Proceedings of the Edinburgh Mathematical Society (1987) **30**, 201–205 ©

ON THE AUTOMORPHISMS OF THE GROUP RING OF A UNIQUE PRODUCT GROUP

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(Received 26th June 1985)

1. Introduction

Let R be a ring with an identity and a nilpotent ideal N. Let G be a group and let $R(G)$ be the group ring of G over R. The aim of this paper is to study the relationships between the automorphisms of G and R-linear automorphisms of $R(G)$ which either preserve the augmentation or do so modulo the ideal N. We shall show, for example, that if G is a unique product group ([**6**], Chapter 13, Section 1) then every automorphism of $R(G)$ is modulo N induced from some automorphism of G. This result, which is immediate if, for instance, R is an integral domain, is here requiring of proof since $R(G)$ has non-trivial units (e.g. if $N \neq 0$, $1+n(g-h)$, $\forall n \in N$, $\forall g$, $h \in G$ is a unit of augmentation 1), the existence of which is responsible for some of the difficulties inherent in the present investigation. We are obliged to the referee for several helpful suggestions and, in particular, for the proof of Lemma 2.2 whose use obviates our previous combinatorial arguments.

Every automorphism of G extends naturally and R-linearly to an augmentation-preserving automorphism of $R(G)$ but, in general, it cannot be expected that every such automorphism is so induced. We should wish to restrict our attention to augmentation-preserving automorphisms but it is here more convenient to widen considerations to include automorphisms which are augmentation-preserving modulo N. If we seem to labour the point it is because the matter is one of choice and the particular paper [**5**] that motivated our work does not invoke the augmentation. Henceforth we shall understand that an automorphism θ of $R(G)$ is also R-linear (i.e. $\theta(\alpha g+\beta h)=\alpha\theta(g)+\beta\theta(h)$, $\forall g$, $h \in G$, $\forall \alpha$, $\beta \in R$) and if $\varepsilon: R(G) \to R$ is the augmentation map given by $\varepsilon(\sum_{g \in G}\alpha_g g)=\sum_{g \in G}\alpha_g$ then θ will be said to preserve the augmentation (of $R(G)$) if $\varepsilon(\theta(x))=\varepsilon(x)$, $\forall x \in R(G)$, and to be augmentation-preserving modulo N if $\varepsilon(\theta(x))-\varepsilon(x) \in N$, $\forall x \in R(G)$. If θ is an automorphism of G then we denote its natural extension to an automorphism of $R(G)$ also by θ. We note that $N(G)=\{\sum \alpha_g g: \alpha_g \in N\}$ is a nilpotent ideal of $R(G)$ and that $R(G)/N(G)$ is isomorphic to $(R/N)(G)$. We shall speak of reducing $R(G)$ modulo N when we replace an expression in $R(G)$ by the corresponding expression in $(R/N)(G)$ obtained by reducing each coefficient in the expression by its residue class modulo N. If θ is an automorphism of $R(G)$ which is augmentation-preserving modulo N then [Theorem 3.2] θ induces canonically a mapping $\tilde{\theta}:(R/N)(G) \to (R/N)(G)$ and $\tilde{\theta}$ is an automorphism of $(R/N)(G)$ which is, in an obvious sense, augmentation-preserving on $(R/N)(G)$.

2. Preliminary lemmas

For completeness we begin with a lemma which is part of the folklore but for which there appears to be no precise reference (see [7], p. 63).

Lemma 2.1. *Let $\theta:R(G)\to R(G)$ be an automorphism of $R(G)$. Define θ^* by*

$$\theta^*(g)=\frac{\theta(g)}{\varepsilon(\theta(g))} \qquad (\forall g\in G),$$

where ε is the augmentation map, and extend θ^* linearly to $R(G)$. Then θ^* is an augmentation-preserving automorphism of $R(G)$.

Proof. It is obvious that $\varepsilon(\theta^*(g))=1$ $(\forall g\in G)$ and the remaining assertions are easily verified. □

The following lemma is used to establish the subjectivity of a particular endomorphism.

Lemma 2.2. *Let T be a ring, let M be a nilpotent ideal of T and let S be a subring of T. Suppose that $T=S+M$ and that $(S\cap M)T=M$. Then $S=T$.*

Proof. We have

$$\begin{aligned}M&=(S\cap M)T=(S\cap M)(S+M)\\&=(S\cap M)S+(S\cap M)M\\&=(S\cap M)+(S\cap M)M\end{aligned}$$

By induction on $i\geqq 1$ we show that

$$M=(S\cap M)+(S\cap M)^iM$$

and we have established this result for $i=1$. We now assume the truth for i and deduce the result for $i+1$ as follows. We have

$$\begin{aligned}M&=(S\cap M)+(S\cap M)^i[(S\cap M)+(S\cap M)M]\\&=(S\cap M)+(S\cap M)^{i+1}+(S\cap M)^{i+1}M\\&=(S\cap M)+(S\cap M)^{i+1}M,\end{aligned}$$

which gives the result. But $S\cap M$ is nilpotent and so we conclude that $M=S\cap M$. Hence $M\subseteq S$ and thus $T=S+M=S$. □

3. Main results

We recall that R is called a domain if R has no proper divisors of zero (as always R has an identity but commutativity is not assumed). For convenience we make the following definition.

Definition. Let R be a domain. Let $\mathscr{C}(R)$ denote the class of all groups G such that $R(G)$ has no non-trivial units and no proper divisors of zero.

Certainly if $G \in \mathscr{C}(R)$ then G is torsion-free and well-known facts imply that $\mathscr{C}(R)$ contains all free and free-abelian groups. If K is a field of characteristic zero then $\mathscr{C}(K)$ contains all unique product groups ([**6**], p. 591); in passing we remark that every unique product group is a two unique product group and conversely ([**8**], Theorem 1). In his review of [**8**] Andreadakis [MR82j: 20060] states, as does Lichtman ([**4**], p. 533), that an unpublished result of Strojnowski proves that for any field K and for any unique product group G the units of $K(G)$ are trivial. We may observe however that our definition is over-determined, since if for some domain R and group G the units of $R(G)$ are trivial then $R(G)$ has no proper divisors of zero and, consequently, $\mathscr{C}(K)$ contains all unique product groups for any field K. Finally we remark that Lichtman ([**4**], p. 549) raises the question as to whether a torsion-free one-relator group G is a unique product group with a view to determining the units of $R(G)$; we should report that Brodskii has announced [**1**] and Howie has shown ([**3**], Corollary 4.3) that such a group G is locally indicable in the sense of Higman and so $G \in \mathscr{C}(R)$([**2**], Theorems 12, 13; [**6**], p. 638).

The next lemma follows directly from the definition above.

Lemma 3.1. *Let R be a domain and let $G \in \mathscr{C}(R)$. Let θ be an augmentation-preserving automorphism of $R(G)$. Then θ is an extension to $R(G)$ of an automorphism θ of G.*

We now consider how automorphisms may be "lifted" modulo N.

Theorem 3.2. *Let R/N be a domain. Let $G \in \mathscr{C}(R/N)$ and let θ be an automorphism of $R(G)$ which is augmentation-preserving modulo N. Then there exists an automorphism ϕ of G such that*

$$\theta(g) \equiv \phi(g) \bmod N(G) \qquad (\forall g \in G).$$

Proof. $N(G)$ is a nilpotent ideal of $R(G)$ and since $G \in \mathscr{C}(R/N)$ $(R/N)(G)$ is a domain. It follows that $N(G)$ is the unique minimal prime ideal of $R(G)$ and hence is a characteristic ideal of $R(G)$. Thus θ induces an automorphism $\tilde{\theta}$ on $(R/N)(G)$ and, by our assumption, $\tilde{\theta}$ preserves the augmentation of $(R/N)(G)$. Since $G \in \mathscr{C}(R/N)$ we may apply Lemma 3.1 to obtain the result. □

We now consider a converse to the above.

Theorem 3.3. (*No assumptions on R/N or on G*). *Let $\theta: G \to R(G)$ be a homomorphism such that there exists an automorphism ϕ of G for which*

$$\theta(g) \equiv \phi(g) \bmod N(G) \qquad (\forall g \in G).$$

Then θ extends to an automorphism of $R(G)$ which is augmentation-preserving modulo N.

Proof. Let $\theta: R(G) \to R(G)$ be the extended R-linear homomorphism. We prove first that θ is surjective and we use the notation of Lemma 2.2. Let $T = R(G)$, $S = \theta(T)$, $M = N(G)$. Then M is a nilpotent ideal of T. Since the composite map $R(G) \xrightarrow{\theta} R(G) \to (R/N)(G)$ is clearly surjective we have $T = S + M$. Further $N \subseteq R \subseteq S$ and so $M = NT \subseteq (S \cap M)T \subseteq M$. By Lemma 2.2 we conclude that $S = T$ and so θ is surjective.

We now prove that θ is injective. By hypothesis there exists an automorphism ϕ of G such that

$$\theta(g) \equiv \phi(g) \bmod N(G) \qquad (\forall g \in G).$$

For the sake of argument suppose there exist distinct elements $g_1, g_2, \ldots, g_r \in G$ and $\lambda_1, \lambda_2, \ldots, \lambda_r \in R$ such that

$$\sum_{i=1}^{r} \lambda_i \theta(g_i) = \theta\left(\sum_{i=1}^{r} \lambda_i g_i\right) = 0.$$

Now we have

$$\theta(g_i) = \phi(g_i) + n_i$$

where $n_i \in N(G)$ $(i = 1, 2, \ldots, r)$ and so

$$\sum_{i=1}^{r} \lambda_i(\phi(g_i) + n_i) = 0. \qquad (*)$$

Modulo N we have

$$\sum_{i=1}^{r} \bar{\lambda}_i \overline{\phi(g_i)} = \bar{0}.$$

But ϕ is an automorphism of G and so in $(R/N)(G)$ we have $\bar{\lambda}_i = \bar{0}$ $(i = 1, 2, \ldots, r)$ and thus $\lambda_i \in N$ $(i = 1, 2, \ldots, r)$. Now reconsider (*) but this time modulo N^2. On omitting terms with coefficients modulo N^2 we again obtain what, formally, is the same relation, namely in $(R/N^2)(G)$ we have

$$\sum_{i=1}^{r} \bar{\lambda}_i \overline{\phi(g_i)} = \bar{0}$$

where $(\bar{\ })$ denotes a residue class modulo N^2. Hence, similarly as before, we conclude that

$$\lambda_i \in N^2 \qquad (i = 1, 2, \ldots, r).$$

Continuing in this manner we deduce, as N is nilpotent, that

$$\lambda_i = 0 \qquad (i = 1, 2, \ldots, r).$$

Hence θ is injective.

Finally θ is clearly augmentation-preserving modulo N. □

We conclude by stating a particular result, of independent interest, which follows from Theorems 3.2 and 3.3.

Theorem 3.4. *Let G be a free abelian group of rank n on the n free generators $g_1, g_2, \ldots, g_n$. Let θ be an automorphism of $R(G)$ which is augmentation-preserving modulo N. Then*

(1) *$\theta(g_i)$ is a unit of $R(G)$ $(i=1,2,\ldots,n)$ and*
(2) *if $\theta(g_i)=\sum a_i(r_1,r_2,\ldots,r_n)g_1^{\alpha_{ir_1}}g_2^{\alpha_{ir_2}}\ldots g_n^{\alpha_{ir_n}}$ where*

$$a_i(r_1,r_2,\ldots,r_n)\in R \quad (\forall(r_1,r_2,\ldots,r_n))$$

then $a_i(r_1,r_2,\ldots,r_n)\in N$ $\forall(r_1,r_2,\ldots,r_n)$ with one exception, say $(r_{i_1},r_{i_2},\ldots,r_{i_n})$ for which $a_i(r_{i_1},r_{i_2},\ldots,r_{i_n})=1$. Furthermore the selection of these n-tuples $(r_{i_1},r_{i_2},\ldots,r_{i_n})$, one for each i, yields an $n\times n$ matrix $(\alpha_{ir_{ij}})$ such that $\det(\alpha_{ir_{ij}})=\pm 1$.

Conversely if there is defined a mapping $\theta:\{g_1,g_2,\ldots,g_n\}\to R(G)$ satisfying (1) and (2) then θ extends to an automorphism of $R(G)$ which is augmentation-preserving modulo N.

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