PHRAGMÈN-LINDELÖF AND COMPARISON THEOREMS FOR ELLIPTIC-PARABOLIC DIFFERENTIAL EQUATIONS

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1. Introduction. Theorems of Phragmèn–Lindelöf type and other related results for solutions of elliptic–parabolic equations have been given by numerous authors in recent years. Many of these results are based upon the maximum principle and the use of auxiliary comparison functions which are constructed as supersolutions of the equations under various conditions on the coefficients. In this paper we present an axiomatized treatment of these topics, replacing specific hypotheses on the nature of the coefficients of the equations by a single assumption concerning the maximum principle and another concerning the existence of positive supersolutions, in terms of which the theorems are stated. Since the first assumption is valid under very mild conditions, the application of these results to the solutions of any particular elliptic–parabolic equation essentially requires only the determination of supersolutions for that equation. In this way the theorems may be tailored to fit individual equations although of course when supersolutions are available for an entire class of equations (as in **(3)** and **(6)**, for example), the theorems apply to the class as a whole. In this connection let us mention that for uniformly elliptic equations the determination of optimum supersolutions may often be accomplished with the use of maximizing operators **(9)**.

Our main result (Theorem 4.4) is a comparison theorem which, in its simplest form, may be illustrated by the following example.

Let $u(x, y)$ be a harmonic function in the half strip $-\pi/2 \leqslant x \leqslant \pi/2, y \geqslant 0$ with zero Dirichlet data on the vertical sides. If for some constant β, $0 < \beta < 1$, we have $u(x, y) = o(e^{\beta y})$ as $y \rightarrow +\infty$, then in fact $u(x, y) = O(e^{-\beta y})$ as $y \rightarrow +\infty$.

Besides providing explicit bounds for the possible rates of growth and decay of solutions, such estimates find application in the approximation of solutions and the investigation of stability with respect to changes in the data or in the coefficients of the differential equation.

2. Notations and basic conditions. Let D be an open set in E^n with boundary ∂D. D may be unbounded, in which case we shall consider infinity

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to be part of the boundary. At various times it will be convenient to distinguish certain parts Γ, Γ_1, Γ_2 of the boundary and to write $B = \partial D - \Gamma$ or $B = \partial D - \Gamma_1 \cup \Gamma_2$. For example these distinguished sets may be an isolated point, the point at infinity, or a surface of singularities of the coefficients of our equations.

We shall be concerned with the linear differential operator

$$L \equiv L^* + c = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x)$$

where $x = (x_1, \ldots, x_n)$ and the coefficients are defined in the set D. Unless otherwise stated we shall assume throughout this paper that L has the following two basic properties in D:

(1) *Given any real-valued function $u(x)$, twice differentiable in an open set $D_1 \subset D$, such that $Lu(x) \leqslant 0$ for all $x \in D_1$, and*

$$\liminf_{\substack{x \to \partial D_1 \\ x \in D_1}} u(x) \geqslant 0,$$

then $u(x) \geqslant 0$ for all $x \in D_1$.

(2) *There exists a real-valued function $V(x)$, twice differentiable in D, such that $V(x) \geqslant 0$ and $LV(x) \leqslant 0$ for all $x \in D$.*

Remark 2.1. Property (1) is usually stated as a corollary to the (weak) maximum principle. In fact we have the following

LEMMA 1. *Suppose that the matrix (a_{ij}) is symmetric and positive semi-definite in D and that either:*

(i) *$c(x) < 0$ for all $x \in D$;*

or

(ii) *$c(x) \leqslant 0$ for all $x \in D$; and for every bounded open set Ω with $\bar{\Omega} \subset D$ there exists a real-valued function $h^{\Omega}(x)$, twice differentiable in Ω and continuous on $\bar{\Omega}$ such that $L^*h^{\Omega}(x) < 0$ for all $x \in \Omega$.*

Then L has the property (1) *in D.*

Case (i) is well known; cf. **(7**, p. 4**)**. The proof in case (ii) (and under weaker hypotheses) may be found in **(8)**.

Remark 2.2. No further hypothesis on the behaviour of $V(x)$ will be necessary. However, it will be clear from the theorems that better results are obtained when it is possible to select a $V(x)$ which is unbounded as $x \to \Gamma$ (or $x \to \Gamma_1$). Such a function, if it exists, may be called an anti-barrier for L at $\Gamma(\Gamma_1)$; see **(6)**.

Remark 2.3. While the positive semi-definiteness of the matrix (a_{ij}) is basic to property (1), the condition $c \leqslant 0$ is not. The permissible size of $c(x)$ is connected with the question of the eigenvalues of the operator L^* which in turn is related to property (2). The following lemma bears upon this relation.

Here and elsewhere in the paper we make use of the definitions

$$f^+(x) = \max\{0, f(x)\} \text{ and } f^-(x) = \max\{0, -f(x)\}.$$

LEMMA 2. *Suppose that the following conditions hold:*
(i) *the matrix* (a_{ij}) *is symmetric and positive semi-definite in* D*;*
(ii) *for every bounded open set* Ω *with* $\bar{\Omega} \subset D$ *there exists a real-valued function* $h^\Omega(x)$, *twice differentiable in* Ω *and continuous on* $\bar{\Omega}$, *such that* $L^*h^\Omega(x) < 0$ *for all* $x \in \Omega$*;*
(iii) *there exists a real-valued function* $V(x)$, *twice differentiable in* D, *such that* $\inf_{x\in D} V(x) > 0$ *and* $(L^* + c^+(x))V(x) \leqslant 0$ *for all* $x \in D$.
Then the operator L *has properties* (1) *and* (2) *in* D.

Proof. Property (2) follows immediately since

$$LV(x) = (L^* + c^+ - c^-)V(x) \leqslant -c^-V \leqslant 0 \qquad \text{for all } x \in D.$$

To establish property (1), suppose that $u(x)$ is a real-valued function, twice differentiable in the open set $D_1 \subset D$, such that

$$Lu(x) \leqslant 0 \text{ for all } x \in D_1 \quad \text{and} \quad \liminf_{\substack{x\to\partial D_1\\ x\in D_1}} u(x) \geqslant 0.$$

Define $w(x) = u(x)/V(x)$ if $x \in D_1$ and note that

$$\liminf_{\substack{x\to\partial D_1\\ x\in D_1}} w(x) \geqslant 0$$

and

$$\bar{L}w(x) \equiv \frac{1}{V(x)} Lu(x) = \sum_{i,j=1}^{n} a_{ij}\, w_{i,j} + \sum_{i=1}^{n} \Big(b_i + 2\sum_{j=1}^{n} a_{ij} V_j/V\Big) w_i + (LV/V)w \leqslant 0$$

if $x \in D_1$. Furthermore, defining $H^\Omega(x) = h^\Omega(x)/V(x)$ for each bounded open set Ω with $\bar{\Omega} \subset D$, we have for all $x \in \Omega$

$$\bar{L}^*H^\Omega(x) \equiv \sum_{i,j=1}^{n} a_{ij}\, H_{i,j}^{\ \Omega} + \sum_{i=1}^{n} \Big(b_i + 2\sum_{j=1}^{n} a_{ij}\, V_j/V\Big) H_i^{\ \Omega}$$
$$= \frac{1}{V(x)} L^*h^\Omega(x) - \frac{h^\Omega(x)}{V^2(x)} L^*V(x).$$

Since we may assume without loss of generality that $h^\Omega(x) < 0$, and we have $L^*V(x) \leqslant -c^+V \leqslant 0$ for all $x \in D$, the above equation yields $\bar{L}^*H^\Omega(x) < 0$ for all $x \in \Omega$. Applying Lemma 1, we conclude that $\bar{L}$ has property (1) in D_1, from which we obtain $w(x) \geqslant 0$ in D_1, and then $u(x) \geqslant 0$ in D_1.

3. Phragmèn–Lindelöf theorems. In this section we distinguish a part Γ of the boundary of D and write $B = \partial D - \Gamma$.

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THEOREM 3.1. (Phragmèn–Lindelöf). *Assume that L has properties* (1) *and* (2) *in D. Let $u(x)$ be a real-valued function, twice differentiable in D, such that*

$$Lu(x) \leqslant 0 \text{ for all } x \in D \quad \text{and} \quad \liminf_{\substack{x \to B \\ x \in D}} u(x) \geqslant 0.$$

If in addition $u^{-}(x) = o(V(x))$ as $x \to \Gamma$, $x \in D$, then in fact $u(x) \geqslant 0$ for all $x \in D$.

Proof. We use the standard Phragmèn–Lindelöf technique. Choose $\epsilon > 0$ and consider $w(x) = u(x) + \epsilon V(x)$ for $x \in D$. We have

$$Lw(x) \leqslant 0 \text{ if } x \in D \quad \text{and} \quad \liminf_{\substack{x \to B \\ x \in D}} w(x) \geqslant 0$$

by the hypotheses on $u(x)$ and $V(x)$. Furthermore the growth limitation on $u^{-}(x)$ implies that

$$\liminf_{\substack{x \to \Gamma \\ x \in D}} w(x) \geqslant 0.$$

We conclude from property (1) that $w(x) \geqslant 0$ if $x \in D$.

This result holds for every $\epsilon > 0$. If we now fix any arbitrary point $x^0 \in D$ and let ϵ tend to zero, we have the result $u(x^0) \geqslant 0$ of the theorem.

Replacing u by $-u$ in the theorem yields another result which we do not state. The two theorems may be combined to give the following

COROLLARY 3.2 (Uniqueness). *Assume that L has properties* (1) *and* (2) *in D. Let $u(x)$ be a real-valued function, twice differentiable in D, such that*

$$Lu(x) = 0 \text{ for all } x \in D \quad \text{and} \quad \lim_{\substack{x \to B \\ x \in D}} u(x) = 0.$$

If in addition $u(x) = o\{V(x)\}$ as $x \to \Gamma$, $x \in D$, then in fact $u(x) \equiv 0$ in D.

Remark 3.3. The corollary may be related to the problem of isolated singularities of solutions. In fact suppose that $U(x)$, $\tilde{U}(x)$ are two solutions of $Lu = f$ in a punctured ball about the origin, taking the same boundary values on the surface of the ball. Here Γ is the single point at the origin and B is the surface of the punctured ball D. If $U - \tilde{U} = o\{V\}$ as $x \to 0$, we conclude that U and $\tilde{U}$ are identical in D and thus may be defined to have the same singularity at the origin. In particular, if $\tilde{U}$ is a solution in the entire ball, then the singularity of U at the origin is removable.

As an immediate application of the Phragmèn–Lindelöf theorem we have

THEOREM 3.4 (Extended Minimum Principle). *Assume that L has properties* (1) *and* (2) *in D and that $c(x) \leqslant 0$ for all $x \in D$. Let $u(x)$ be a real-valued*

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function, twice differentiable in D, such that $Lu(x) \leqslant 0$ for all $x \in D$ and $u^-(x) = o\{V(x)\}$ as $x \to \Gamma$, $x \in D$. Then

$$u(x) \geqslant \min(0, m) \text{ for all } x \in D \quad \text{where } m = \liminf_{\substack{x \to B \\ x \in D}} u(x).$$

Proof. If $m = -\infty$ there is nothing to prove. We may therefore assume that $m > -\infty$ and define $m^* = \min(0, m)$. Consider now $w(x) = u(x) - m^*$ for $x \in D$. We have $Lw(x) \leqslant -c(x)m^* \leqslant 0$ if $x \in D$ and

$$\liminf_{\substack{x \to B \\ x \in D}} w(x) = \liminf_{\substack{x \to B \\ x \in D}} u(x) - m^* \geqslant 0.$$

Furthermore $w^-(x) \leqslant u^-(x) = o\{V(x)\}$ as $x \to \Gamma$, $x \in D$. By Theorem 3.1 we conclude that $w(x) \geqslant 0$ if $x \in D$, which is the desired result.

Remark 3.5. If $c \equiv 0$ in D and $V(x)$ is an anti-barrier for L at Γ we may conclude in Theorem 3.4 that $u(x) \geqslant m$ for all $x \in D$.

A corresponding extended maximum principle may also be obtained by merely replacing u by $-u$ in Theorem 3.4.

In the case that $c(x) \leqslant 0$, the ordinary maximum principle is said to hold for the operator L if any non-constant function $u(x)$ satisfying $Lu(x) \geqslant 0$ cannot assume an interior non-negative maximum **(5)**. If in the previous theorems we suppose the ordinary maximum principle to hold for L (with $c \leqslant 0$) on bounded open subsets of D (which implies property (1) for L in D), then the results may be sharpened to yield strict inequalities in (each component of) D unless u is constant there. If in addition Γ is isolated from B, then the conclusion

$$\liminf_{\substack{x \to \Gamma \\ x \in D}} u(x) > \min(0, m)$$

may also be obtained in Theorem 3.4 in some cases; see **(3)**.

When L enjoys both the ordinary maximum principle and an anti-barrier at infinity, it is possible to obtain theorems of Liouville type, as has already been observed in **(6**, p. 523**)** and **(3**, pp. 333–334**)**. Specifically we have

COROLLARY 3.6 (Liouville). *Suppose that the ordinary maximum principle is valid for L (with $c \equiv 0$) on bounded open subsets of E^n and that in some neighbourhood D of infinity there exists an anti-barrier $V(x)$ for L at infinity. Let $u(x)$ be a real-valued function, twice differentiable in E^n, such that $Lu(x) \leqslant 0$ for all $x \in E^n$ and $u^-(x) = o\{V(x)\}$ as $x \to \infty$. Then u is identically constant.*

Proof. Here $\Gamma = \{\infty\}$ and B is the finite boundary of D. If u is not identically constant the ordinary maximum principle implies the existence of a point $x^0 \in D$ such that $u(x^0) < \min_{x \in B} u(x)$. Remark 3.5 then yields an immediate contradiction.

Before leaving this section we shall consider another, slightly different approach to the Phragmèn–Lindelöf theorem which was used by Gilbarg **(2)** and Hopf **(4)**. For simplicity in notation we shall restrict the discussion to the case when D is unbounded, B is the finite boundary, and $\Gamma = \{\infty\}$. We define the sets

$$D_R = D \cap \{||x|| < R\}, \quad B_R = B \cap \{||x|| \leqslant R\}, \quad \text{and } C_R = D \cap \{||x|| = R\}$$

where

$$||x|| = \left(\sum_{i=1}^{n} x_i^{\,2}\right)^{\frac{1}{2}}.$$

In place of the property (2) we now assume the following *property* (2′) in D: *For sufficiently large values of R there exists a function $V_R(x)$, defined on $D_R \cup C_R$ and twice differentiable in D_R, such that*

(i) $V_R(x) \geqslant 0$ *and* $LV_R(x) \leqslant 0$ *for all* $x \in D_R$;

(ii) $\inf_{x \in C_R} V_R(x) = 1$;

(iii) *there exists a positive function* $\mu(R)$ *such that* $V_R(x) = O\{1/\mu(R)\}$ *as* $R \to \infty$, *for each fixed* $x \in D_R$. For the construction of such functions $V_R(x)$ we refer to **(1, 2,** and **4)**. Theorem 3.1 now takes the following form.

THEOREM 3.7 (Phragmèn–Lindelöf). *Assume that L has properties* (1) *and* (2′) *in D. Let $u(x)$ be a real-valued function, twice differentiable in D, such that*

$$Lu(x) \leqslant 0 \textit{ for all } x \in D \quad \textit{and} \quad \liminf_{\substack{x \to B \\ x \in D}} u(x) \geqslant 0.$$

If in addition $u^-(x) = o\{\mu(||x||)\}$ as $x \to \infty$, $x \in D$, then $u(x) \geqslant 0$ for all $x \in D$.

Proof. Let x^0 be an arbitrary but fixed point of D. We shall prove that $u(x^0) \geqslant 0$.

Given any $\epsilon > 0$ the growth condition on $u^-(x)$ implies that there exists an R such that $u(x) \geqslant -\epsilon\mu(||x||)$ for $x \in D$, $||x|| \geqslant R$. We may also assume that R is so large that $x^0 \in D_R$. Define

$$w_R(x) = u(x) + \epsilon\mu(R)V_R(x) \qquad \text{for } x \in D_R \cup C_R.$$

We have

$$Lw_R(x) \leqslant 0 \text{ if } x \in D_R \quad \text{and} \quad \liminf_{\substack{x \to B_R \\ x \in D_R}} w_R(x) \geqslant 0.$$

Furthermore, for $x \in C_R$, we have $w_R(x) \geqslant -\epsilon\mu(R) + \epsilon\mu(R) = 0$. Property (1) then implies that $w_R(x) \geqslant 0$ for all $x \in D_R$. In particular this holds at x^0, and since $\mu(R)V_R(x^0)$ is bounded independent of R, we have

$$u(x^0) \geqslant -\epsilon\mu(R)V_R(x^0) \geqslant -\epsilon M(x^0).$$

Now let $\epsilon \to 0$ to obtain the desired result.

4. Comparison theorems. In this section we distinguish two parts, Γ_1 and Γ_2, of the boundary of D and write $B = \partial D - \Gamma_1 \cup \Gamma_2$. Again D

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may be unbounded. However, we now require that B consist of only finite points.

In addition to the function $V(x)$ we shall require a second function $v(x)$ whose existence we assume as a property (3) of the operator L in D.

(3) *There exists a real valued function* $v(x)$, *twice differentiable in* D *and continuous on* $D \cup B$, *such that* $v(x) \geqslant 0$ *and* $Lv(x) \leqslant 0$ *for all* $x \in D$.

For the comparison theorems it is desirable, when possible, to select functions with the behaviour $V(x) \to \infty$ and $v(x) \to 0$ as $x \to \Gamma_1$; i.e., to choose V and v to be an anti-barrier and a barrier, respectively, for L at Γ_1. Such a situation has occurred already in the example in the Introduction (where $\Gamma_1 = +\infty$, $V = e^{\beta y} \cos \beta x$, and $v = e^{-\beta y} \cos \beta x$).

Let us define the non-negative function $\alpha(x) = -Lv(x)$ for $x \in D$. Then we have the following comparison theorem.

THEOREM 4.1. *Assume that* L *has the properties* (1), (2), *and* (3) *in* D. *Let* $u(x)$ *be a real-valued function, twice differentiable in* D *and continuous on* $D \cup B$, *such that* $Lu(x) = f(x)$ *for all* $x \in D$ *and* $u(x) = g(x)$ *for all* $x \in B$. *Suppose that there exist non-negative constants* F, G *such that* $f(x) \geqslant -F\alpha(x)$ *if* $x \in D$ *and* $g(x) \leqslant Gv(x)$ *if* $x \in B$. *If, in addition,* $u^+(x) = o\{V(x)\}$ *as* $x \to \Gamma_1$, $x \in D$ *and there exists a constant* H *such that* $u^+(x) \leqslant Hv(x)$ *in some neighbourhood of* Γ_2 *in* D, *then* $u(x) \leqslant Mv(x)$ *for all* $x \in D \cup B$ *where* $M = \max\{F, G, H\}$.

Remark 4.2. If the stronger hypothesis $u^+(x) = o\{v(x)\}$ as $x \to \Gamma_2$, $x \in D$ holds, then the conclusion is valid with $M = \max\{F, G\}$.

Proof. Consider the function $w(x) = Mv(x) - u(x)$ for $x \in D \cup B$. We have

$$Lw(x) = -M\alpha(x) - f(x) \leqslant -[F\alpha(x) + f(x)] \leqslant 0 \qquad \text{for } x \in D$$

and

$$w(x) \geqslant Gv(x) - g(x) \geqslant 0 \qquad \text{for } x \in B.$$

Furthermore the growth conditions on $u^+(x)$ imply that

$$\liminf_{\substack{x \to \Gamma_2 \\ x \in D}} w(x) \geqslant 0 \quad \text{and} \quad w^-(x) \leqslant u^+(x) = o\{V(x)\}$$

as $x \to \Gamma_1$, $x \in D$. The result $w(x) \geqslant 0$ for $x \in D$ then follows from Theorem 3.1 with Γ_1 replacing Γ and $B \cup \Gamma_2$ replacing B.

COROLLARY 4.3. *If in addition to the hypotheses of Theorem* 4.1 *we assume that* $f(x) \leqslant 0$ if $x \in D$, $g(x) \geqslant 0$ *if* $x \in B$, *and* $u^-(x) = o\{V(x)\}$ *as* $x \to \Gamma_1 \cup \Gamma_2$, $x \in D$, *then* $0 \leqslant u(x) \leqslant Mv(x)$ *for all* $x \in D \cup B$.

Proof. The additional conclusion $u(x) \geqslant 0$ follows directly from Theorem 3.1 with $\Gamma_1 \cup \Gamma_2$ replacing Γ.

Bounds in the opposite direction can be obtained by replacing u by $-u$. We state only the combined result.

THEOREM 4.4. *Assume that L has properties* (1), (2), *and* (3) *in D. Let $u(x)$ be a real-valued function, twice differentiable in D and continuous on $D \cup B$, such that $Lu(x) = f(x)$ for $x \in D$ and $u(x) = g(x)$ for $x \in B$. Suppose that there exist constants F, G such that $|f(x)| \leqslant F\alpha(x)$ for all $x \in D$ and $|g(x)| \leqslant Gv(x)$ for all $x \in B$. If, in addition, $u(x) = o\{V(x)\}$ as $x \to \Gamma_1$, $x \in D$ and there exists a constant H such that $|u(x)| \leqslant Hv(x)$ in some neighbourhood of Γ_2 in D, then $|u(x)| \leqslant Mv(x)$ for all $x \in D \cup B$ where $M = \max\{F, G, H\}$.*

Remark 4.5. If the stronger hypothesis $u(x) = o\{v(x)\}$ as $x \to \Gamma_2$, $x \in D$ holds, then the conclusion is valid with $M = \max\{F, G\}$. The obvious uniqueness result which follows when $f = g = 0$ can also be obtained from Corollary 3.2, if Γ is replaced by $\Gamma_1 \cup \Gamma_2$ and V by $V + v$.

5. An analogous treatment of these same topics, with our Dirichlet-type boundary conditions replaced by other boundary data, requires a corresponding modification in the definitions of the properties (1), (2), and (3) of the operator L.

Let us also note here that all results of this paper remain valid for the case $n = 1$, when L becomes an ordinary differential operator.

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