ON THE APPROXIMATION OF π BY SPECIAL ALGEBRAIC NUMBERS

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1. Introduction and result. Suppose that m_0 is an integer, $m_0 \geq 3$, $\rho = \exp(2\pi i/m_0)$, $K = \mathbb{Q}(\rho, i)$, ν denotes the degree of K, $\xi \in K$ has degree N over $\mathbb{Q}$. The length $L(\xi) = \sum_{j=0}^{N} |a_j|$, where $P(z) = \sum_{j=0}^{N} a_j z^j$ is the (irreducible) minimal polynomial for ξ with relatively prime integer coefficients. Feldman [**2**, p. 49] proved that there is an absolute constant $c_0 > 0$ such that

$$|\pi - \xi| > \exp\{-c_o \nu^2(1 + N^{-1} \log L)\}. \tag{1}$$

From [**2**, p. 49, Notes 1 and 2] we know that $\nu = \varphi(m_0)$ or $\nu = 2\varphi(m_0)$, and $\varphi(m_0) \geq c_1 m_0 (\log\log m_0)^{-1}$ ($c_1 > 0$ an absolute constant), where $\varphi(m_0)$ denotes Euler's function.

P. L. Cijsouw has developed some new refinements of the Gelfond–Baker method to derive an improved approximation measure for π [**1**]. In this note we use these refinements and two simple lemmas of [**2**] to prove the following result.

THEOREM. *There exists a positive absolute constant c_2 such that*

$$|\pi - \xi| > \exp\{-c_2 \nu^2(1 + (N \log \nu)^{-1} \log L)\} \tag{2}$$

for all algebraic numbers $\xi \in K = \mathbb{Q}(\rho, i)$, where $\nu \geq 2$ denotes the degree of K, and N and L the degree and the length of ξ, respectively.

It is clear that (2) improves upon (1); the following proof is simpler than that in [**2**].

COROLLARY. *If ξ has large degree, i.e. $N \gg \nu$, then*

$$\log |\pi - \xi| \gg -\left(N^2 + \frac{N}{\log \nu} \log L\right).$$

2. Lemmas.

LEMMA 1 [**2**, p. 49, Lemma 1]. *Let C be a natural number and let a_{ij} be real numbers, where*

$$\sum_{j=1}^{\tilde{t}} |a_{ij}| \leq A, \quad i = 1, \ldots, r \ r < \tilde{t}.$$

Then there exists a nontrivial collection of rational integers $x_1, \ldots, x_{\tilde{t}}$ for which

$$|a_{i1}x_1 + \ldots + a_{i\tilde{t}}x_{\tilde{t}}| \leq 2AC((C+1)^{\tilde{t}/r} - 2)^{-1}, \quad i = 1, \ldots, r,$$
$$|x_j| \leq C, \quad j = 1, \ldots, \tilde{t}.$$

LEMMA 2 [**2**, p. 50, Lemma 3]. *Let* ρ *be a root of unity,* ν *the degree of the field* $K=\mathbb{Q}(\rho, i)$ *and* $n=n(\xi)$ *the degree of* ξ, *where* $\xi\in K$; *further let* T, L, M *be non-negative integers. If the rational integers* A_{tlm} *satisfy the inequality*

$$\sum_{t=0}^{T}\sum_{l=0}^{L}\sum_{m=0}^{M}|A_{tlm}|\leq B,$$

then either $\delta=0$ *or*

$$|\delta|=\left|\sum_{t=0}^{T}\sum_{l=0}^{L}\sum_{m=0}^{M}A_{tlm}\xi^t i^l \rho^m\right|\geq B^{1-\nu}L(\xi)^{-\nu T/n}.$$

LEMMA 3 [**1**, p. 96, Lemma 3]. *Let* F *be an entire function, let* S *and* T *be positive integers, and let* R *and* $\tilde{A}$ *be real numbers such that* $R\geq 2S$ *and* $\tilde{A}>2$. *Then*

$$\max_{|z|\leq R}|F(z)|\leq 2(2/\tilde{A})^{TS}\max_{|z|\leq \tilde{A}R}|F(z)|+(9R/S)^{TS}\max_{s,t}|F^{(t)}(s)|/t!,$$

where the last maximum is taken over all integers s, t *with* $0\leq s<S$, $0\leq t<T$.

In the following, let T and M denote fixed positive integers and define the polynomials $g_m(z)$ $(m=0, 1, \ldots, M-1)$ as follows:

$$g_0(z)=1, \quad g_m(z)=\left(\left[\frac{m}{T}\right]!\right)^{-T}\prod_{j=1}^{m}(z+j) \quad (m=1, 2, \ldots, M-1),$$

where $T-1$ is the highest order of the derivatives one has to use.

LEMMA 4 [**1**, p. 96, Lemma 4]. *For* $t=0, 1, \ldots, T-1$,

$$|g_m^{(t)}(z)|/t!\leq \exp(|z|+2m+3m\log T).$$

There exists a positive integer d *such that all the numbers* $(d/t!)g_m^{(t)}(s)$ $(m=0, 1, \ldots, M-1, t=0, 1, \ldots, T-1, s=0, 1, 2, \ldots)$ *are integers and*

$$d\leq \exp(4M\log T).$$

LEMMA 5 [**1**, p. 96, Lemma 5]. *Let* a *be a non-zero complex number. Let* F *be the exponential polynomial*

$$F(z)=\sum_{k=0}^{K-1}\sum_{m=0}^{M-1}C_{km}g_m(z)e^{akz},$$

where the C_{km} *are complex numbers and* K, M *positive integers. Put*

$$\tilde{C}=\max_{k,m}|C_{km}|, \quad \Omega=\max(1, (K-1)\,|a|), \quad \omega=\min(1, |a|),$$

let S' *be a positive integer and define* E *by*

$$E=\max_{s,t}|F^{(t)}(s)|/t!,$$

where s, t are integers with $0 \leq s < S'$, $0 \leq t < T$. *If*

$$TS' \geq 2KM + 15\Omega S',$$

then

$$\tilde{C} \leq \tfrac{3}{2}(2e)^M (6/(\omega K))^{KM} 18^{TS'} E\{\max(6\Omega, 3KM/(4S'))\}^{KM}.$$

3. Proof of the theorem. Put $Y = \log \nu + \dfrac{\log L}{N}$ and suppose that

$$|\pi - \xi| \leq \exp\left(-x^{13} \frac{\nu^2}{\log \nu} Y\right); \tag{3}$$

we shall show that (3) leads to a contradiction if x (an integer) is large enough. Choose the following integers:

$$K = [x^3 \nu], \qquad S = \left[x^2 \frac{\nu}{\log \nu} Y\right], \qquad T = \left[x^6 \frac{\nu}{\log \nu}\right],$$

$$M = \left[x^6 \frac{\nu}{\log^2 \nu} Y\right], \qquad S' = x^2 S, \qquad C = \left[\exp\left\{x^7 \frac{\nu}{\log \nu} Y\right\}\right].$$

Let

$$F(z) = \sum_{k=0}^{K-1} \sum_{m=0}^{M-1} \sum_{j=0}^{\varphi(m_0)-1} C_{kmj} \rho^j g_m(z) e^{2k\pi i z/m_0},$$

where the C_{kmj} are rational integers with $|C_{kmj}| \leq C$, specified later.

In the following adopt the convention: $\binom{a}{b} = 0$ if $b > a$. For all non-negative integers s, t we have

$$F^{(t)}(s) = \sum_k \sum_m \sum_j C_{kmj} \rho^j \sum_{\tau=0}^{m} \binom{t}{\tau} g_m^{(\tau)}(s)(2\pi i/m_0)^{t-\tau} k^{t-\tau} \rho^{ks};$$

define

$$\phi_{ts} = \sum_k \sum_m \sum_j C_{kmj} \rho^j \sum_{\tau=0}^{m} \binom{t}{\tau} g_m^{(\tau)}(s)(2\xi i/m_0)^{t-\tau} k^{t-\tau} \rho^{ks}.$$

There are six steps in the proof.

(a) $|\pi^{t-\tau} - \xi^{t-\tau}| \leq T(\pi+1)^T |\pi - \xi| \leq \exp\left(-\dfrac{1}{2} x^{13} \dfrac{\nu^2}{\log \nu} Y\right)$,

and for $0 \leq t < T$, $0 \leq s < S'$,

$$|F^{(t)}(s) - \phi_{ts}| \leq KM^2 \nu C (4\pi TK)^T \exp\left(S' + 5M \log T - \frac{1}{2} x^{13} \frac{\nu^2}{\log \nu} Y\right)$$

$$\leq \exp\left(-x^{12} \frac{\nu^2}{\log \nu} Y\right). \tag{4}$$

(b) Let d be the integer introduced in Lemma 4 and put $C_{km}=\sum_{j=0}^{\varphi(m_0)-1} C_{kmj}\rho^j$. Define

$$\tilde{\phi}_{ts}=dm_0^T\phi_{ts}=dm_0^T\sum_{k,m} C_{km}\sum_{\tau=0}^{m}\binom{t}{\tau}g_m^{(\tau)}(s)(2\xi i/m_0)^{t-\tau}k^{t-\tau}\rho^{ks}$$

for $s=0,1,\ldots,S-1$, $t=0,1,\ldots,T-1$. Then $\text{Re}\,\tilde{\phi}_{ts}$, $\text{Im}\,\tilde{\phi}_{ts}$ can be considered as a system of linear forms of the $\tilde{t}=\varphi(m_0)KM$ parameters C_{kmj}. By Lemma 1 with $r=2TS$, there exist rational integers, not all zero, such that $|C_{kmj}|\leq C$ and

$$|\text{Re}\,\tilde{\phi}_{ts}|+|\text{Im}\,\tilde{\phi}_{ts}|\leq 4AC((C+1)^{\tilde{t}/r}-2)^{-1}.$$

In our case we get

$$\begin{aligned}A&\leq dm_0^T(M+1)(2T)^T(\exp(S+5M\log T))(2K)^T(1+\pi)^T\\&\leq\exp\left(x^6(\log^2 x)\frac{\nu}{\log\nu}Y\right).\end{aligned}$$

Hence

$$|\tilde{\phi}_{ts}|\leq\exp\left(2x^7\frac{\nu}{\log\nu}Y-\frac{3}{8}x^8\frac{\nu^2}{\log\nu}Y\right)\leq\exp\left(-x^7(\log x)\frac{\nu^2}{\log\nu}Y\right), \tag{5}$$

since $\tilde{t}/r\geq\frac{3}{8}x\nu$.

$\tilde{\phi}_{ts}$ is polynomial in ξ, i, ρ of degrees T, T, $(K-1)(S-1)+\phi(m_0)-1$, respectively, and with rational integer coefficients A_{kmj} satisfying

$$\sum_k\sum_m\sum_j|A_{kmj}|\leq\exp\left\{x^6(\log^2 x)\frac{\nu}{\log\nu}Y+x^7\frac{\nu}{\log\nu}Y\right\}\leq\exp\left(2x^7\frac{\nu}{\log\nu}Y\right).$$

Define $B=\exp\left(2x^7\frac{\nu}{\log\nu}Y\right)$. Then, applying Lemma 2, we obtain either $\tilde{\phi}_{ts}=0$, or

$$|\tilde{\phi}_{ts}|\geq\exp\left\{-\left((1-\nu)\log B+\frac{\nu T}{N}\log L\right)\right\}>\exp\left\{-\left(3x^7\frac{\nu^2}{\log\nu}Y\right)\right\},$$

which contradicts (5) for x sufficiently large. Hence $\tilde{\phi}_{ts}=0$, $\phi_{ts}=0$ and by (4)

$$|F^{(t)}(s)|\leq\exp\left(-x^{12}\frac{\nu^2}{\log\nu}Y\right)\quad(0\leq t<T,0\leq s<S). \tag{6}$$

(c) Now we apply Lemma 3 to $F(z)$ with $R=S'$ and choose $\tilde{A}$ comparatively large, namely $\tilde{A}=6\nu$. It follows, because

$$\max_{|z|\leq 6\nu S'}|F(z)|\leq KM\nu C\exp(6\nu S'+5M\log T+24\pi S'K)\leq\exp\left(x^7(\log x)\frac{\nu^2}{\log\nu}Y\right),$$

that, by (6),

$$\max_{|z|\le S'}|F(z)|\le 2(1/3\nu)^{TS}\max_{|z|\le 6\nu S'}|F(z)|+(9S'/S)^{TS}\max_{s,t}|F^{(t)}(s)|/t!$$

$$\le \exp\left\{-\frac{1}{2}x^8\frac{\nu^2}{\log\nu}Y+2x^7(\log x)\frac{\nu^2}{\log\nu}Y\right\}$$

$$+\exp\left\{x^8(\log^2 x)\left(\frac{\nu}{\log\nu}\right)^2 Y-x^{12}\frac{\nu^2}{\log\nu}Y\right\}\le\exp\left(-\frac{1}{4}x^8\frac{\nu^2}{\log\nu}Y\right). \quad (7)$$

Cauchy's theorem implies that, for $0\le t<T$, $0\le s<S'$,

$$|F^{(t)}(s)|\le T^TS'\max_{|z|\le S'}|F(z)|\le\exp\left(-\frac{1}{5}x^8\frac{\nu^2}{\log\nu}Y\right).$$

Using (4) and $d\le\exp(4M\log T)$ (with Lemma 3), we obtain

$$|\tilde{\phi}_{ts}|\le\exp\left(-\frac{1}{6}x^8\frac{\nu^2}{\log\nu}Y\right) \quad (8)$$

for $0\le t<T$, $0\le s<S'$.

(d) Applying Lemma 2 for $0\le t<T$, $0\le s<S'$, we see by similar considerations to those in (b) with S' replacing S that either $\tilde{\phi}_{ts}=0$ or

$$|\tilde{\phi}_{ts}|\ge\exp\left(-x^7(\log x)\frac{\nu^2}{\log\nu}Y\right),$$

a contradiction to (8) for x large. Hence $\phi_{ts}=0$ and by (4)

$$|F^{(t)}(s)|\le\exp\left(-x^{12}\frac{\nu^2}{\log\nu}Y\right) \quad (9)$$

for $0\le t<T$, $0\le s<S'$.

(e) We can now apply Lemma 5 with $a=2\pi i/m_0$. We have

$$\Omega=\max(1,(K-1)|a|)<1+(2\pi K/m_0)\le x^3\log x.$$

Hence

$$TS'\ge 2KM+15\Omega S'.$$

Further from

$$\frac{m_0}{\log\log m_0}\le c_3\varphi(m_0)\le c_4\nu$$

with c_3, c_4 absolute constants, we obtain $m_0\le x\nu^2$, Also

$$\omega=\min(1,|a|)=\min(1,2\pi/m_0)\ge(x\nu^2)^{-1}$$

and so

$$\omega K \geq \frac{1}{2} x^3 \nu (x\nu^2)^{-1} \geq (2\nu)^{-1}.$$

Hence

$$(6/\omega K)^{KM} \leq (12\nu)^{KM} \leq \exp\left(x^9 (\log x) \frac{\nu^2}{\log \nu} Y\right).$$

Therefore, by (9), it follows that

$$\begin{aligned} \tilde{C} &\leq \exp\left\{x^6 (\log x) \frac{\nu}{\log^2 \nu} Y + x^9 (\log x) \frac{\nu^2}{\log \nu} Y + x^{10} (\log x) \left(\frac{\nu}{\log \nu}\right)^2 Y \right. \\ &\quad \left. + x^9 \left((\log^2 x)\left(\frac{\nu}{\log \nu}\right)^2 \log \nu\right) Y - x^{12} \frac{\nu^2}{\log \nu} Y\right\} \\ &\leq \exp\left(-x^{11} \frac{\nu^2}{\log \nu} Y\right). \end{aligned} \tag{10}$$

(f) Finally, the C_{km} are polynomials in ρ with rational integer coefficients; hence we have $C_{km} = 0$ or, by Lemma 2,

$$|C_{km}| \geq (\varphi(m_0) C)^{1-\nu} \geq \exp\left(-x^7 (\log x) \frac{\nu^2}{\log \nu} Y\right),$$

which contradicts (10) for x sufficiently large. So

$$C_{km} = 0 \quad (k = 0, 1, \ldots, K-1,\ m = 0, 1, \ldots, M-1).$$

But ρ is an algebraic number of degree $\varphi(m_0)$; hence

$$C_{kmj} = 0 \quad (0 \leq k < K,\ 0 \leq m < M,\ 0 \leq j \leq \varphi(m_0) - 1),$$

which gives a contradiction to the choice of the integers C_{kmj}. Thus (3) is impossible for x large enough and the theorem is proved.

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