

## THE AVERAGE DISTANCE BETWEEN TWO POINTS

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### Abstract

We provide bounds on the average distance between two points uniformly and independently chosen from a compact convex subset of the  $s$ -dimensional Euclidean space.

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Let  $X$  be a compact convex subset of the  $s$ -dimensional Euclidean space  $\mathbb{R}^s$  and assume that we choose uniformly and independently two points from  $X$ . How large is the expected Euclidean distance  $\|\cdot\|$  between these two points? In other words, we require the quantity

$$a(X) := \mathbb{E}[\|x - y\|] = \frac{1}{\lambda(X)^2} \int_X \int_X \|x - y\| d\lambda(x) d\lambda(y),$$

where  $\lambda$  denotes the  $s$ -dimensional Lebesgue measure. This problem was stated in [1, 2, 4, 5]. Note that there is a close connection between this problem and that of finding the moments of the length of random chords (see [8, Ch. 4, Section 2] or [9, Ch. 2]).

Trivially  $a(X) \leq d(X)$ , where  $d(X) = \max\{\|x - y\| : x, y \in X\}$  is the diameter of  $X$ . The following results are well known from the literature.

### EXAMPLE 1.

- (1) For all compact convex subsets of  $\mathbb{R}$  (the intervals) we have  $a(X) = d(X)/3$ .
- (2) If  $X \subseteq \mathbb{R}^s$  is a ball with diameter  $d(X)$ , then

$$a(X) = \frac{s}{2s + 1} \beta_s d(X),$$

where

$$\beta_s = \begin{cases} \frac{2^{3s+1}((s/2)!)^2 s!}{(s+1)(2s)!\pi} & \text{for even } s, \\ \frac{2^{s+1}(s!)^3}{(s+1)((s-1)/2!)^2(2s)!} & \text{for odd } s. \end{cases}$$

For a proof see [4] or [8]. In particular, if  $X$  is a disc in  $\mathbb{R}^2$  with diameter  $d(X)$ , then

$$a(X) = 64d(X)/(45\pi) = 0.45271 \dots d(X).$$

- (3) If  $X \subseteq \mathbb{R}^2$  is a rectangle of sides  $a \geq b$ , then (see [8])

$$a(X) = \frac{1}{15} \left[ \frac{a^3}{b^2} + \frac{b^3}{a^2} + d \left( 3 - \frac{a^2}{b^2} - \frac{b^2}{a^2} \right) + \frac{5}{2} \left( \frac{b^2}{a} \log \frac{a+d}{b} + \frac{a^2}{b} \log \frac{b+d}{a} \right) \right],$$

where  $d = d(X) = \sqrt{a^2 + b^2}$ . In particular, if  $X$  is a square, then

$$a(X) = (2 + \sqrt{2} + 5 \log(\sqrt{2} + 1)) \frac{d(X)}{15\sqrt{2}} = 0.36869 \dots d(X).$$

- (4) If  $X$  is a cube in  $\mathbb{R}^s$ , then

$$a(X) = \frac{1}{\sqrt{6}} \left( 1 - \frac{7}{40s} - \frac{65}{869s^2} + \dots \right) d(X)$$

and

$$a(X) \leq \frac{1}{\sqrt{6}} \left( \frac{1 + 2\sqrt{1 - 3/(5s)}}{3} \right)^{1/2} d(X).$$

For a proof of the asymptotic formula see [5], and for a proof of the upper bound see [2].

- (5) If  $X \subseteq \mathbb{R}^2$  is an equilateral triangle of side  $a$ , then (see [8])

$$a(X) = \frac{3a}{5} \left( \frac{1}{3} + \frac{\log 3}{4} \right).$$

In the following we prove a general bound on  $a(X)$  for  $X \subseteq \mathbb{R}^s$  with fixed diameter  $d(X) = 1$ . Furthermore, we present two results which may be useful to give upper and lower bounds on  $a(X)$ .

Denote by  $\mathcal{M}(X)$  the space of all regular Borel probability measures on  $X$ . It is well known that  $\mathcal{M}(X)$  equipped with the  $w^*$ -topology becomes a compact convex space. For  $x \in X$ , let  $\delta_x \in \mathcal{M}(X)$  be the point measure concentrated on  $x$ . It is easy to show that the set  $\{\delta_x \mid x \in X\}$  is the set of all extreme points of  $\mathcal{M}(X)$  and hence from

the Krein–Milman theorem we find that  $\mathcal{M}(X)$  is the  $w^*$ -closure of the convex hull of  $\{\delta_x \mid x \in X\}$ . Let  $\mathcal{F} = \{(1/n) \sum_{i=1}^n \delta_{x_i} \mid x_1, \dots, x_n \in X, n \in \mathbb{N}\}$ . Then one can show that  $\mathcal{F}$  is the set of all convex combinations with rational coefficients of extreme points of  $\mathcal{M}(X)$ . Now, since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , we deduce from the above considerations that  $\mathcal{F}$  is dense in  $\mathcal{M}(X)$ .

For any  $\mu \in \mathcal{M}(X)$ , we define

$$I(\mu) := \int_X \int_X \|x - y\| d\mu(x) d\mu(y).$$

It is known that the mapping  $I : \mathcal{M}(X) \rightarrow \mathbb{R}$  is continuous with respect to the  $w^*$ -topology on  $\mathcal{M}(X)$  (see [10, Lemma 1]). Note that  $a(X) = I(\lambda')$  where  $\lambda'$  is the normalized Lebesgue measure on  $X$ .

**REMARK 2.** Let  $X$  be a compact subset of  $\mathbb{R}^s$  and let  $(x_n)_{n \geq 0}$  be a sequence which is uniformly distributed in  $X$  with respect to the normalized Lebesgue measure  $\lambda'$  on  $X$ , that is,  $\mu_N := N^{-1} \sum_{i=0}^{N-1} \delta_{x_i} \rightarrow \lambda'$  with respect to  $w^*$ -topology on  $\mathcal{M}(X)$ . Then by continuity of  $I$  we obtain

$$\frac{1}{N^2} \sum_{i,j=0}^{N-1} \|x_i - x_j\| = I(\mu_N) \rightarrow I(\lambda') = a(X) \quad \text{as } N \rightarrow \infty.$$

**THEOREM 3.** Let  $X$  be a compact subset of  $\mathbb{R}^s$  with diameter  $d(X) = 1$ . Then

$$a(X) \leq \sqrt{\frac{2s}{s+1} \frac{2^{s-2} \Gamma(s/2)^2}{\Gamma(s-1/2) \sqrt{\pi}}},$$

where  $\Gamma$  denotes the gamma function. For  $s = 2$  this bound can be improved to

$$a(X) \leq \frac{229}{800} + \frac{44}{75} \sqrt{2 - \sqrt{3}} + \frac{19}{480} \sqrt{5} = 0.678442 \dots$$

**PROOF.** We have

$$a(X) = I(\lambda') \leq \sup_{\mu \in \mathcal{M}(X)} I(\mu).$$

Since  $I : \mathcal{M}(X) \rightarrow \mathbb{R}$  is continuous with respect to the  $w^*$ -topology on  $\mathcal{M}(X)$  and  $\mathcal{F}$  is dense in  $\mathcal{M}(X)$  we obtain

$$\sup_{\mu \in \mathcal{M}(X)} I(\mu) = \sup_{n \in \mathbb{N}, x_1, \dots, x_n \in X} \frac{1}{n^2} \sum_{i,j=1}^n \|x_i - x_j\|.$$

It was shown by Nickolas and Yost [6] that, for all  $x_1, \dots, x_n \in X \subseteq \mathbb{R}^s$  with  $d(X) = 1$ ,

$$\frac{1}{n^2} \sum_{i,j=1}^n \|x_i - x_j\| \leq \sqrt{\frac{2s}{s+1} \frac{2^{s-2} \Gamma(s/2)^2}{\Gamma(s-1/2) \sqrt{\pi}}}.$$

For  $s = 2$  it was shown by Pillichshammer [7] that, for all  $x_1, \dots, x_n \in \mathbb{R}^2$  with  $\|x_i - x_j\| \leq 1$ ,

$$\frac{1}{n^2} \sum_{i,j=1}^n \|x_i - x_j\| \leq \frac{229}{800} + \frac{44}{75} \sqrt{2 - \sqrt{3}} + \frac{19}{480} \sqrt{5} = 0.678442 \dots$$

The result follows from these bounds. □

**REMARK 4.** Note that it is not true in general that  $X \subseteq Y$  implies  $a(X) \leq a(Y)$ . For example, for  $h > 0$ , let  $A_h$  denote the right triangle with vertices  $\{(0, 0), (1, 0), (1, h)\}$ . Then

$$\begin{aligned} a(A_h) &= \frac{4}{h^2} \int_0^1 \int_0^{hx_1} \int_0^1 \int_0^{hx_2} \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} dy_2 dx_2 dy_1 dx_1 \\ &\geq 4 \int_0^1 \int_0^1 \frac{1}{h^2} \int_0^{hx_1} \int_0^{hx_2} |x_1 - x_2| dy_2 dy_1 dx_2 dx_1 \\ &= 4 \int_0^1 \int_0^1 |x_1 - x_2| x_1 x_2 dx_2 dx_1 = \frac{4}{15}. \end{aligned}$$

On the other hand,

$$a(A_h) \leq 4 \int_0^1 \int_0^1 x_1 x_2 \sqrt{(x_1 - x_2)^2 + h^2} dx_2 dx_1$$

and hence  $\lim_{h \rightarrow 0^+} a(A_h) = 4/15$ . Thus for any  $\varepsilon > 0$  there is a  $h_0 > 0$  such that, for all  $0 < h < h_0$ ,  $|a(A_h) - 4/15| < \varepsilon$ .

For  $l > 0$ , let  $B_l$  be the rectangle with vertices  $\{(0, 0), (1, 0), (1, -l), (0, -l)\}$ . Then from Example 1 we obtain  $\lim_{l \rightarrow 0^+} a(B_l) = 1/3$ . Thus for any  $\varepsilon > 0$  there is a  $l_0 > 0$  such that, for all  $0 < l < l_0$ ,  $|a(B_l) - 1/3| < \varepsilon$ .

Now let  $\varepsilon, \delta > 0$ . Choose  $0 < h < \min\{1, h_0\}$ , and  $0 < l < \min\{1, l_0\}$  small enough such that  $\lambda(B_l) < \delta \lambda(A_h)$  and let  $C_{h,l} := A_h \cup B_l$ . Then

$$\begin{aligned} a(C_{h,l}) &= \frac{\lambda(A_h)^2}{(\lambda(A_h) + \lambda(B_l))^2} a(A_h) + \frac{\lambda(B_l)^2}{(\lambda(A_h) + \lambda(B_l))^2} a(B_l) \\ &\quad + \frac{2}{(\lambda(A_h) + \lambda(B_l))^2} \int_{A_h} \int_{B_l} \|x - y\| d\lambda(x) d\lambda(y) \\ &< a(A_h) + \left(\frac{\delta}{1 + \delta}\right)^2 a(B_l) + \frac{3\delta}{1 + \delta} < \frac{4}{15} + \varepsilon + \delta^2 \left(\frac{1}{3} + \varepsilon\right) + 3\delta. \end{aligned}$$

Hence if we choose  $1/60 > \varepsilon > 0$  and  $\delta > 0$  small enough we can obtain  $a(C_{h,l}) < 3/10$ . Of course  $B_l \subseteq C_{h,l}$ , but

$$a(B_l) \geq \frac{1}{3} - \varepsilon \geq \frac{19}{60} > \frac{3}{10} > a(C_{h,l}).$$

**LEMMA 5.**

(1) Let  $X$  and  $Y$  be compact sets in  $\mathbb{R}^s$  with  $\lambda(X \cap Y) = 0$ . Then

$$\lambda(X \cup Y)a(X \cup Y) \geq \lambda(X)a(X) + \lambda(Y)a(Y).$$

(2) Let  $X \subseteq Y$  be compact sets in  $\mathbb{R}^s$ . Then

$$\lambda(X)a(X) \leq \lambda(Y)a(Y).$$

**PROOF.** (1) We have

$$\begin{aligned} a(X \cup Y) &= \frac{\lambda(X)^2}{(\lambda(X) + \lambda(Y))^2} a(X) + \frac{\lambda(Y)^2}{(\lambda(X) + \lambda(Y))^2} a(Y) \\ &\quad + 2 \frac{\lambda(X)\lambda(Y)}{(\lambda(X) + \lambda(Y))^2} \frac{1}{\lambda(X)\lambda(Y)} \int_X \int_Y \|x - y\| d\lambda(x) d\lambda(y). \end{aligned}$$

For any regular Borel probability measures  $\mu$  and  $\nu$  on a subset  $A$  of the Euclidean space  $\mathbb{R}^s$  we have (see [10, Equation (\*\*)])

$$2 \int_A \int_A \|x - y\| d\mu(x) d\nu(y) \geq I(\mu) + I(\nu).$$

Now let  $A = X \cup Y$ , let  $\mu$  be the probability measure on  $A$  which is the normalized Lebesgue measure on  $X$  and which is zero on  $Y$  and let  $\nu$  be the probability measure on  $A$  which is the normalized Lebesgue measure on  $Y$  and which is zero on  $X$ . Then

$$\begin{aligned} \frac{2}{\lambda(X)\lambda(Y)} \int_X \int_Y \|x - y\| d\lambda(x) d\lambda(y) &= 2 \int_A \int_A \int_Y \|x - y\| d\mu(x) d\nu(y) \\ &\geq \int_{X \cup Y} \int_{X \cup Y} \|x - y\| d\mu(x) d\mu(y) + \int_{X \cup Y} \int_{X \cup Y} \|x - y\| d\nu(x) d\nu(y) \\ &= a(X) + a(Y). \end{aligned}$$

Hence

$$\begin{aligned} (\lambda(X) + \lambda(Y))^2 a(X \cup Y) &\geq \lambda(X)^2 a(X) + \lambda(Y)^2 a(Y) + \lambda(X)\lambda(Y)(a(X) + a(Y)) \\ &= (\lambda(X)a(X) + \lambda(Y)a(Y))(\lambda(X) + \lambda(Y)). \end{aligned}$$

(2) This assertion follows from the first one. □

**COROLLARY 6.** Let  $X \subseteq \mathbb{R}^s$  be compact and convex and let  $r = r(X)$  be the in-radius and  $R = R(X)$  be the circumradius of  $X$ . Then

$$\frac{\pi^{s/2}}{\Gamma(s/2 + 1)} \frac{2s}{2s + 1} \beta_s r^{s+1} \leq \lambda(X)a(X) \leq \frac{\pi^{s/2}}{\Gamma(s/2 + 1)} \frac{2s}{2s + 1} \beta_s R^{s+1}$$

with equality if  $X$  is a ball. In particular, for  $s = 2$  we have

$$\frac{128}{45} r^3 \leq \lambda(X)a(X) \leq R^3 \frac{128}{45}$$

with equality if  $X$  is a disc.

**PROOF.** Let  $K_{\text{in}}$  be the in-ball and let  $K_{\text{circ}}$  be the circumscribed ball of  $X$ . From Lemma 5 we obtain  $\lambda(K_{\text{in}})a(K_{\text{in}}) \leq \lambda(X)a(X) \leq \lambda(K_{\text{circ}})a(K_{\text{circ}})$  and the result follows from Example 1 (note that the volume of an  $s$ -dimensional ball of radius  $t > 0$  is given by  $\pi^{s/2}t^s / \Gamma(s/2 + 1)$ ).  $\square$

**REMARK 7.** It follows from a result of Blaschke [3] that, for any plane compact convex  $X \subseteq \mathbb{R}^2$ ,

$$a(X) \geq \frac{128}{45\pi} \sqrt{\frac{\lambda(X)}{\pi}}$$

with equality if  $X$  is a disc. In many cases this bound yields better results than the lower bound from Corollary 6 in the plane case (see Examples 8 and 10 below). For more information see [8, Ch. 4, Section 2] or [9, Ch. 2, Equation (2.55)].

**EXAMPLE 8.** For  $n \in \mathbb{N}$ ,  $n \geq 3$ , let  $X_n \subseteq \mathbb{R}^2$  be the regular  $n$ -gon with vertices on the unit circle. Then  $\lambda(X_n) = (n/2) \sin(2\pi/n)$ ,  $R = 1$  and  $r = \cos(\pi/n)$ . Hence we obtain

$$\frac{256}{45} \frac{\cos^3(\pi/n)}{n \sin(2\pi/n)} \leq a(X_n) \leq \frac{256}{45} \frac{1}{n \sin(2\pi/n)}.$$

From Remark 7 we even obtain the lower bound

$$a(X_n) \geq \frac{128}{45\pi} \sqrt{\frac{n}{2\pi} \sin \frac{2\pi}{n}}$$

which is slightly better than the lower bound above. Note that

$$\lim_{n \rightarrow \infty} \frac{128}{45\pi} \sqrt{\frac{n}{2\pi} \sin \frac{2\pi}{n}} = \lim_{n \rightarrow \infty} \frac{256}{45} \frac{\cos^3(\pi/n)}{n \sin(2\pi/n)} = \lim_{n \rightarrow \infty} \frac{256}{45} \frac{1}{n \sin(2\pi/n)} = \frac{128}{45\pi}.$$

In some cases the following easy lemma gives better estimates than Corollary 6.

**LEMMA 9.** Let  $X$  be a compact subset of  $\mathbb{R}^s$  and let  $T : \mathbb{R}^s \rightarrow \mathbb{R}^s$  be a linear mapping with norm  $\|T\|_2$ . Then we have  $a(T(X)) \leq a(X)\|T\|_2$ .

**EXAMPLE 10.** Let  $X$  be an ellipse  $x^2 + y^2/b^2 \leq 1$  in the Euclidean plane with  $0 < b \leq 1$ . Then  $X = T(K)$  where  $K$  is the disc with diameter 2 and center in the origin and where  $T = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$ . It is easy to see that  $\|T\|_2 = \max\{1, |b|\} = 1$  and  $\|T^{-1}\|_2 = 1/b$ . Then from Lemma 9 we obtain

$$b \frac{128}{45\pi} = ba(K) \leq a(X) \leq a(K) = \frac{128}{45\pi}$$

whereas from Corollary 6 we would just obtain

$$b^2 \frac{128}{45\pi} \leq a(X) \leq \frac{1}{b} \frac{128}{45\pi}.$$

From Remark 7 we obtain the lower bound  $a(X) \geq \sqrt{b}(128/45\pi)$ .

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