathbb{1}$. When considering conformal transformations of the spinor operator, we must keep in mind that the transformed Witten spinor should again be a solution of the Dirac equation. According to (3.1), this leads us to the following definition.

Definition 3.3 On the manifold $(M, \tilde{g})$ with $\tilde{g} = \lambda^2 g$ we define the *conformally transformed spinor operator* $\tilde{\Pi} \in \Gamma(\tilde{\Sigma} \boxtimes \tilde{\Sigma}^*)$ by

$$\tilde{\Pi}(x, y) := \lambda^{\frac{1-n}{2}}(x) \lambda^{\frac{1-n}{2}}(y) \sum_{i=1}^{N} \langle \tilde{\psi}_i(y), \cdot \rangle \tilde{\psi}_i(x),$$

where $\psi_1, \ldots, \psi_N$ are the solutions of (3.2).

In Euclidean space, the Witten spinors are constant, and therefore $\Pi(x) \equiv \mathbb{1}$. Applying the above definition to the metric (2.1), we obtain for the spinor operator Π_{Sch} in the Schwarzschild metric the explicit expression

$$\Pi_{\text{Sch}}(x) = \left(1 + \frac{1}{r(x)^{n-2}}\right)^{-2\frac{n-1}{n-2}} \mathbb{1}_{\Sigma_x}. \tag{3.3}$$

In the remainder of this section we will establish a connection between the conformally transformed spinor operator on $(\overline{M}, \tilde{g})$ and the Green's function of the Dirac operator.

Definition 3.4 Let X be a spin manifold and Δ the diagonal of $X \times X$,

$$\Delta = \{(x, x) \text{ with } x \in M\} \subset X \times X.$$

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We let S_X be a smooth section in the bundle $\Sigma \boxtimes \Sigma^*|((X \times X) \setminus \Delta)$,

$$S_X : X \times X \setminus \Delta \longrightarrow \Sigma_X \boxtimes \Sigma_X^*,$$

and we also consider S_X as the integral kernel of a corresponding operator acting on the compactly supported, smooth spinors on X by

$$(S_X \psi)(x) = \int_{X \setminus \{x\}} S_X(x, y) \psi(y)\, dy.$$

Then S_X is called the *Green's function of the Dirac operator* $\mathcal{D}$ on X if it satisfies the distributional equation $\mathcal{D}_{X,x} S_X(x, y) = \delta(x, y)$.

This distributional equation can be stated equivalently by the condition that

$$\int_X S_X(x, y) \mathcal{D}_x \psi(x)\, d^n x = \psi(y)$$

for all compactly supported smooth sections in the spinor bundle. As is easily verified by a direct computation, the Green's function on $\mathbb{R}^n$ with the Euclidean metric is given by

$$S_{\mathbb{R}^n}(x, y) = -\frac{1}{\omega_{n-1}} \frac{x - y}{|x - y|^n}, \tag{3.4}$$

where ω_{n-1} is the volume of S^{n-1}. The Green's functions of conformally flat spaces can be computed using the following transformation law for the Green's function under conformal changes.

Lemma 3.5 *Let (X, g) and $(\tilde{X}, \tilde{g})$ be two manifolds with conformally equivalent metrics, $\tilde{g} = \lambda^2 g$. Then the corresponding Green's functions S and $\tilde{S}$ are related by*

$$S_{\tilde{X}}(x, y) = \lambda^{\frac{1-n}{2}}(x) \lambda^{\frac{1-n}{2}}(y) S_X(x, y).$$

Proof Let S be the Green's function of the Dirac operator (X, g), *i.e.,*

$$\mathcal{D}_x \int_x S_X(x, y) \psi(y)\, dy = \psi(x).$$

Then from (3.1),

$$\tilde{\mathcal{D}}_x \lambda^{-\frac{n-1}{2}}(x) \Big(\int_{\tilde{X}} S_X(x, \tilde{y}) \psi(\tilde{y}) \lambda^{-n}(\tilde{y})\, d\tilde{y} \Big)^{\sim} = \lambda^{-\frac{n+1}{2}}(x) \tilde{\psi}(x)$$

and thus

$$\tilde{\mathcal{D}}_x \Big(\int_{\tilde{X}} \lambda^{-\frac{n-1}{2}}(x) S_X(x, \tilde{y}) \lambda^{-\frac{n-1}{2}}(\tilde{y}) \lambda^{-\frac{n+1}{2}}(\tilde{y}) \psi(\tilde{y})\, d\tilde{y} \Big)^{\sim} = \lambda^{-\frac{n+1}{2}}(x) \tilde{\psi}(x),$$

where $\tilde{\mathcal{D}}$ and $d\tilde{y}$ denote the Dirac operator and the volume element on $\tilde{X}$, respectively. ∎

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Theorem 3.6 *Let (M, g) be an asymptotically Schwarzschild manifold and $(\overline{M}, \tilde{g})$ its conformal compactification. Then the conformally transformed spinor operator $\tilde{\Pi}$ and the Green's function $\tilde{S}_{\overline{M}}$ of the Dirac operator on $(\overline{M}, \tilde{g})$ satisfy the following identity,*

$$\tilde{\Pi}(x, y) = \omega_{n-1}^2 (2\sigma^2)^{n-1} \tilde{S}_{\overline{M}}(x, \mathfrak{n}) \tilde{S}_{\overline{M}}(\mathfrak{n}, y),$$

where $\mathfrak{n} \in C \subset (\overline{M}, \tilde{g})$ is the north pole in the spherical cap of the compactification.

Proof Let $B_\varepsilon(\mathfrak{n}) \subset C$ be a ball of radius ε around $\mathfrak{n}$, $\varepsilon < \delta$. Then

$$\begin{aligned}\int_{\overline{M}\setminus B_\varepsilon(\mathfrak{n})} \tilde{\Pi}(x, y) \left(\tilde{\mathcal{D}}\, \tilde{\psi}\right)(y)\, dy &= \sum_{i=1}^{N} \int_{\overline{M}\setminus B_\varepsilon(\mathfrak{n})} \operatorname{div} V_{\tilde{\psi},\psi_i}(x)\, dx \hat{\psi}_i(y) \\ &= \sum_{i=1}^{N} \int_{\partial B_\varepsilon(\mathfrak{n})} \langle \tilde{\psi}, n_\varepsilon \cdot \hat{\psi}_i \rangle(x)\, dx \hat{\psi}_i(y),\end{aligned}$$

where $\hat{\psi}_i = \lambda^{-\frac{(n-1)}{2}} \cdot \tilde{\psi}_i$, and n_ε is the outer normal on $\partial B_\varepsilon(\mathfrak{n})$. Here the vector field V_{ψ,ψ_i} is defined by

$$\tilde{g}(V_{\psi,\psi_i}, w) = \langle \tilde{\psi}, w \cdot \tilde{\psi}_i \rangle\, \lambda^{-\frac{(n-1)}{2}}(x).$$

Similarly, we obtain

$$\begin{aligned}(3.5)\qquad &\int_{\overline{M}\setminus B_\varepsilon(\mathfrak{n})} \int_{\overline{M}\setminus B_{\varepsilon'}(\mathfrak{n})} \langle \tilde{\mathcal{D}}\tilde{\varphi}_1(x), \tilde{\Pi}(x, y) \tilde{\mathcal{D}}\tilde{\varphi}_2(y) \rangle\, dxdy \\ &\qquad = \sum_{i=1}^{N} \Big(\int_{\partial B_\varepsilon(\mathfrak{n})} \langle \tilde{\varphi}_1, n_\varepsilon \cdot \hat{\psi}_i \rangle(x)\, dx \Big) \cdot \Big(\int_{\partial B_{\varepsilon'}(\mathfrak{n})} \langle n_{\varepsilon'} \cdot \hat{\psi}_i, \tilde{\varphi}_2 \rangle(y)\, dy \Big).\end{aligned}$$

In order to investigate the limit $\varepsilon \to 0$, we consider the trivialization of the spinor bundle in a neighborhood of the north pole given by stereographic projection from the south pole $-\mathfrak{n}$. The change of the charts of $S_\sigma^n \setminus (\mathfrak{n} \cup -\mathfrak{n})$ given by the stereographic projections from the north and south pole is given by $\omega(x) = \sigma^2 x/|x|^2$ with differential $d\omega_x(v) = \frac{\sigma^2}{|x|^2} S(x/|x|)v$, where $S(x/|x|)$ denotes the reflection at $(\mathbb{R} \cdot x)^\perp$. Thus, for an orthonormal basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$, the vector fields $\tilde{e}_i \colon \mathbb{R}^n \setminus 0 \to \mathbb{R}^n$, $x \mapsto \frac{\sigma^2 + |x|^2}{2\sigma^2} S(x/|x|)e_i$ are extendable to the sphere $S_\sigma^n \setminus (-\mathfrak{n})$ as orthonormal vector fields. The lift of $S(x/|x|) \in \mathrm{O}(n)$ to $\mathrm{Pin}(n)$ is given by Clifford multiplication with

$$\frac{x}{|x|} = \frac{2\sigma^2}{\sigma^2 + |x|^2} d\rho_{\rho^{-1}(x)}(n_\varepsilon)$$

for $x \in \rho(B_\varepsilon(\mathfrak{n}))$, and therefore the Clifford products $(n_\varepsilon \cdot \tilde{\psi}_i)_{i=1,\ldots,N}$ are extendable to the north pole as orthonormal basis $\tilde{\psi}_{0,1}(\mathfrak{n}), \ldots, \tilde{\psi}_{0,2^{\lceil \frac{n}{2} \rceil}}(\mathfrak{n})$.

From the asymptotic behavior of the solutions of (3.2) it follows that

$$n_\varepsilon(x) \cdot \hat{\psi}_i(x) = (2\sigma^2)^{\frac{1-n}{2}} r'(x)^{1-n} \cdot \tilde{\psi}_{0,i}(\mathfrak{n}) + \mathcal{O}(|r'(x)|^{2-n})$$

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with $\langle \tilde{\psi}_{0,i}(\mathfrak{n}), \tilde{\psi}_{0,j}(\mathfrak{n}) \rangle = \delta_{ij}$, where $(r', \Omega)\colon S^n_\sigma \to \mathbb{R}^n$ denotes the coordinates of the stereographic projection from the south pole. Therefore,

$$\lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(\mathfrak{n})} \langle \tilde{\varphi}(x), n_\varepsilon \cdot \hat{\psi}_i(x) \rangle \, dx = \lim_{\varepsilon\to 0} \varepsilon^{1-n} (2\sigma^2)^{\frac{n-1}{2}} \operatorname{vol}(S^{n-1}_\varepsilon) \langle \tilde{\varphi}(\mathfrak{n}), \tilde{\psi}_{0,i}(\mathfrak{n}) \rangle + \mathcal{O}(\varepsilon)$$
$$= \omega_{n-1} (2\sigma^2)^{\frac{n-1}{2}} \langle \tilde{\varphi}(\mathfrak{n}), \tilde{\psi}_{0,i}(\mathfrak{n}) \rangle.$$

Now in (3.5) we can take the limit $\varepsilon \to 0$ to obtain

$$\int_{\overline{M}\times\overline{M}} \langle \tilde{\mathcal{D}} \tilde{\varphi}_1(x), \tilde{\Pi}(x,y) \tilde{\mathcal{D}} \tilde{\varphi}_2(y) \rangle \, dxdy = (2\sigma^2)^{n-1} \langle \tilde{\varphi}_1(\mathfrak{n}), \tilde{\varphi}_2(\mathfrak{n}) \rangle \cdot \omega^2_{n-1}$$
$$= (2\sigma^2)^{n-1} \int_{\overline{M}\times\overline{M}} \langle S_{\overline{M}}(\mathfrak{n}, x)\, \tilde{\mathcal{D}}\, \varphi_1(x), S_{\overline{M}}(\mathfrak{n}, y) \tilde{\mathcal{D}} \varphi_2(y) \rangle \, dxdy \cdot \omega^2_{n-1}.$$

■

4 The Green's Function of the Square of the Dirac Operator

Let us outline our strategy. Our goal is to derive weighted L^2-estimates of the Witten spinors. Since the spinor operator is composed of the Witten spinors (see Definition 3.3), the expression $\int_{\overline{M}} \operatorname{Tr} \tilde{\Pi}(x)\, dx$ is of interest, where Tr denotes the trace on the N-dimensional vector space Σ_x (for details see Section 7). Using the formula from Theorem 3.6 and the cyclicity of the trace, we are led to the integral

$$\int_{\overline{M}} \tilde{S}_{\overline{M}}(\mathfrak{n}, x) \tilde{S}_{\overline{M}}(x, \mathfrak{n})\, dx. \tag{4.1}$$

If the last argument of the second factor $\tilde{S}_{\overline{M}}$ were different from $\mathfrak{n}$, we could immediately carry out the integral,

$$\int_{\overline{M}} \tilde{S}_{\overline{M}}(\mathfrak{n}, x) \tilde{S}_{\overline{M}}(x, y)\, dx = G(\mathfrak{n}, y) \quad (y \neq \mathfrak{n}), \tag{4.2}$$

where G denotes the Green's function of the Dirac operator squared,

$$\tilde{\mathcal{D}}^2_x G(x, y) = \delta(x - y). \tag{4.3}$$

This simple argument suffers from the problem that the integrals in (4.1) and (4.2) diverge. Namely, since the order of the pole of $\tilde{S}_{\overline{M}}$ is expected to be the same as in Euclidean space (3.4), we find that the product of the Green's functions should have a non-integrable pole of the form

$$\tilde{S}_{\overline{M}}(\mathfrak{n}, x) \tilde{S}_{\overline{M}}(x, \mathfrak{n}) \sim \frac{1}{d^{2n-2}},$$

where d denotes the geodesic distance from the north pole (2.6). Despite this problem, one can hope that the above argument works if suitable functions are subtracted

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from the integrands in order to compensate for the singularities. This leads us to conjecture a relation of the following form,

$$\int_{\overline{M}} \big(\tilde{S}_{\overline{M}}(\mathfrak{n}, x)\tilde{S}_{\overline{M}}(x, \mathfrak{n}) - (\text{counter terms}) \big)\, dx = \lim_{\mathfrak{n} \neq y \to \mathfrak{n}} \big(G(\mathfrak{n}, y) - (\text{counter terms}) \big).$$

In order to specify the counter terms, we let $\chi_\delta := \chi_C$ be the characteristic function of the spherical cap. Now we take the Dirac Green's function of the sphere and multiply it by χ_δ, so that it is supported inside the cap C around the north pole. Using the isometry with the spherical cap of $\overline{M}$, we can lift S_δ to $\overline{M}$,

$$S_\delta \colon \overline{M} \times \overline{M} \setminus \Delta \to \Sigma_{\overline{M}} \boxtimes \Sigma^*_{\overline{M}} \quad \text{with } S_\delta(x, y) = \chi_\delta(x) S_{S^n_\sigma}(x, y) \chi_\delta(y).$$

Finally, we set

$$G_\delta(y) = \int_{\overline{M}} S_\delta(\mathfrak{n}, x) S_\delta(x, y)\, dx. \tag{4.4}$$

Theorem 4.1 *The Dirac Green's function $\tilde{S}$ on the compact manifold $(\overline{M}, \tilde{g})$ satisfies the relation*

$$\int_{\tilde{M}} \big(\tilde{S}_{\overline{M}}(\mathfrak{n}, x)\tilde{S}_{\overline{M}}(x, \mathfrak{n}) - S_\delta(\mathfrak{n}, x) S_\delta(x, \mathfrak{n}) \big)\, dx = \lim_{\mathfrak{n} \neq y \to \mathfrak{n}} [G(\mathfrak{n}, y) - G_\delta(y)].$$

In Section 6 the counter terms are computed more explicitly, see Lemma 6.1. The remainder of this section is devoted to the proof of the above theorem. For any $y \in B_{\delta/4}(\mathfrak{n})$, we introduce the function $f_y \colon \overline{M} \to \mathbb{R}$ by

$$f_y(x) = \tilde{S}_{\overline{M}}(x, y) - S_\delta(x, y). \tag{4.5}$$

Lemma 4.2 *There is a constant c such that*

$$\|f_y\|_\infty < c \text{ for all } y \in B_{\frac{\delta}{4}}(\mathfrak{n}) \text{ and } \lim_{y \to \mathfrak{n}} f_y(x) = f_{\mathfrak{n}}(x) \text{ uniformly in } x \in \overline{M}.$$

Proof We let $\mu \in C^\infty_0(\mathbb{R})$ be a non-negative test function $\mu \in C^\infty_0(\mathbb{R})$ with $\mu|_{[0, \frac{1}{2}]} \equiv 1$, $0 \leq \mu \leq 1$ and $\operatorname{supp} \mu \subset (-1, 1)$, and we define the function $\eta \colon S^n_\sigma \to \mathbb{R}$ by

$$\eta(x) = \mu\Big(\frac{d(x)}{\delta} \Big), \tag{4.6}$$

where d is again the geodesic distance from the north pole (2.6). This function is supported inside the spherical cap and is identically equal to one in the neighborhood $B_{\delta/2}(\mathfrak{n})$ of the north pole. For $y \in B_{\delta/4}(\mathfrak{n})$ we set $g_y(x) = \tilde{S}_{\overline{M}}(x, y) - \eta(x) S_\delta(x, y)$. Then $\tilde{\mathcal{D}} g_y(x) = -(\operatorname{grad} \eta(x)) S_{S^n_\sigma}(x, y)$. Since the function grad η is supported in the annulus $B_\delta(\mathfrak{n}) \setminus B_{\delta/2}(\mathfrak{n})$, whereas $y \in B_{\delta/4}(\mathfrak{n})$, and using that the Green's function on the sphere is smooth away from the pole $x = y$ (see Section 6 for details), we conclude that $\tilde{\mathcal{D}}\, g_y \in C^\infty(\tilde{M})$. Applying standard elliptic regularity theory, we obtain that

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$g_y \in C^\infty(\overline{M})$ and that it is uniformly bounded in y. Again using that $S_{S^n_\sigma}$ is smooth away from the diagonal, we conclude that $\tilde{S}_\delta(x, y)$ is smooth for $x \in B_\delta(\mathfrak{n}) \setminus B_{\delta/2}(\mathfrak{n})$, and is bounded uniformly in x and y. We finally note that

$$f_y(x) = g_y(x) + (1 - \eta_\varepsilon(x))\tilde{S}(x, y). \qquad \blacksquare$$

Using (4.5), we decompose the product of Green's functions as follows,

$$\begin{aligned} \tilde{S}_{\overline{M}}(\mathfrak{n}, x)\tilde{S}_{\overline{M}}(x, y) &= \tilde{S}_{\overline{M}}(x, \mathfrak{n})^* \tilde{S}_{\overline{M}}(x, y) \\ &= (S_\delta(x, \mathfrak{n}) + f_\mathfrak{n}(x))^* (S_\delta(x, y) + f_y(x)) \\ &= S_\delta(\mathfrak{n}, x) S_\delta(x, y) + f_\mathfrak{n}(x)^* S_\delta(x, y) \\ &\quad + S_\delta(x, \mathfrak{n})^* f_y(x) + f_\mathfrak{n}(x)^* f_y(x), \end{aligned} \tag{4.7}$$

where again $y \in B_{\delta/2}(\mathfrak{n})$ and $x \in \tilde{M}$. Let us analyze the x-integrals of the expressions obtained. The integral over the first term gives precisely G_δ, (4.4). For the last three expressions we can interchange the integral with the limit $y \to \mathfrak{n}$, as the next lemma shows.

Lemma 4.3 *The following limits can be taken inside the integral,*

$$\lim_{\mathfrak{n} \neq y \to \mathfrak{n}} \int_{\tilde{M}} f_\mathfrak{n}(x)^* S_\delta(x, y)\, dx = \int_{\tilde{M}} f_\mathfrak{n}(x)^* S_\delta(x, \mathfrak{n})\, dx, \tag{4.8}$$

$$\lim_{\mathfrak{n} \neq y \to \mathfrak{n}} \int_{\tilde{M}} S_\delta(x, \mathfrak{n})^* f_y(x)\, dx = \int_{\tilde{M}} S_\delta(x, \mathfrak{n}) f_\mathfrak{n}(x)\, dx, \tag{4.9}$$

$$\lim_{\mathfrak{n} \neq y \to \mathfrak{n}} \int_{\tilde{M}} f_\mathfrak{n}(x)^* f_y(x)^*\, dx = \int_{\tilde{M}} f_\mathfrak{n}(x) f_\mathfrak{n}(x)^*\, dx. \tag{4.10}$$

Proof Equations (4.9) and (4.10) follow immediately from Lebesgue's dominated convergence theorem using Lemma 4.2 and the fact that the pole of the Green's function on the sphere is integrable (see Section 6 for details). The proof of (4.8) is a bit harder, and we use the symmetry of S_{S^n} on S^n_σ. Let φ_y be an isometry on S^n_σ with $\varphi_y(y) = \mathfrak{n}$. Then since the Lebesgue integral over S^n_σ is invariant under φ,

$$\int_{\tilde{M}} f_\mathfrak{n}(x)^* S_\delta(\tilde{x}, \tilde{y})\, d\tilde{x} = \int_{S^n_\sigma} f_\mathfrak{n}(x)^* S_\delta(x, y)\, dx = \int_{S^n_\sigma} f_{\varphi_y(\mathfrak{n})}(x)^* S_\delta(x, \mathfrak{n})\, dx.$$

Now we can again apply Lebesgue's dominated convergence theorem:

$$\begin{aligned} \lim_{\mathfrak{n} \neq y \to \mathfrak{n}} \int_{\tilde{M}} f_\mathfrak{n}(x)^* S_\delta(x, y)\, dx &= \lim_{\mathfrak{n} \neq y \to \mathfrak{n}} \int_{S^n_\sigma} f_{\varphi_y(\mathfrak{n})}(x)^* S_\delta(x, \mathfrak{n})\, dx \\ &= \int_{S^n_\sigma} f_\mathfrak{n}(x)^* S_\delta(x, \mathfrak{n})\, dx = \int_{\tilde{M}} f_\mathfrak{n}(x)^* S_\delta(x, \mathfrak{n})\, dx. \end{aligned} \qquad \blacksquare$$

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We write (4.7) in the form

$$\begin{aligned}&\tilde{S}_{\overline{M}}(\mathfrak{n},x)\tilde{S}_{\overline{M}}(x,y) - S_\delta(\mathfrak{n},x)S_\delta(x,y)\\ &\quad = \quad F_y(x) := f_{\mathfrak{n}}(x)^* S_\delta(x,y) + S_\delta(x,\mathfrak{n})^* f_y(x) + f_{\mathfrak{n}}(x)^* f_y(x).\end{aligned}$$

According to Lemma 4.2 and Lemma 4.3, we may commute the integral over x with the limit $y \to \mathfrak{n}$ as follows,

$$\begin{aligned}&\int_{\tilde{M}} (\tilde{S}_{\overline{M}}(\mathfrak{n},x)\tilde{S}_{\overline{M}}(x,\mathfrak{n}) - S_\delta(\mathfrak{n},x)S_\delta(x,\mathfrak{n}))\,dx\\ &\qquad = \int_{\tilde{M}} \lim_{y\to\mathfrak{n}} F_y(x)\,dx = \lim_{y\to\mathfrak{n}} \int_{\tilde{M}} F_y(x)\,dx\\ &\qquad = \lim_{y\to\mathfrak{n}} \int_{\tilde{M}} (\tilde{S}_{\overline{M}}(\mathfrak{n},x)\tilde{S}_{\overline{M}}(x,y) - S_\delta(\mathfrak{n},x)S_\delta(x,y))\,dx.\end{aligned}$$

This concludes the proof of Theorem 4.1.

5 Pointwise Estimate of G near the Pole

In this section we shall estimate the quantity $\lim_{y\to\mathfrak{n}}(G(\mathfrak{n},y) - G_\delta(y))$ appearing in Theorem 4.1. The first difficulty is that that G_δ is defined by an integral (4.4) and is thus rather complicated; it would be more convenient to work instead with the Green's function on the sphere $G_{S^n_\sigma}(\mathfrak{n},y)$. Therefore, we introduce the function H_δ by

$$H_\delta(y) = G_{S^n_\sigma}(\mathfrak{n},y) - G_\delta(y). \tag{5.1}$$

This function depends only on the Green's functions on the sphere and can be computed explicitly (for details see Lemma 6.1 below). Thus it remains to control the difference of the Green's functions G and $G_{S^n_\sigma}$. First we need to localize the last Green's function inside the spherical cap, so that we can lift it to $\overline{M}$. To this end, we multiply it with the function η (4.6), which is supported inside the spherical cap and is identically equal to one in a neighborhood of the north pole. Then our task is to estimate the limit

$$\lim_{y\to\mathfrak{n}} \gamma(y) \quad \text{with } \gamma(y) := G(\mathfrak{n},y) - \eta(y)G_{S^n_\sigma}(\mathfrak{n},y). \tag{5.2}$$

Our strategy is as follows. When we apply the Dirac operator squared to γ, the δ-contributions cancel,

$$h(y) := \tilde{\mathcal{D}}^2\gamma(y) = \tilde{\mathcal{D}}^2(\eta(y)G_{S^n_\sigma}(\mathfrak{n},y)) - (\tilde{\mathcal{D}}^2\eta(y))G_{S^n_\sigma}(\mathfrak{n},y). \tag{5.3}$$

The resulting terms all involve derivatives of η and are thus supported in the annulus $\frac{\delta}{2} < d < \delta$, where $G_{S^n_\sigma}$ is smooth (for details see Section 6). We conclude that $h \in C_0^\infty(C)$. We consider the equation

$$\tilde{\mathcal{D}}^2\gamma = h \quad \text{on } \overline{M} \tag{5.4}$$

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as an elliptic equation for γ. Standard elliptic regularity theory yields that $\gamma \in C^\infty(\overline{M})$. Furthermore, the operator $\tilde{\mathcal{D}}^2$ is essentially self-adjoint on the Hilbert space $L^2(\overline{M})$ of square-integrable spinors with domain $C^\infty(\overline{M})$. According to the Lichnerowicz–Weitzenböck formula and the fact that the scalar curvature is non-negative on $\overline{M}$, we know that the operator $\tilde{\mathcal{D}}^2$ is strictly positive, and we obtain

$$\|\gamma\|_{L^2(\overline{M})} \leq \frac{1}{\inf \sigma(\tilde{\mathcal{D}}^2)} \|h\|_{L^2(\overline{M})}. \tag{5.5}$$

This L^2-estimate is clearly not good enough, we need a *pointwise* estimate. The general method is to derive integral estimates for the derivatives of γ and then to apply the Sobolev imbedding theorem. The Sobolev imbedding theorem on a manifold involves the isoperimetric constant (see [5,9]). The basic problem is that the isoperimetric constant on $\overline{M}$ depends on the unknown geometry in the compact set K and is therefore not under control. In order to bypass this problem, we shall always work with functions which are supported inside the spherical cap, so that we can use the Sobolev imbedding on S^n_σ. More specifically, we work with the Sobolev inequality [9]

$$|\gamma(\mathfrak{n})| \leq \sup_{S^n_\sigma} |\eta^k \gamma| \leq c_S \|\eta^k \gamma\|_{H^{k,2}(S^n_\sigma)} \quad \text{for } k > \frac{n}{2}. \tag{5.6}$$

Here c_S is the Sobolev constant on the unit sphere S^n, and the Sobolev norm $\|\cdot\|_{H^{k,2}(S^n_\sigma)}$ is defined by

$$\|f\|^2_{H^{k,2}(S^n_\sigma)} = \sum_{\kappa \text{ with } |\kappa| \leq k} \sigma^{2|\kappa|-n} \int_{S^n_\sigma} \|\nabla^\kappa f(x)\|^2 \, dx.$$

We inserted the factor $\sigma^{2|\kappa|-n}$ for convenience; it makes the Sobolev norm invariant under scalings of σ.

It remains to get estimates for the Sobolev norms in (5.6). The next lemma shows that we can equivalently consider the L^2-norms of higher powers of Dirac operator.

Lemma 5.1 *There is a constant c which depends only on n and the quotient δ/σ (but is independent of γ) such that*

$$\|\eta^{k+1}\gamma\|^2_{H^{k,2}(S^n_\sigma)} \leq c \sum_{l=1}^{k} \sigma^{2l-n} \|\eta^{l+1} \tilde{\mathcal{D}}^l \gamma\|^2_{L^2(S^n_\sigma)}. \tag{5.7}$$

Proof Since both sides of the inequality have the same scaling in σ, we may assume that $\sigma = 1$. Using the Leibniz rule and the boundedness of η and its derivatives, we immediately get

$$\|\eta^{k+1}\gamma\|^2_{H^{k,2}(S^n)} \leq c \sum_{|\kappa| \leq k} \int_{S^n_\sigma} \|\nabla^\kappa (\eta^{|\kappa|+1} f(x))\|^2 \, dx. \tag{5.8}$$

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Thus it suffices to show that the right side of (5.8) is controlled by the right side of (5.7). We proceed by induction in k. For $k = 0$, there is nothing to prove. Suppose that the inequality holds for given k. Then by the Leibniz rule and the Schwarz inequality,

$$\|\eta^{k+2}\tilde{\mathcal{D}}^{k+1}\gamma\|^2_{L^2(S^n)} = \|\tilde{\mathcal{D}}^{k+1}(\eta^{k+2}\gamma)\|^2_{L^2(S^n)} + (\text{l.o.t}),$$

where "(l.o.t.)" stands for L^2-norms of $\nabla^\kappa(\eta^{|\kappa|}\gamma)$ for lower orders $|\kappa| < k+1$. Using the induction hypothesis, we obtain

$$\sum_{l=1}^{k+1} \|\eta^{l+1}\tilde{\mathcal{D}}^l\gamma\|^2_{L^2(S^n)} \geq \frac{1}{c}\sum_{l=1}^{k+1} \|\tilde{\mathcal{D}}^l(\eta^{l+1}\gamma)\|^2_{L^2(S^n)}.$$

The Lichnerowicz–Weitzenböck formula together with the fact that the scalar curvature is non-negative imply that $\tilde{D}^2 \geq \Delta$. Hence setting $\chi = \eta^{k+1}$,

$$\langle\tilde{\mathcal{D}}^{k+1}\chi\gamma, \tilde{\mathcal{D}}^{k+1}\chi\gamma\rangle_{L^2(S^n)} = \langle\tilde{\mathcal{D}}^{2k+2}\chi\gamma, \chi\gamma\rangle_{L^2(S^n)} \geq \langle\Delta^{k+1}\chi\gamma, \chi\gamma\rangle_{L^2(S^n)}.$$

Integrating by parts, for each Laplacian we bring one derivative on each side of the scalar product. Commuting covariant derivatives we pick up curvature terms which are clearly bounded on S^n. We thus obtain

$$\langle\Delta^{k+1}\chi\gamma, \chi\gamma\rangle_{L^2(S^n)} = \sum_{|\kappa|=k+1} \langle\nabla^\kappa\chi\gamma, \nabla^\kappa\chi\gamma\rangle_{L^2(S^n)} + (\text{l.o.t.}).$$

Again using the induction hypothesis, the result follows. ∎

The L^2-norms on the right side of (5.7) can easily be estimated similar to (5.5),

$$\|\eta^l\tilde{\mathcal{D}}^l\gamma\|^2_{L^2(S^n_\sigma)} \leq \|\tilde{\mathcal{D}}^l\gamma\|^2_{L^2(\overline{M})} \leq \frac{1}{\inf\sigma(\tilde{\mathcal{D}}^2)}\|\tilde{\mathcal{D}}^{l+2}\gamma\|^2_{L^2(\overline{M})} = \frac{1}{\inf\sigma(\tilde{\mathcal{D}}^2)}\|\tilde{\mathcal{D}}^l h\|^2_{L^2(S^n_\sigma)}.$$

Notice that the norm on the very right depends only on the geometry of the spherical cap. Combining this last estimate with (5.6) and Lemma 5.1, we obtain the following result.

Corollary 5.2 There is a constant c depending only on n and δ/σ such that

$$\lim_{y\to\mathfrak{n}} \gamma(y) \leq \frac{c\sigma^{-n}}{\inf\sigma(\tilde{\mathcal{D}}^2)}.$$

Proof Putting together the above estimates, we obtain

$$\lim_{y\to\mathfrak{n}} \gamma(y)^2 = \gamma(\mathfrak{n})^2 \leq c\sum_{l=1}^{k} \sigma^{2l-n}\|\eta^{l+1}\tilde{\mathcal{D}}^l\gamma\|^2_{L^2(S^n_\sigma)} \tag{5.9}$$

$$\leq \frac{c\sigma^{-2n}}{\inf\sigma(\tilde{\mathcal{D}}^2)}\sum_{l=1}^{k} \sigma^{2l+n}\|\tilde{\mathcal{D}}^l h\|^2_{L^2(S^n_\sigma)}.$$

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Since h is a given function on the spherical cap, the summands in the last sum can be bounded by a constant depending only on n, δ and σ. In order to determine the scaling in σ, we first note that

$$\tilde{\mathcal{D}}_x^2 G_{S^n_\sigma}(x,y) = \delta_{S^n_\sigma}(x,y) = \sigma^{-n}\delta_{S^n_1}(\sigma^{-1}x,\sigma^{-1}y) = \sigma^{-n}\tilde{\mathcal{D}}_x^2 G_{S^n_1}(\sigma^{-1}x,\sigma^{-1}y),$$

and thus

$$G_{S^n_\sigma}(x,y) = \sigma^{-n+2} G_{S^n_1}(\sigma^{-1}x,\sigma^{-1}y).$$

Using this in (5.3), one sees that h scales like σ^{-n}. We conclude that the last sum in (5.9) is scaling invariant, and therefore this sum can be bounded by a constant which depends only on the quotient δ/σ. ∎

6 The Green's Functions on the Sphere

In the previous constructions we used the Green's functions on the sphere $S_{S^n_\sigma}$ and $G_{S^n_\sigma}$. We shall now compute these Green's functions and estimate the composite expressions $S_\delta(x,\mathfrak{n})S_\delta(\mathfrak{n},x)$ and $G_\delta - G_{S^n_\sigma}$.

Under the conformal transformation of $S^n_\sigma\backslash\{\mathfrak{n}\}$ to Euclidean $\mathbb{R}^n$, the Green's function $S_{S^n_\sigma}$ clearly goes over to the Green's function of Euclidean space (3.4). Applying Lemma 3.5, we thus obtain an explicit formula for the Green's function on S^n_σ in the coordinates of the stereographic projection from the north pole:

$$S_{S^n_\sigma}(x,y) = -\frac{1}{\omega_{n-1}}\frac{x-y}{|x-y|^n}\Big(\frac{2\sigma^2}{\sigma^2+|x|^2}\Big)^{\frac{1-n}{2}}\Big(\frac{2\sigma^2}{\sigma^2+|y|^2}\Big)^{\frac{1-n}{2}}. \tag{6.1}$$

In particular, one sees that the Green's function on the sphere is smooth away from the diagonal, and that the pole at $x = y$ is integrable.

In the next lemma we compute the product $S_\delta(x,\mathfrak{n})S_\delta(\mathfrak{n},x)$ as well as G_δ and H_δ defined by (4.4), (5.1). It is most convenient to work on S^n_σ in the coordinate system obtained by stereographic projection from the south pole. The corresponding radial coordinate r' is related to the radial coordinate r in the stereographic projection from the north pole by

$$r' = \frac{\sigma^2}{r}. \tag{6.2}$$

Lemma 6.1 *Setting $R' = \sigma^2/R$, the following identities hold inside the spherical cap C,*

$$S_\delta(x,\mathfrak{n})S_\delta(\mathfrak{n},x) = \frac{(2r'^2)^{1-n}}{\omega_{n-1}^2}\Big(\frac{2\sigma^2}{\sigma^2+r'^2}\Big)^{1-n}\mathbb{1}, \tag{6.3}$$

$$G_\delta(x) = \frac{1}{\omega_{n-1}}\Big(\frac{4\sigma^2}{\sigma^2+r'^2}\Big)^{\frac{1-n}{2}}\int_{r'}^{R'}\frac{2\sigma^2}{\sigma^2+\tau^2}\frac{1}{\tau^{n-1}}d\tau, \tag{6.4}$$

$$H_\delta(x) = \frac{1}{\omega_{n-1}}\Big(\frac{4\sigma^2}{\sigma^2+r'^2}\Big)^{\frac{1-n}{2}}\int_{R'}^{\infty}\frac{2\sigma^2}{\sigma^2+\tau^2}\frac{1}{\tau^{n-1}}\,d\tau. \tag{6.5}$$

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Proof The identity (6.3) can be obtained in two ways. Either one computes the product $S_\delta(x,y)S_\delta(y,x)$ with S_δ in the "north pole chart" (6.1) and takes the limit $y \to \infty$, or, alternatively, one can compute the product $S_\delta(x,\mathfrak{n})S_\delta(\mathfrak{n},x)$ in the chart of the stereographic projection from the south pole.

For the rest of the proof we work with the stereographic projection from the south pole, which we denote by $\pi\colon S^n_c \to \mathbb{R}^n$. Using the explicit formulas (3.4), (6.1), (2.2), we obtain

$$\begin{aligned} G_\delta(y) &= \int_{S^n_\sigma} S_\delta(\mathfrak{n},x)S_\delta(x,y)\,dx = \int_{B_\delta(\mathfrak{n})} S_{S^n_\sigma}(\mathfrak{n},x)S_{S^n_\sigma}(x,y)\,dx \\ &= \int_{\pi(B_\delta(\mathfrak{n}))} S_{\mathbb{R}^n}(0,x)S_{\mathbb{R}^n}(x,y)\frac{2\sigma^2}{\sigma^2+|x|^2}\Big(\frac{4\sigma^2}{\sigma^2+|y|^2}\Big)^{\frac{1-n}{2}}\,dx \\ &= \Big(\frac{4\sigma^2}{\sigma^2+|y|^2}\Big)^{\frac{1-n}{2}} \int_{\pi(B_\delta(\mathfrak{n}))} (\mathcal{D}_{\mathbb{R}^n,x}F(x))S_{\mathbb{R}^n}(x,y)\,dx \end{aligned}$$

with

$$F(x) := \frac{1}{\omega_{n-1}}\int_{r'(x)}^{R'} \frac{2\sigma^2}{\sigma^2+\tau^2}\frac{1}{\tau^{n-1}}\,d\tau. \tag{6.6}$$

We now integrate by parts. The boundary terms drop out because F vanishes on $\partial B_\delta(\mathfrak{n})$ (note that the pole at the origin is of order $(r')^{n-2}$, and so we do not need to worry about boundary terms there). We conclude that

$$G_\delta(y) = \Big(\frac{4\sigma^2}{\sigma^2+|y|^2}\Big)^{\frac{1-n}{2}} \int_{B_\delta(\mathfrak{n})} F(x)\mathcal{D}_{\mathbb{R}^n_x}S_{\mathbb{R}^n}(x,y)\,dx = \Big(\frac{2\sigma^2}{1+|y|^2}\Big)^{\frac{1-n}{2}}F(y),$$

proving (6.4); H_δ can be computed similarly,

$$\begin{aligned} H_\delta(y) &= \int_{S^n_\sigma\setminus B_\delta(\mathfrak{n})} S_{S^n_\sigma}(\mathfrak{n},x)S_{S^n_\sigma}(x,y)\,dx \\ &= \Big(\frac{4\sigma^2}{\sigma^2+|y|^2}\Big)^{\frac{1-n}{2}} \int_{\mathbb{R}^n\setminus\pi(B_\delta(\mathfrak{n}))} (\mathcal{D}_{\mathbb{R}^n,x}F(x))S_{\mathbb{R}^n}(x,y)\,dx. \end{aligned}$$

Now we integrate by parts. Since $\mathcal{D}_{\mathbb{R}^n,x}S_{\mathbb{R}^n}(x,y)$ gives a contribution only for $x = y \notin \mathbb{R}^n \setminus \pi(B_\delta(\mathfrak{n}))$, only the boundary integrals contribute. The boundary terms on $\partial(\pi(B_\delta(\mathfrak{n})))$ again drop out because F vanishes there. Thus it remains to consider the boundary terms at infinity,

$$H_\delta(y) = \Big(\frac{4\sigma^2}{\sigma^2+|y|^2}\Big)^{\frac{1-n}{2}} \lim_{L\to\infty}\int_{S^{n-1}_L} \nu\cdot F(x)S_{\mathbb{R}^n}(x,y)\,dx,$$

where $\nu\cdot$ denotes Clifford multiplication with the outer normal. Putting in (3.4), (6.6), we obtain (6.5). ∎

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Substituting (6.4), (6.5) into (5.1), we also obtain an explicit expression for the Green's function $G_{S^n_\sigma}$,

$$G_{S^n_\sigma}(\mathfrak{n}, x) = \frac{1}{\omega_{n-1}} \Big(\frac{4\sigma^2}{\sigma^2 + r'^2} \Big)^{\frac{1-n}{2}} \int_{r'}^{\infty} \frac{2\sigma^2}{\sigma^2 + \tau^2} \frac{1}{\tau^{n-1}}\, d\tau.$$

This formula shows that $G_{S^n_\sigma}(x, y)$ is indeed smooth away from the diagonal.

7 A Weighted L^1-Estimate of the Deviation Operator

In this section, we first combine the previous results to derive an integral estimate for the trace of the spinor operator (Theorem 7.1). We then introduce the so-called deviation operator, which gives us information on how much the spinor operator differs in the asymptotic end from the spinor operator in Schwarzschild. Using a positivity argument (Lemma 7.3), we can then prove the main results of this paper: a weighted L^1-estimate of the deviation operator (Theorem 7.5) and a weighted L^2-estimate of the Witten spinors (Theorem 7.6).

Let μ be a function which in the asymptotic end coincides with the norm of the spinor operator in Schwarzschild (3.3) and vanishes otherwise,

$$\mu\colon M \to \mathbb{R}, \quad \mu(x) = \chi_{M\setminus K}(x) \Big(1 + \frac{1}{r(x)^{n-2}} \Big)^{-2\frac{n-1}{n-2}}. \tag{7.1}$$

In the next Theorem we compute an integral involving the trace of the matrix $\Pi - \mu \mathbb{1}$.

Theorem 7.1 *The spinor operator on M satisfies the following identity,*

$$\int_M \mathrm{Tr}(\Pi(x) - \mu(x)\mathbb{1})\lambda(x)\, dx = \omega_{n-1}^2 (2\sigma^2)^{n-1} \lim_{y \to \mathfrak{n}} \mathrm{Tr}(\gamma(y)) + N\alpha, \tag{7.2}$$

with γ according to (5.2)*, and where α is given by explicit integrals in Euclidean space,*

$$\alpha = \int_{B_R(0)} \frac{2\sigma^2}{\sigma^2 + |x|^2}\, d^n x - \int_{B_R(0)\setminus\overline{B_\rho(0)}} \Big(1 + \frac{1}{|x|^{n-2}} \Big)^{\frac{2}{n-2}} \lambda(\phi^{-1}(x))\, d^n x. \tag{7.3}$$

Here $d^n x$ is the Lebesgue measure in $\mathbb{R}^n$, and λ is to be chosen as in Lemma 2.2.

We point out that the parameters γ and α clearly depend on the conformal compactification, but they are (for a given function λ) independent of R. This is obvious for γ because its definition (5.2) involves only the Green's functions on $\overline{M}$ and S^n_σ. For the parameter α it follows from the fact that for $r > R$, λ is given by (2.4), so that the second integrand in (7.3) reduces to the first. Hence the integrals in (7.3) remain unchanged if R is increased.

https://doi.org/10.4153/CJM-2007-040-1 Published online by Cambridge University Press

Proof of Theorem 7.1 Using Definition 3.3 and Theorem 3.6, we get

$$\begin{aligned}
&\int_M \operatorname{Tr}(\Pi(x) - \mu(x)\mathbb{1})\lambda(x)\,dx \\
&\quad= \int_{\tilde{M}} \operatorname{Tr}\big(\tilde{\Pi}(x) - \mu(x)\lambda^{1-n}(x)\mathbb{1}\big)\,dx \\
&\quad= \int_{\tilde{M}} \operatorname{Tr}\big(\omega_{n-1}^2(2\sigma^2)^{n-1}\tilde{S}_{\overline{M}}(x,\mathfrak{n})\tilde{S}_{\overline{M}}(\mathfrak{n},x) - \mu(x)\lambda^{1-n}(x)\mathbb{1}\big)\,dx \\
&\quad= \omega_{n-1}^2(2\sigma^2)^{n-1}\int_{\tilde{M}} \operatorname{Tr}\big(\tilde{S}_{\overline{M}}(\mathfrak{n},x)\tilde{S}_{\overline{M}}(x,\mathfrak{n}) - S_\delta(\mathfrak{n},x)S_\delta(x,\mathfrak{n})\big)\,dx \\
&\qquad+ \int_{\tilde{M}} \operatorname{Tr}\big(\omega_{n-1}^2(2\sigma^2)^{n-1}S_\delta(x,\mathfrak{n})S_\delta(\mathfrak{n},x) - \mu(x)\lambda^{1-n}(x)\mathbb{1}\big)\,dx.
\end{aligned}$$

According to (2.4), (7.1) and (6.3), (6.2),

$$\omega_{n-1}^2(2\sigma^2)^{n-1}S_\delta(x,\mathfrak{n})S_\delta(\mathfrak{n},x) - \mu(x)\lambda^{1-n}(x) \equiv 0 \quad \text{on } C,$$

and thus the last integral reduces to an integral over the annular region $\overline{M}\setminus(K\cup C)$. Furthermore, we can apply Theorem 4.1 as well as (5.1), (5.2) to obtain

$$\begin{aligned}
\int_M \operatorname{Tr}\big(\Pi(x) - \mu(x)\mathbb{1}\big)\,\lambda(x)\,dx &= \omega_{n-1}^2(2\sigma^2)^{n-1}\lim_{y\to\mathfrak{n}}\operatorname{Tr}(\gamma(y)) \\
&\quad+ \omega_{n-1}^2(2\sigma^2)^{n-1}\operatorname{Tr}(H_\delta(\mathfrak{n})) - \int_{M\setminus(K\cup C)} \operatorname{Tr}\big(\mu(x)\lambda(x)\mathbb{1}\big)\,dx.
\end{aligned}$$

The last two terms can be computed explicitly with (6.5) and (2.1),

$$\begin{aligned}
\omega_{n-1}^2(2\sigma^2)^{n-1}H_\delta(\mathfrak{n}) &= \omega_{n-1}\sigma^{2(n-1)}\int_{R'}^{\infty}\frac{2\sigma^2}{\sigma^2+\tau^2}\tau^{n-1}\,d\tau \\
&= \omega_{n-1}\int_0^R \frac{2\sigma^2}{\sigma^2+r^2}r^{n-1}\,dr = \int_{B_R(0)}\frac{2\sigma^2}{\sigma^2+|x|^2}\,d^nx, \\
\int_{M\setminus(K\cup C)}\mu(x)\lambda(x)\,dx &= \int_{B_R(0)\setminus\overline{B_\rho(0)}}\Big(1+\frac{1}{|x|^{n-2}}\Big)^{\frac{2}{n-2}}\lambda(\phi^{-1}(x))\,d^nx.
\end{aligned}$$

■

We now analyze the integral in (7.2) in more detail, with the aim of getting a connection to an L^1-norm. Inside the compact set K, the function λ vanishes. Thus according to Definition 3.2, for every spinor $\psi\in\Sigma_x$,

$$\langle\psi,(\Pi(x)-\mu(x)\mathbb{1})\psi\rangle = \langle\psi,\Pi(x)\psi\rangle = \sum_{i=1}^{N}|\langle\psi_i(x),\psi\rangle|^2 \geq 0. \tag{7.4}$$

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Hence the matrix $\Pi - \mu\mathbb{1}$ is positive, and we can control the sup-norm by the trace,

$$\|\Pi(x) - \mu(x)\mathbb{1}\| \leq \mathrm{Tr}(\Pi(x) - \mu(x)\mathbb{1}) \quad \text{for all } x \in K. \tag{7.5}$$

In the asymptotic end, where $\mu > 0$, we cannot expect that the operator $\Pi - \mu\mathbb{1}$ is still positive. Nevertheless, the next lemma shows that the integral over the trace is indeed positive and can be identified with the trace of a positive operator, which we call the deviation operator.

Definition 7.2 Working in the asymptotic end in the chart $(\phi, M\setminus K)$, we introduce for every solution ψ_i of the boundary value problem (3.2) the *Witten deviation* $\delta\psi_i$ by

$$\delta\psi_i = \psi_i - \left(1 + \frac{1}{r(x)^{n-2}}\right)^{-\frac{n-1}{n-2}} \psi_{0,i}.$$

For every $x \in M \setminus K$, the *deviation operator* $\delta\Pi$ is defined by

$$\delta\Pi(x)\colon \Sigma_x \to \Sigma_x : \psi \mapsto \sum_{i=1}^{N} \langle \delta\psi_i(x), \psi\rangle \delta\psi_i(x).$$

Thus the deviation operator is defined similarly to the spinor operator; one only replaces the Witten spinors by the corresponding Witten deviations. Repeating the argument in (7.4), one sees that the deviation operator is also positive.

Lemma 7.3

$$\int_{M\setminus K} \mathrm{Tr}(\Pi(x) - \mu(x)\mathbb{1})\lambda(x)\, dx = \int_{M\setminus K} \mathrm{Tr}(\delta\Pi)\lambda(x)\, dx.$$

Proof We work in the chart $(\phi, M \setminus K)$ and in $\phi(M \setminus K) = \mathbb{R}^n \setminus \overline{B_\rho(0)}$ choose polar coordinates (r, ω) with $\omega \in S^{n-1}$. According to the behavior of the Dirac operator under conformal transformations (3.1), every Witten spinor ψ can be written in the form

$$\psi(x) = \left(1 + \frac{1}{r(x)^{n-2}}\right)^{-\frac{n-1}{n-2}} \Psi(x)$$

with Ψ a harmonic spinor on $\mathbb{R}^n \setminus \overline{B_\rho(0)}$ endowed with the Euclidean metric. We now expand Ψ in partial waves,

$$\Psi(r, \omega) = \Psi_0 + \sum_{l=1}^{\infty} \Psi^l(\omega) \frac{1}{r^{l+n-2}},$$

where l are the angular quantum numbers, and the $\Psi^l(\omega)$ are linear combinations of the corresponding spin-weighted spherical harmonics. From the smoothness of ψ and the asymptotics at infinity, it is clear that the sum converges in $L^2\big(\mathbb{R}^n \setminus \overline{B_\rho(0)}\big)^N$.

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Consequently, the Witten spinors and the Witten deviations have the following partial wave expansions,

$$\psi_i(r,\omega) = \left(1+\frac{1}{r(x)^{n-2}}\right)^{-\frac{n-1}{n-2}} \left(\psi_{0,i} + \sum_{l=1}^{\infty} \Psi_i^l(\omega)\frac{1}{r^{l+n-2}}\right),$$

$$\delta\psi_i(r,\omega) = \left(1+\frac{1}{r(x)^{n-2}}\right)^{-\frac{n-1}{n-2}} \sum_{l=1}^{\infty} \Psi_i^l(\omega)\frac{1}{r^{l+n-2}}.$$

Using that the spin-weighted spherical harmonics for different l are orthogonal on $L^2(S^{n-1})$, a short calculation shows that for all $r > R$,

$$\int_{S^{n-1}} \mathrm{Tr}(\Pi(x) - \mu(x)\mathbb{1})(r,\omega)d\omega = \int_{S^{n-1}} \mathrm{Tr}(\delta\Pi)(r,\omega)\,d\omega. \qquad \blacksquare$$

Combining Theorem 7.1 with the above lemma, we immediately obtain the following identity for the weighted L^1-norm of the spinor operator and the deviation operator.

Corollary 7.4 The spinor operator on M satisfies the following identity,

$$\int_K \mathrm{Tr}(\Pi(x))\,dx + \int_{M\setminus K} \mathrm{Tr}(\delta\Pi(x))\lambda(x)\,dx = \omega_{n-1}^2(2\sigma^2)^{n-1} \lim_{y\to\mathfrak{n}} \mathrm{Tr}(\gamma(y)) + N\alpha,$$

where γ and α are given by (5.2) and (7.3).

Putting in the estimate of Corollary 5.2 gives the following result.

Theorem 7.5 *There is a constant c depending only on the dimension such that*

$$\int_K \|\Pi(x)\|\,dx + \int_{M\setminus K} \|\delta\Pi(x)\|\lambda(x)\,dx \le c\frac{(\rho+1)^n}{\sigma^2 \inf(\tilde{\mathcal{D}}^2)}, \tag{7.6}$$

where λ is to be chosen according to Lemma 2.2.

Proof According to Lemma 7.3 and the positivity of $\Pi(x)$ and $\delta\Pi(x)$, we know that

$$\int_K \|\Pi(x)\|\,dx + \int_{M\setminus K} \|\delta\Pi(x)\|\lambda(x)\,dx \le \int_M \mathrm{Tr}(\Pi(x) - \mu(x)\mathbb{1})\lambda(x)\,dx.$$

We now apply Corollary 5.2 and use that, according to (2.5), σ scales like the radius ρ. Furthermore, it is obvious from (7.3), (2.5) that α scales like ρ^n. This gives the estimate

$$\int_K \|\Pi(x)\|\,dx + \int_{M\setminus K} \|\delta\Pi(x)\|\lambda(x)\,dx \le \frac{c(\rho+1)^n}{\sigma^2 \inf(\tilde{\mathcal{D}}^2)} + c(\rho+1)^n.$$

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It remains to bound the lowest eigenvalue of $\tilde{\mathcal{D}}^2$ from above. To this end, we choose a smooth wave function ψ which is supported in the spherical cap, and consider its Rayleigh quotient,

$$\inf \sigma(\tilde{\mathcal{D}}^2) \leq \frac{\langle \tilde{\mathcal{D}}\psi, \tilde{\mathcal{D}}\psi \rangle_{L^2(\overline{M})}}{\langle \psi, \psi \rangle_{L^2(\overline{M})}} \leq \frac{c}{\sigma^2}. \qquad \blacksquare$$

From this theorem one obtains weighted L^2-estimates for all Witten spinors.

Corollary 7.6 There is a constant c depending only on the dimension such that every Witten spinor ψ satisfies the weighted L^2-estimate

$$\int_K \|\psi(x)\|^2 \, dx + \int_{M \setminus K} \|\delta\psi(x)\|^2 \lambda(x) \, dx \leq c \frac{(\rho+1)^n}{\sigma^2 \inf(\tilde{\mathcal{D}}^2)}$$

with λ according to Lemma 2.2.

Proof We choose a basis of Witten spinors $\psi_1, \ldots, \psi_n$ such that $\psi_1 = \psi$. Then for all $\phi \in \Sigma_x$,

$$\langle \phi, \Pi(x)\phi \rangle = \sum_{i=1}^{N} |\langle \psi_i(x), \phi \rangle|^2 \geq |\langle \psi(x), \phi \rangle|^2.$$

Taking the supremum over all unit spinors ϕ, we conclude that $\|\psi(x)\|^2 \leq \|\Pi(x)\|$. In the same way, one sees that $\|\delta\psi(x)\|^2 \leq \|\delta\Pi(x)\|$. $\blacksquare$

Theorem 1.1 and Theorem 1.2 are special cases of Corollary 7.6 and Corollary 7.4, respectively.

Acknowledgments We would like to thank the Erwin-Schrödinger-Institut, Wien, and the Centre de Recherches Mathématiques, Montréal, for hospitality and support.

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