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A commutativity theorem for rings

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Let R be a ring with an identity and for each x, y in R, $(xy)^k = x^k y^k$ for three consecutive positive integers k. It is shown in this note that R is a commutative ring.

It is well known that each of the following conditions on any group G insures that G is commutative:

(i) for each x, y in G, $(xy)^2 = x^2 y^2$;

(ii) for each x, y in G, $(xy)^k = x^k y^k$ for three consecutive positive integers k.

Several authors have considered the ring-theoretic analogues of the above group-theoretic results [*1*, *2*, *3*, *4*, *5*, *6*, *7*]. Johnsen, Outcalt and Yaqub have shown in [5] that if R is any nonassociative ring with 1 such that $(xy)^2 = x^2 y^2$ for all x, y in R, then R is commutative. Furthermore, they provided examples showing that for any integer $k > 2$, there exists a noncommutative ring R with 1 satisfying the identity $(xy)^k = x^k y^k$ for all x, y in R. For the ring-theoretic analogue of (ii), a partial solution was given by Luh [*6*]. He showed that any primary ring having the condition $(xy)^k = x^k y^k$ for three consecutive positive integers k is commutative. The purpose of this note is to furnish a complete, but elementary, solution of the ring-theoretic analogue of (ii).

THEOREM. *If R is a ring with 1 which satisfies the identities $(xy)^k = x^k y^k$, $k = n, n+1, n+2$, where n is a positive integer, then R is commutative.*

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Proof. Let x, y be in R. From $x^{n+1}y^{n+1} = x^n y^n xy$, it follows that

(1) $$x^n(xy^n-y^nx)y = 0 .$$

Since (1) holds for all x, y in R, substitute $(x+1)$ for x and simplify, to get

(2) $$(x+1)^n(xy^n-y^nx)y = 0 .$$

Multiply (2) on the left by x^{n-1} and expand $(x+1)^n$ by the binomial theorem, keeping in mind the identity (1); it follows that

(3) $$x^{n-1}(xy^n-y^nx)y = 0 .$$

Since (3) is valid for each x, y in R, continue the above process, that is, replace x by $(x+1)$ and multiply (3) on the left by x^{n-2} eventually one gets

(4) $$x(xy^n-y^nx)y = 0 .$$

Again substitute $(x+1)$ for x and use (4), to get

(5) $$(xy^n-y^nx)y = 0 .$$

Now from the identity $x^{n+2}y^{n+2} = x^{n+1}y^{n+1}xy$, we have

(6) $$x^{n+1}(xy^{n+1}-y^{n+1}x)y = 0 .$$

Employing the same technique used to get (5) from (1), one obtains

(7) $$(xy^{n+1}-y^{n+1}x)y = 0 .$$

Multiply both sides of (5) on the left by y, to get

(8) $$yxy^{n+1} = y^{n+1}xy .$$

From (7) and (8) we have

(9) $$(xy-yx)y^{n+1} = 0 .$$

Now apply the same technique used to get (5) from (1), this time substituting $(y+1)$ for y; we then have

(10) $$(xy-yx)y = 0 .$$

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Finally replace y by $(y+1)$ and use (10), to obtain $xy - yx = 0$. Thus R is commutative.

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