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Strichartz estimates for orthonormal families of initial data and weighted oscillatory integral estimates

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Abstract

We establish new Strichartz estimates for orthonormal families of initial data in the case of the wave, Klein–Gordon and fractional Schrödinger equations. Our estimates extend those of Frank–Sabin in the case of the wave and Klein–Gordon equations, and generalize work of Frank et al. and Frank–Sabin for the Schrödinger equation. Due to a certain technical barrier, except for the classical Schrödinger equation, the Strichartz estimates for orthonormal families of initial data have not previously been established up to the sharp summability exponents in the full range of admissible pairs. We obtain the optimal estimates in various notable cases and improve the previous results.

The main novelty of this paper is our derivation and use of estimates for weighted oscillatory integrals, which we combine with an approach due to Frank and Sabin. Our weighted oscillatory integral estimates are, in a certain sense, rather delicate endpoint versions of known dispersive estimates with power-type weights of the form $|\xi|^{-\lambda}$ or $(1 + |\xi|^2)^{-\lambda/2}$, where $\lambda \in \mathbb{R}$. We achieve optimal decay rates by considering such weights with appropriate $\lambda \in \mathbb{C}$. For the wave and Klein–Gordon equations, our weighted oscillatory integral estimates are new. For the fractional Schrödinger equation, our results overlap with prior work of Kenig–Ponce–Vega in a certain regime. Our contribution to the theory of weighted oscillatory integrals has also been influenced by earlier work of Carbery–Ziesler, Cowling et al., and Sogge–Stein.

Finally, we provide some applications of our new Strichartz estimates for orthonormal families of data to the theory of infinite systems of Hartree type, weighted velocity averaging lemmas for kinetic transport equations, and refined Strichartz estimates for data in Besov spaces.

Contents

1. Introduction

For a given dispersion relation ϕ, the function $U_\phi f$ denotes the solution to the initial value problem

$$\begin{cases} i\partial_t u + \phi(D)u = 0, & (t,x) \in \mathbb{R}^{1+d}, \ \ d \geq 1, \\ u(0,\cdot) = f. \end{cases}$$

This paper is concerned with extended versions of Strichartz estimates taking the form

$$\left\| \sum_j \nu_j |U_\phi f_j|^2 \right\|_{L_t^{\frac{q}{2}} L_x^{\frac{r}{2}}} \lesssim \|\nu\|_{\ell^\beta} \tag{1.1}$$

for families of orthonormal functions $(f_j)_j$ in a given Hilbert space $\mathcal{H}$, which we shall take to be homogeneous or inhomogeneous Sobolev spaces. This particular line of investigation originated in recent work of Frank et al. [29] for the Schrödinger propagator ($\phi(\xi) = |\xi|^2$). The idea to generalize classical inequalities from a single-function input to an orthonormal family traces back further, with pioneering work of Lieb–Thirring [57] establishing extended versions of certain Gagliardo–Nirenberg–Sobolev inequalities and applications to the stability of matter. We also mention Lieb's extended version of Sobolev inequalities in [56], which will be of use to us in the current paper.

    Motivation to study estimates of the form (1.1) is plentiful. Applications to the Hartree equation modelling infinitely many fermions in a quantum system can be found in work of Chen–Hong–Pavlovic [19, 20], Frank–Sabin [30], Lewin–Sabin [54, 55] and Sabin [69]. Physically, the quantity $\sum_{j=1}^N |e^{it\Delta} f_j|^2$ gives a representation of the density of a quantum system of N fermions (at time t), where f_j represents the jth fermion at the initial time, and thus it is desirable to have optimal control over such a quantity in terms of the number of particles N; this corresponds to obtaining bounds of the form (1.1) with β as large as possible. Other applications include consequences for the wave operator for time-dependent potentials, which may be found in [29]. In a somewhat different direction, one may obtain refined versions of the classical (single-function) Strichartz estimates for data in certain Besov spaces rather quickly from (1.1) via the Littlewood–Paley inequality; this observation may be found in [31] and provides an approach to refined Strichartz inequalities, which is distinct and rather simpler than the more well-known approach via bilinear Fourier (adjoint) restriction estimates. In addition to the papers cited already, we refer the reader forward to Section 7, where we present several applications of the nature described above; these applications will be consequences of the new estimates we obtain in the current paper and on which we now begin to focus.

The classical (single-function) Strichartz estimates enjoy a rather general theory, which allows for a fairly unified approach to a wide class of dispersive equations such as the Schrödinger equation $\phi(\xi) = |\xi|^2$ or, more generally, the fractional Schrödinger equation $\phi(\xi) = |\xi|^\alpha$ ($\alpha \neq 0, 1$), the wave equation $\phi(\xi) = |\xi|$ and the Klein–Gordon equation $\phi(\xi) = \langle\xi\rangle$, where $\langle\xi\rangle := (1 + |\xi|^2)^{1/2}$. In the extended framework (1.1), matters are significantly more complicated, and a comparable general theory seems rather distant. In this paper, we provide substantial progress in this direction, generalizing work in [29] to the fractional Schrödinger case and significantly extending the results in [30] for the wave and Klein–Gordon equations. Our approach is, to a certain extent, based on an abstract framework, and we expect that similar results can be obtained by our approach for broader classes of dispersion relations. To facilitate the presentation of prior results and our new results in the extended framework (1.1), we first review the classical (single-function) case.

1.1. Classical Strichartz estimates

Let $\mathcal{H}$ be a Hilbert space, such as $\dot{H}^s$ or H^s, the homogeneous or inhomogeneous Sobolev spaces of order s built over $L^2(\mathbb{R}^d)$. (For the definitions of $\dot{H}^s$, H^s and $\phi(D)$, we refer the reader forward to Section 2.) The classical Strichartz estimates usually take the form

$$\|U_\phi f\|_{L_t^q L_x^r} \lesssim \|f\|_{\mathcal{H}}. \tag{1.2}$$

Since ϕ is typically smooth away from the origin, a commonly used technique to prove the estimate (1.2) is first to consider the case that the Fourier (frequency) support of the initial data f is localized to an annulus and then apply Littlewood–Paley theory to extend to general data in the class $\mathcal{H}$. This strategy allows us to avoid complications that arise from singular behavior of the dispersive relation ϕ near the origin and at infinity. For initial data with such localized frequency, one can prove estimates of the form (1.2) if a dispersive estimate such as

$$\sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{i(x\cdot\xi + t\phi(\xi))} \chi_0(|\xi|) \, d\xi \right| \lesssim (1 + |t|)^{-\sigma} \tag{1.3}$$

holds, where χ_0 is a bump function with support away from zero and $\sigma > 0$. More precisely, for $\sigma > 0$, we say that $(q, r) \in [2, \infty) \times [2, \infty)$ is σ-admissible if

$$\frac{1}{q} \leq \sigma\left(\frac{1}{2} - \frac{1}{r}\right).$$

In the case of equality

$$\frac{1}{q} = \sigma\left(\frac{1}{2} - \frac{1}{r}\right),$$

we say that (q, r) is *sharp σ-admissible*, and in the case of strict inequality

$$\frac{1}{q} < \sigma\left(\frac{1}{2} - \frac{1}{r}\right)$$

we say that (q, r) is *non-sharp σ-admissible*. It follows from work of Keel–Tao [48] that (1.2) holds for frequency localized f whenever we have the dispersive estimate (1.3) and (q, r) is σ-admissible.

We remark that pairs (q, ∞) and (∞, r) are not included in our definition of admissible pairs. Estimates of the form (1.2) are available for certain pairs of this type; indeed, such an estimate obviously holds if $\mathcal{H} = L^2$ and $(q, r) = (\infty, 2)$. The case $r = \infty$, in particular, requires special attention. For example, in the case of the wave equation $\phi(\xi) = |\xi|$, it was shown by Fang–Wang [28] that (1.2) fails with $\mathcal{H} = \dot{H}^{\frac{3}{4}}$ when $(q, r, d) = (4, \infty, 2)$, and it was shown in [59] and [36] that (1.2) fails with $\mathcal{H} = \dot{H}^{\frac{d-1}{2}}$

when $(q, r) = (2, \infty)$, for $d = 3$ and $d \geq 4$, respectively. For the Schrödinger equation $\phi(\xi) = |\xi|^2$, the situation is slightly different; (1.2) holds with $\mathcal{H} = L^2$ when $(q, r, d) = (4, \infty, 1)$ and fails with $\mathcal{H} = \dot{H}^{\frac{d-2}{2}}$ when $(q, r) = (2, \infty)$ for $d \geq 2$ (see [59] and [36]).

The fractional Schrödinger equation

First, we consider the case of the fractional Schrödinger equation $\phi(\xi) = |\xi|^\alpha$, where $\alpha \neq 0, 1$:

$$i\partial_t u = (-\Delta)^{\alpha/2} u, \qquad u(0) = u_0. \tag{1.4}$$

If $\alpha = 2$, this corresponds to the well-known Schrödinger equation. The fractional Schrödinger equation (1.4) with $\alpha \in (0, 2) \setminus \{1\}$ appears in [51, 52] as a consequence of generalizing Feynman's path integral arising in Brownian motion to the one generated by Lévy motion, as well as in [44] as a model of water waves. For further studies on the fractional Schrödinger equation, we refer the reader to [24, 21, 38, 42, 47, 49]; and for the case $\alpha = 4$, to which special attention has been paid, see [45, 46, 65, 66].

If $\alpha \in \mathbb{R} \setminus \{0, 1\}$, for (q, r) that are $\frac{d}{2}$-admissible, the classical Strichartz estimate

$$\|e^{it(-\Delta)^{\alpha/2}} f\|_{L_t^q L_x^r} \lesssim \|f\|_{\dot{H}^s}, \quad s = \frac{d}{2} - \frac{d}{r} - \frac{\alpha}{q} \tag{1.5}$$

holds (see, for example, [25]). For $s \leq -\frac{d}{2}$, the space $\dot{H}^s$ does not admit natural classes of dense functions such as the Schwartz class, so we restrict our attention to the case where $\frac{d}{r} + \frac{\alpha}{q} < d$. Note that for the classical Schrödinger equation with $\alpha = 2$, we have $s \in [0, \frac{d}{2})$ whenever (q, r) is $\frac{d}{2}$-admissible, so the additional assumption $\frac{d}{r} + \frac{\alpha}{q} < d$ is automatically satisfied in this case.

The wave equation

It is well-known that the solution of the wave equation

$$\partial_{tt} u = \Delta u, \qquad (u(0), \partial_t u(0)) = (u_0, u_1)$$

can be written as $u = u_+ + u_-$, where u_+ and u_- are given by

$$u_\pm(x, t) = e^{\pm it\sqrt{-\Delta}} f_\pm(x),$$

and f_+ and f_- satisfy $(f_+ + f_-, i\sqrt{-\Delta}(f_+ - f_-)) = (u_0, u_1)$. As a result, the Strichartz estimates for the wave equation are usually given by those for the one-sided propagator $e^{\pm it\sqrt{-\Delta}}$. For $d \geq 2$, it is known that the estimate

$$\|e^{it\sqrt{-\Delta}} f\|_{L_t^q L_x^r} \lesssim \|f\|_{\dot{H}^s}, \quad s = \frac{d}{2} - \frac{d}{r} - \frac{1}{q} \tag{1.6}$$

holds if (q, r) is $\frac{d-1}{2}$-admissible. By an elementary scaling argument, one may easily verify that (1.6) fails for other values of s. We also remark that if (q, r) is sharp $\frac{d-1}{2}$-admissible, then we have $s = \frac{d+1}{2}\left(\frac{1}{2} - \frac{1}{r}\right)$. In the case where (q, r) is sharp $\frac{d-1}{2}$-admissible and $q = r$, the Strichartz estimate (1.6) becomes

$$\|e^{it\sqrt{-\Delta}} f\|_{L_{x,t}^{\frac{2(d+1)}{d-1}}} \lesssim \|f\|_{\dot{H}^{1/2}}, \tag{1.7}$$

which basically corresponds to the Stein–Tomas adjoint restriction estimate for the cone.

The Klein–Gordon equation

In a similar manner to the wave equation, the solution of the Klein–Gordon equation

$$\partial_{tt}u + u = \Delta u, \qquad (u(0), \partial_t u(0)) = (u_0, u_1)$$

decomposes into a sum of waves given by the propagators $e^{\pm it\sqrt{1-\Delta}}$, and therefore the Strichartz estimates usually take the form

$$\|e^{it\sqrt{1-\Delta}} f\|_{L_t^q L_x^r} \lesssim \|f\|_{H^s}. \tag{1.8}$$

The propagator $e^{it\sqrt{1-\Delta}}$ possesses properties of a mixed nature. With high frequency initial data, the behavior resembles the wave propagator $e^{it\sqrt{-\Delta}}$, whereas the mapping property of $e^{it\sqrt{1-\Delta}}$ is similar to that of $e^{it\Delta}$ in the low-frequency regime. This is related to the fact that the surface $(\xi, \langle\xi\rangle)$ has nonvanishing Gaussian curvature near the origin while the surface gets close to the cone $(\xi, |\xi|)$ as $|\xi| \to \infty$. In fact, if $s = \frac{d}{2} - \frac{d}{r} - \frac{1}{q}$, it is possible to deduce the estimate (1.6) from (1.8) via Littlewood–Paley theory, scaling and a limiting argument. Similarly, it is not difficult to obtain the Strichartz estimate for $e^{it\Delta}$ if (1.8) holds for sharp $\frac{d}{2}$-admissible (q, r).

For $d \geq 2$, it is known that (1.8) holds if (q, r) is sharp $\frac{d-1}{2}$-admissible and

$$s \geq \frac{d+1}{2}\left(\frac{1}{2} - \frac{1}{r}\right). \tag{1.9}$$

Testing (1.8) on a Knapp-type example reveals that this range of s cannot be enlarged. Also, for $d \geq 1$, it is known that (1.8) holds if (q, r) is sharp $\frac{d}{2}$-admissible and

$$s \geq \frac{d+2}{2}\left(\frac{1}{2} - \frac{1}{r}\right), \tag{1.10}$$

which, again, may be shown to be the optimal range of s for such (q, r).

1.2. Strichartz estimates for orthonormal functions – known results

In this work, we are concerned with Strichartz estimates for orthonormal families of initial data (1.1). As mentioned before, the key point is to make the exponent β as large as possible. Indeed, the case $\beta = 1$ is equivalent to the classical Strichartz estimates: clearly, (1.2) implies (1.1) via the triangle inequality, and conversely, (1.1) trivially implies (1.2) by taking the family $(f_j)_j$ to consist of a single function f with unit norm in $\mathcal{H}$.

For the Schrödinger equation, contributions in [29, 30, 31] mean the sharp value of β has been obtained whenever (q, r) is sharp $\frac{d}{2}$-admissible.

Theorem 1 [29, 30, 31]. *Let $d \geq 1$, and suppose (q, r) is sharp $\frac{d}{2}$-admissible.*

(i) *If $2 \leq r < \frac{2(d+1)}{d-1}$, then*

$$\left\|\sum_j \nu_j |e^{it\Delta} f_j|^2\right\|_{L_t^{\frac{q}{2}} L_x^{\frac{r}{2}}} \lesssim \|\nu\|_{\ell^\beta} \tag{1.11}$$

holds for all families of orthonormal functions $(f_j)_j$ in L^2 and $\beta = \frac{2r}{r+2}$. This estimate is sharp in the sense that the estimate fails for $\beta > \frac{2r}{r+1}$.

(ii) *If $d = 2$ and $6 \leq r < \infty$, or if $d \geq 3$ and $\frac{2(d+1)}{d-1} \leq r \leq \frac{2d}{d-2}$, then (1.11) holds for all families of orthonormal functions $(f_j)_j$ in L^2 and $\beta < \frac{q}{2}$. This estimate is sharp in the sense that the estimate fails for $\beta > \frac{q}{2}$.*

A remarkable phenomena here is that, for $d \geq 3$, the sharp value of β for the estimate (1.11) coincides with one (i.e. the trivial case in light of the remarks prior to the above theorem) at the endpoints of the sharp $\frac{d}{2}$-admissible line: that is, $(q, r) = (\infty, 2)$ and $(q, r) = (2, \frac{2d}{d-2})$, which, respectively, correspond to the trivial energy conservation estimate and the Keel–Tao endpoint estimate [48]. Also, the sharp value of β reaches its maximum at the point $(q, r) = (\frac{2(d+1)}{d}, \frac{2(d+1)}{d-1})$. In fact, as pointed out in [31], the estimates in (*ii*) follow from those in (*i*) by interpolation between $(q, r) = (2, \frac{2d}{d-2})$ and points arbitrarily close to $(q, r) = (\frac{2(d+1)}{d}, \frac{2(d+1)}{d-1})$; in this sense, $(q, r) = (\frac{2(d+1)}{d}, \frac{2(d+1)}{d-1})$ may be considered as an endpoint case of the estimate (1.11). It remains open whether one can establish a suitable estimate in weaker form with $(q, r) = (\frac{2(d+1)}{d}, \frac{2(d+1)}{d-1})^1$ so that other sharp estimates can be recovered from it by interpolation. See [5] for discussion on failure of such endpoint estimates in Lorentz spaces.

We also note that the sharp value of β has been obtained for the Schrödinger equation whenever (q, r) is non-sharp $\frac{d}{2}$-admissible; see [5]. In this case, f_j are assumed to be contained in $\dot{H}^s$ with $s = \frac{d}{2} - \frac{d}{r} - \frac{2}{q}$. See, for example, Theorem 4. As we shall see below, in general, the sharp admissible case will play a crucial role in establishing the estimates in the non-sharp admissible case. Therefore, in this introductory section, we only present our new results for the sharp admissible cases; the statements for the non-sharp cases will appear in Section 5.

The work of Frank–Sabin [30] also contains results for estimates of the form (1.1) for the wave equation and the Klein–Gordon equation; however, as we shall see below, these are not as advanced as the results contained in Theorem 1 for the Schrödinger equation.

For the wave equation, Frank–Sabin [30] obtained a substantial generalization of (1.7) for orthonormal functions of initial data in $\dot{H}^{1/2}$. By interpolation with a trivial estimate in the case $(q, r) = (\infty, 2)$, we state their result as follows.

Theorem 2 [30]. *Let $d \geq 2$. Suppose (q, r) is sharp $\frac{d-1}{2}$-admissible and $2 \leq r \leq \frac{2(d+1)}{d-1}$. Then for all families of orthonormal functions $(f_j)_j$ in $\dot{H}^s$, with $s = \frac{d+1}{2}(\frac{1}{2} - \frac{1}{r})$ and $\beta = \frac{2r}{r+2}$, the following estimate holds:*

$$\left\| \sum_j \nu_j |e^{it\sqrt{-\Delta}} f_j|^2 \right\|_{L_t^{\frac{q}{2}} L_x^{\frac{r}{2}}} \lesssim \|\nu\|_{\ell^\beta}. \tag{1.12}$$

For the Klein–Gordon equation, Frank and Sabin [30] established the following.

Theorem 3 [30]. *Let $\sigma > 0$, and suppose (q, r) is sharp σ-admissible.*

(i) *If $\sigma = \frac{d-1}{2}$ and $2 \leq r \leq \frac{2(d+1)}{d-1}$, for all families of orthonormal functions $(f_j)_j$ in H^s, $s = \frac{d+1}{2}(\frac{1}{2} - \frac{1}{r})$, the following estimate holds:*

$$\left\| \sum_j \nu_j |e^{it\sqrt{1-\Delta}} f_j|^2 \right\|_{L_t^{\frac{q}{2}} L_x^{\frac{r}{2}}} \lesssim \|\nu\|_{\ell^\beta}, \quad \beta = \frac{2r}{r+2}. \tag{1.13}$$

(ii) *If $\sigma = \frac{d}{2}$ and $2 \leq r \leq \frac{2(d+2)}{d}$, then (1.13) holds for all families of orthonormal functions $(f_j)_j$ in H^s, with $s = \frac{d+2}{2}(\frac{1}{2} - \frac{1}{r})$ and $\beta = \frac{2r}{r+2}$.*

In both Theorems 2 and 3, the range of exponents only goes up to the diagonal case $q = r$, and the other estimates with $q < r$ are not sharp with respect to the summability exponent β. This is due to the fact that their argument relies on a special property that is available only when $q = r$, and it is clear that their argument does not extend beyond the diagonal case. More precisely, in [30] the authors followed the original idea of Strichartz [74], and they regarded the evolution operators as adjoint restriction operators given by a measure supported on the associated surface (the cone and hyperboloid, respectively, for the

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[1]When $d = 1$, we have $(q, r) = (4, \infty)$. We refer the reader to [6] for some recent results in such a case.

wave and Klein–Gordon equations). For instance, let S denote the upward cone

$$S = \{(\xi, \tau) \in \mathbb{R}^d \times \mathbb{R} : \tau = |\xi|\}.$$

Then $e^{it\sqrt{-\Delta}} f(x)$ can be written in the form $\widehat{g\,d\mu}(x, t)$, where $d\mu(\xi, \tau) = \frac{\delta(\tau - |\xi|)}{|\xi|}$ and $g(\xi, \tau) = |\xi| \widehat{f}(\xi)$. With this notation, Strichartz [74] proved

$$\|\widehat{g\,d\mu}\|_{L_{x,t}^{\frac{2(d+1)}{d-1}}} \lesssim \|g\|_{L^2(S, d\mu)}, \tag{1.14}$$

and from this, (1.7) trivially follows. The proof of (1.14) in [74] used an argument based on interpolation involving an analytic family of operators of the form $T_z f = {}^\vee G_z * f$, where G_z is chosen suitably depending on $d\mu$. The key oscillatory integral estimates on the kernel of these operators relies on a rather delicate identity for ${}^\vee G_z$ whose validity seems tightly connected to the choice of measure $d\mu$ above. Frank and Sabin's clever observation was that the same basic ingredients could be used to derive (1.12) in Theorem 2. For (q, r) beyond the diagonal case $q = r$ (where $s = \frac{1}{2}$), however, it is necessary to handle data with higher regularity, and thus, roughly speaking, one would like to replace $d\mu$ above with a more singular version of the form $\frac{\delta(\tau - |\xi|)}{|\xi|^a}$ $(a > 1)$. This causes significant technical difficulty in getting the appropriate kernel estimates; for example, no explicit identity seems available away from the case $a = 1$.

Our main contribution to overcome the aforementioned difficulty is to find an appropriate measure and establish corresponding weighted oscillatory integral estimates of so-called *damped–type* with optimal decay rates (see Section 3 for further details). These oscillatory integral estimates then yield kernel estimates for a suitable analytic family of operators. Our new idea is sufficiently robust to allow us to significantly improve upon the above results for the wave and Klein–Gordon equations, as well as the fractional Schrödinger equation. Below, we describe our main new results for the sharp admissible case; we also obtain new results in the non-sharp admissible case, but for reasons we have already mentioned, we postpone our presentation of these results to Section 5.

Remark 1. As an alternative approach to obtain the Strichartz estimate (1.1) for orthonormal data, one might attempt to adapt the typical strategy for the single-function case based on Littlewood–Paley theory. That is to say, first prove the estimate for data with localized frequency on dyadic annuli and then put together the estimates for each dyadic piece. However, such an approach does not seem to be effective in the case of orthonormal data. Indeed, in contrast with the classical Strichartz estimate, for (1.1) there are cases where the frequency localized estimate cannot be upgraded to the frequency global estimate; see [5] for more detail. In particular, our new approach to the estimate (1.1) for the fractional Schrödinger ($\alpha \neq 2$), wave and Klein–Gordon equations provides a novel method of establishing the classical Strichartz estimates (1.4), (1.6) and (1.8) that completely avoids the use of Littlewood–Paley theory.

1.3. Main new results

As already mentioned above, our primary aim is to extend Theorems 2 and 3 to all cases where (q, r) is sharp $\frac{d-1}{2}$-admissible or sharp $\frac{d}{2}$-admissible, as well as generalizing Theorem 1 to the fractional Schrödinger equation.

We first present the result for the fractional Schrödinger equation, which essentially gives the sharp value of β in all cases. As far as we are aware, except for the case $\alpha = 2$ in Theorem 1, there are no estimates of the type (1.1) in the existing literature for the fractional Schrödinger equation. From (1.5), we note that $(f_j)_j$ should be in $\dot{H}^s$ with $s = \frac{d(2-\alpha)}{2}(\frac{1}{2} - \frac{1}{r})$ if (q, r) is sharp $\frac{d}{2}$-admissible.

Theorem 4 (The fractional Schrödinger equation; sharp $\frac{d}{2}$-admissible)**.** *Let* $\alpha \in \mathbb{R} \setminus \{0, 1\}$ *and* $d \geq 1$. *Suppose* (q, r) *is sharp* $\frac{d}{2}$-*admissible with* $\frac{d}{r} + \frac{\alpha}{q} < d$.

(i) If $2 \leq r < \frac{2(d+1)}{d-1}$, for all families of orthonormal functions $(f_j)_j$ in $\dot{H}^s$,

$$\left\| \sum_j \nu_j |e^{it(-\Delta)^{\alpha/2}} f_j|^2 \right\|_{L_t^{\frac{q}{2}} L_x^{\frac{r}{2}}} \lesssim \|\nu\|_{\ell^\beta} \tag{1.15}$$

   holds with $s = \frac{d(2-\alpha)}{2}(\frac{1}{2} - \frac{1}{r})$, and $\beta = \frac{2r}{r+2}$. This estimate is sharp in the sense that the estimate fails for $\beta > \frac{2r}{r+2}$.

(ii) If $d = 2$ and $6 \leq r < \infty$, or if $d \geq 3$ and $\frac{2(d+1)}{d-1} \leq r \leq \frac{2d}{d-2}$, then (1.15) holds for all families of orthonormal functions $(f_j)_j$ in $\dot{H}^s$, with $s = \frac{d(2-\alpha)}{2}(\frac{1}{2} - \frac{1}{r})$, and $\beta < \frac{q}{2}$. This estimate is sharp in the sense that the estimate fails for $\beta > \frac{q}{2}$.

We next give our result for the wave equation, which significantly improves Theorem 2 when $q \leq r$. In this case, we note from (1.6) that it is necessary to consider orthonormal families $(f_j)_j$ in $\dot{H}^s$, $s = \frac{d+1}{2}(\frac{1}{2} - \frac{1}{r})$ whenever (q, r) is sharp $\frac{d-1}{2}$-admissible.

Theorem 5 (The wave equation; sharp $\frac{d-1}{2}$- admissible). *Let $d \geq 2$, and suppose (q,r) is sharp $\frac{d-1}{2}$-admissible.*

(i) *If $2 \leq r < \frac{2d}{d-2}$, then (1.12) holds for all families of orthonormal functions $(f_j)_j$ in $\dot{H}^s$, with $s = \frac{d+1}{2}(\frac{1}{2} - \frac{1}{r})$, and $\beta = \frac{2r}{r+2}$.*

(ii) *If $d = 3$ and $6 \leq r < \infty$, or if $d \geq 4$ and $\frac{2d}{d-2} \leq r \leq \frac{2(d-1)}{d-3}$, then (1.12) holds for all families of orthonormal functions $(f_j)_j$ in $\dot{H}^s$, with $s = \frac{d+1}{2}(\frac{1}{2} - \frac{1}{r})$, and $\beta < \frac{q}{2}$. This estimate is sharp in the sense that the estimate fails for $\beta > \frac{q}{2}$.*

Our main result regarding the Klein–Gordon equation in the cases $\sigma = \frac{d}{2}$ and $\sigma = \frac{d-1}{2}$ are described in the following two theorems.

Theorem 6 (The Klein–Gordon equation; sharp $\frac{d}{2}$-admissible). *Let $d \geq 1$, and suppose (q,r) is sharp $\frac{d}{2}$-admissible.*

(i) *If $2 \leq r < \frac{2(d+1)}{d-1}$, then (1.13) holds for all families of orthonormal functions $(f_j)_j$ in H^s, with $s \geq \frac{d+2}{2}(\frac{1}{2} - \frac{1}{r})$, and $\beta = \frac{2r}{r+2}$. This is sharp in the sense that the estimate fails if $\beta > \frac{2r}{r+2}$.*

(ii) *If $d = 2$ and $6 \leq r < \infty$, or if $d \geq 3$ and $\frac{2(d+1)}{d-1} \leq r \leq \frac{2d}{d-2}$, then (1.13) holds for all families of orthonormal functions $(f_j)_j$ in H^s, with $s \geq \frac{d+2}{2}(\frac{1}{2} - \frac{1}{r})$, and $\beta < \frac{q}{2}$. This estimate is sharp in the sense that the estimate fails for $\beta > \frac{q}{2}$.*

Theorem 7 (The Klein–Gordon equation; sharp $\frac{d-1}{2}$-admissible). *Let $d \geq 2$, and suppose (q,r) is sharp $\frac{d-1}{2}$-admissible.*

(i) *If $2 \leq r < \frac{2d}{d-2}$, then (1.13) holds for all families of orthonormal functions $(f_j)_j$ in H^s, with $s \geq \frac{d+1}{2}(\frac{1}{2} - \frac{1}{r})$, and $\beta = \frac{2r}{r+2}$.*

(ii) *If $d = 3$ and $6 \leq r < \infty$, or if $d \geq 4$ and $\frac{2d}{d-2} \leq r \leq \frac{2(d-1)}{d-3}$, then (1.13) holds for all families of orthonormal functions $(f_j)_j$ in H^s, with $s \geq \frac{d+1}{2}(\frac{1}{2} - \frac{1}{r})$, and $\beta < \frac{q}{2}$. This estimate is sharp in the sense that the estimate fails for $\beta > \frac{q}{2}$.*

It is also possible to unify the cases $\sigma = \frac{d-1}{2}x, \frac{d}{2}$ into the case $\sigma = \frac{d-1}{2} + \rho$, where $\rho \in [0, \frac{1}{2}]$. Indeed, Frank and Sabin [30] obtained a more general result than Theorem 3 corresponding to $\rho \in [0, \frac{1}{2}]$ ($d \geq 2$) and $\rho \in (0, \frac{1}{2}]$ ($d = 1$). For simplicity of the exposition, we only consider the cases $\rho = 0, \frac{1}{2}$ in Theorem 3, Theorem 6, and Theorem 7; we refer the reader forward to Section 4 for discussion of the more general case, including a result for the sharp $(\frac{d-1}{2} + \rho)$-admissible case that completely includes the result in [30].

We shall prove two necessary conditions in Section 6 that justify the claims in the above theorems regarding the sharpness of the range of β. Left open is to show that the exponent $\beta = \frac{2r}{r+2}$ in (*i*) of Theorems 5 and 7 is sharp. Our examples show that this exponent is sharp in the extreme cases $r = 2$ and $r = \frac{2d}{d-2}$, and thus we expect that this is the sharp value for intermediate values of r too.

1.4. Estimates for oscillatory integrals with weights

As we have already mentioned, the new idea in the proof of our main theorems is to make use of estimates for oscillatory integrals with weights, or damping factors, which will be combined with an interpolation argument based on a suitable analytic family of operators. See Proposition 7, where we formalize our approach under the assumption that we already have the desired oscillatory integral estimate. In fact, oscillatory integrals with weights naturally arise by absorbing to new operators the multiplier operators $|D|^s$ and $\langle D \rangle^s$ defining the Sobolev spaces. To obtain the appropriate dispersive estimates for these operators, we need to show the optimal decay estimates for the associated oscillatory integrals with weights. Interestingly, in some cases, such oscillatory integrals turn out to be oscillatory integrals with damping factors [26, 49, 12].

In Section 3, we prove various estimates for oscillatory integrals of the form

$$I^\phi(w) = \int_{\mathbb{R}^d} e^{i(x \cdot \xi + t\phi(\xi))} a(\xi) \, w(\xi) \, d\xi, \tag{1.16}$$

for a suitable choice of the cutoff function a and the weight function w, which we need to choose properly according to our particular purposes. Indeed, it will be crucial for proving Theorems 4–7 that our choice of a and w allows us to recover the optimal decay rate in t. Oscillatory integrals of the form $I^\phi(w)$ are ubiquitous in analysis, and often the weight is chosen to be a power of the determinant of the Hessian matrix of ϕ, denoted by $\det H\phi$, which mitigates bad behavior near degeneracies. For the purpose of our applications to Strichartz estimates for orthonormal families of data, we will choose $w(\xi) = |\xi|^z$ or $\langle \xi \rangle^z$, for certain $z \in \mathbb{C}$, which may not necessarily be the form of $|\det H\phi|^z$. Since the Hessian vanishes in the case of the wave equation, this case provides an example of the case where it will be *necessary* to consider weights not belonging to the typical class of weights that are powers of the Hessian. Our new results concerning such estimates are Propositions 3, 4, 5 and 6. We believe that these weighted oscillatory estimates are of independent interest, and we provide further contextual remarks in the beginning of Section 3.

We end this section with an overview of our paper. In Section 2, we introduce some notation and key facts that will be used throughout the paper. This includes the duality principle from [30], which rephrases Strichartz estimates for orthonormal families of initial data in terms of certain Schatten space estimates. In Section 3, we state and prove the weighted oscillatory integral estimates that are key to our proof of those Schatten space estimates corresponding to the main theorems in this paper. In Section 4, we prove the sufficiency claims in Theorems 4, 5, 6 and 7 (sharp admissible cases), and in Section 5, we prove analogous results in the non-sharp admissible case. The necessity claims in both the sharp and non-sharp admissible cases all follow from the necessary conditions that we will establish in Section 6. Finally, in Section 7, we present several applications of our new estimates to the theory of infinite systems of Hartree type, weighted velocity averaging lemmas for kinetic transport equations, and refined Strichartz estimates.

2. Notation and preliminaries

For $A, B > 0$, and $\rho \in \mathbb{R}$, by $A \lesssim_\rho B$ we mean that $A \le C(1 + |\rho|)^c B$ for some constants $C, c > 0$. We often use the notation $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$. Also, the Hessian matrix of $\phi : \mathbb{R}^d \to \mathbb{C}$ at ξ is given by $H\phi(\xi) = (\partial_i \partial_j \phi(\xi))_{1 \le i, j \le d}$.

2.1. Function spaces

The spaces $\dot{H}^s(\mathbb{R}^d)$, $H^s(\mathbb{R}^d)$, respectively, denote the homogeneous and inhomogeneous Sobolev spaces based on $L^2(\mathbb{R}^d)$, which are equipped with the norms

$$\|f\|_{\dot{H}^s} = \||D|^s f\|_{L^2}, \quad \|f\|_{H^s} = \|\langle D \rangle^s f\|_{L^2}.$$

Here, $\varphi(D)$ will denote the Fourier multiplier operator given by

$$\widehat{\varphi(D)}f(\xi) = \varphi(\xi)\widehat{f}(\xi),$$

where the Fourier transform of a sufficiently nice function $f : \mathbb{R}^d \to \mathbb{C}$ is given by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\,\mathrm{d}x.$$

Also, the inverse Fourier transform is defined by

$$\mathcal{F}^{-1}f(x) = f^\vee(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(\xi)e^{ix\cdot\xi}\,\mathrm{d}\xi.$$

For $p \in [1, \infty)$ and $r \in [1, \infty]$, we write $L^{p,r}(\mathbb{R}^d)$ for the Lorentz space with norm

$$\|f\|_{L^{p,r}} = \left(\int_0^\infty (t^{1/p}f^*(t))^r \frac{\mathrm{d}t}{t} \right)^{1/r}$$

for $r < \infty$, and

$$\|f\|_{L^{p,\infty}} = \sup_{t>0} t^{1/p}f^*(t).$$

For $p \in (1, \infty)$ and $r \in [1, \infty]$, the space $L^{p,r}$ is normable. Strictly speaking, $\|\cdot\|_{L^{p,r}}$ is a true norm when $r \leq p$ and a quasi-norm otherwise; a true norm that is equivalent to $\|\cdot\|_{L^{p,r}}$ is obtained by replacing $f^*(t)$ with $\frac{1}{t}\int_0^t f^*$. For details on fundamental properties of Lorentz spaces, we refer the reader to [73].

Since $L^{p,p} = L^p$ and $L^{p,r_1} \subseteq L^{p,r_2}$ if $r_1 \leq r_2$, Lorentz spaces provide a natural setting to seek refinements of certain classical inequalities for L^p functions. An example is the refined version of the Hardy–Littlewood–Sobolev inequality for functions in Lorentz spaces:

$$\left| \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{g_1(t_1)g_2(t_2)}{|t_1 - t_2|^\lambda} \,\mathrm{d}t_1\mathrm{d}t_2 \right| \lesssim \|g_1\|_{L^{p_1,r_1}}\|g_2\|_{L^{p_2,r_2}}, \tag{2.1}$$

where $\lambda \in (0, 1)$, $p_1, p_2 \in (1, \infty)$ satisfy $\frac{1}{p_1} + \frac{1}{p_2} + \lambda = 2$, and $\frac{1}{r_1} + \frac{1}{r_2} \geq 1$. This can be found in [63].

Finally, we introduce a convenient notation when handling mixed-norm estimates. For (space-time) functions $F : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$ belonging to $L_t^q L_x^r$, we write

$$\|F\|_{q,r} = \|F\|_{L_t^q L_x^r}.$$

In the case of Lorentz spaces, if $F \in L_t^{q,p} L_x^r$ or $F \in L_t^q L_x^{r,p}$, we write

$$\|F\|_{(q,p),r} = \|F\|_{L_t^{q,p} L_x^r}, \quad \|F\|_{q,(r,p)} = \|F\|_{L_t^q L_x^{r,p}},$$

respectively.

2.2. A duality principle

We shall make use of the duality principle originating in the work of Frank and Sabin [30, Lemma 3].[2] In the following statement (and throughout the paper), given an exponent $q \geq 2$, we write $\widetilde{q}$ for the exponent given by

$$\frac{\widetilde{q}}{2} = \left(\frac{q}{2}\right)',$$

or equivalently

$$\frac{1}{\widetilde{q}} = \frac{1}{2} - \frac{1}{q}.$$

For $\beta \in [1, \infty)$, $\mathcal{C}^{\beta} = \mathcal{C}^{\beta}(L^2(\mathbb{R}^d))$ denotes the Schatten space based on $L^2(\mathbb{R}^d)$ that is the space of all compact operators A on $L^2(\mathbb{R}^d)$ such that $\mathrm{Tr}|A|^{\beta} < \infty$, where $|A| = \sqrt{A^*A}$, and its norm is defined by $\|A\|_{\mathcal{C}^{\beta}} = (\mathrm{Tr}|A|^{\beta})^{\frac{1}{\beta}}$. If $\beta = \infty$, we define $\|A\|_{\mathcal{C}^{\infty}} = \|A\|_{L^2 \to L^2}$. Also, the case $\beta = 2$ is special in the sense that $\mathcal{C}^2$ is the Hilbert–Schmidt class and the $\mathcal{C}^2$ norm is given by $\|A\|_{\mathcal{C}^2} = \|K_A\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}$, where K_A is the integral kernel of A. More details on the Schatten classes can be found in [70].

Proposition 1 (Duality principle). *Suppose T is a bounded operator from $L^2(\mathbb{R}^d)$ to $L_t^{q,p} L_x^r$ for some $q > 2$, $p, r \geq 2$ and $\beta \geq 1$. Then*

$$\left\| \sum_j \nu_j |Tf_j|^2 \right\|_{(\frac{q}{2}, \frac{p}{2}), \frac{r}{2}} \lesssim \|\nu\|_{\ell^{\beta}}$$

holds for all families of orthonormal functions $(f_j)_j$ in $L^2(\mathbb{R}^d)$ and all sequences $\nu \in \ell^{\beta}$ if and only if

$$\|WTT^*\overline{W}\|_{\mathcal{C}^{\beta'}} \lesssim \|W\|_{(\widetilde{q}, \widetilde{p}), \widetilde{r}}^2$$

holds for all $W \in L_t^{\widetilde{q}, \widetilde{p}} L_x^{\widetilde{r}}$.

We shall apply the above duality principle in the case $\widetilde{p} = 2\widetilde{r}$. That is to say, we will obtain estimates

$$\left\| \sum_j \nu_j |Tf_j|^2 \right\|_{(\frac{q}{2}, \beta), \frac{r}{2}} \lesssim \|\nu\|_{\ell^{\beta}}, \qquad \beta = \frac{2r}{r+2}$$

from estimates of the form

$$\|WTT^*\overline{W}\|_{\mathcal{C}^{\widetilde{r}}} \lesssim \|W\|_{(\widetilde{q}, 2\widetilde{r}), \widetilde{r}}^2.$$

2.3. Dyadic decomposition

Throughout the paper, χ denotes a fixed function $C_c^{\infty}(-1, 1)$ that satisfies $\chi = 1$ on $(-\frac{1}{2}, \frac{1}{2})$, and we define

$$\chi_0 = \chi(2^{-1} \cdot) - \chi.$$

For $j \in \mathbb{Z}$, we write $\chi_j = \chi_0(2^{-j} \cdot)$ so that χ_j is a smooth cutoff function supported in $(2^{j-1}, 2^{j+1})$. By construction, we have

$$\chi = \sum_{j \leq -1} \chi_j, \quad \sum_{j=-\infty}^{\infty} \chi_j = \chi + \sum_{j=0}^{\infty} \chi_j = 1. \tag{2.2}$$

2.4. Van der Corput's lemma

The following proposition, often referred to as van der Corput's lemma, will be useful throughout Section 3 (see, for example, [72, p. 334]).

Proposition 2. *Let $k \geq 1$ and $\rho_1, \rho_2 \in \mathbb{R}$ with $\rho_1 < \rho_2$. Suppose $\theta : [\rho_1, \rho_2] \to \mathbb{R}$ satisfies*

$$|\theta^{(k)}(\rho)| \geq 1$$

for all $\rho \in [\rho_1, \rho_2]$. Suppose also that $a : [\rho_1, \rho_2] \to \mathbb{R}$ is differentiable and a' is integrable on $[\rho_1, \rho_2]$. Then there exists a constant C_k, depending only on k, such that

$$\left| \int_{\rho_1}^{\rho_2} e^{i\mu\theta(\rho)} a(\rho) \, d\rho \right| \leq C_k \left(|a(\rho_2)| + \int_{\rho_1}^{\rho_2} |a'(\rho)| \, d\rho \right) |\mu|^{-1/k}$$

holds, when $k \geq 2$, or $k = 1$ and θ' is monotonic.

3. Weighted oscillatory integral estimates

In this section, we establish various new weighted oscillatory integral estimates, by which we mean decay estimates for oscillatory integrals of the form $I^\phi(w)$ defined by (1.16). These estimates are related to the dispersive estimates for the associated propagators. If a has compact support, ϕ is smooth on the support a, and ϕ is not degenerate—that is to say, $\det H\phi \neq 0$—then the stationary phase method gives

$$|I^\phi(1)| \lesssim |t|^{-\frac{d}{2}}.$$

This decay estimate is in general best possible under such a non-degeneracy assumption. However, if $\det H\phi$ vanishes, only weaker decay estimates are possible. There have been attempts to recover optimal decay $O(|t|^{-\frac{d}{2}})$ by introducing a suitable damping weight. A typical weight involves powers of the determinant of the Hessian matrix of ϕ. In view of the asymptotic expansion of the oscillatory integral with non-degenerate phase (for example, see [43, (7.7.12), p. 220] and [72, p. 360–361]), it is natural to use the damping factor $w(\xi) = |\det H\phi(\xi)|^{1/2}$ to recover the best possible decay $O(|t|^{-\frac{d}{2}})$.

This type of estimate for $I^\phi(|\det H\phi|^\gamma)$ has been studied by various authors. Early results of this kind go back as far as work of Sogge and Stein [71]. In work of Cowling et al. [26], the damped oscillatory integral estimates were studied with finite-type convex ϕ, but the weight was assumed to have sufficient smoothness and weights with complex exponent γ were not considered. Establishing estimates of the form

$$|I^\phi(|\det H\phi|^{1/2+i\kappa})| \lesssim_\kappa |t|^{-\frac{d}{2}}$$

turned out to be a delicate problem, even without κ. Until now, only results for special classes of ϕ are known. Kenig, Ponce and Vega [49] obtained such an estimate with elliptic polynomial ϕ. In the radial case $\phi(\xi) = \phi_0(|\xi|)$, estimates were obtained by Carbery–Ziesler [12] under the assumption that ϕ_0 and ϕ_0' are convex.

Below, we present our weighted oscillatory integral estimates corresponding to the wave equation, the Klein–Gordon equation and the fractional Schrödinger equation, which we need for proof of the orthonormal Strichartz estimates. In the case of the wave and Klein–Gordon equations, our results are completely new. For the case of the fractional Schrödinger equation, our estimates are new in the case $\alpha < 2$ and overlap with work of Kenig, Ponce and Vega [49] when $\alpha \geq 2$. See also the remark at the end of this section for further discussion about this point. Compared with the previous work, our argument here is significantly simpler. This is mainly due to our efficient use of the bounds that are obtained by making use of the first order derivatives of the phase functions.

3.1. The wave equation

For each $\kappa \in \mathbb{R}$ and $(x, t) \in \mathbb{R}^d \times \mathbb{R}$, we define the oscillatory integral

$$\mathcal{W}_\kappa(x, t) = \int e^{i(x \cdot \xi + t|\xi|)} |\xi|^{-\frac{d+1}{2} + i\kappa} \chi(|\xi|) \, d\xi.$$

Proposition 3. *There exist constants $C < \infty$ and $N > 0$ such that*

$$|\kappa \mathcal{W}_\kappa(x, t)| \leq C(1 + |\kappa|)^N |t|^{-\frac{d-1}{2}} \tag{3.1}$$

for all $\kappa \in \mathbb{R}$ and $(x, t) \in \mathbb{R}^d \times \mathbb{R}$.

To place this estimate in some context, recall the standard dispersive estimate

$$\left| \int e^{i(x \cdot \xi + t|\xi|)} \chi_0(|\xi|) \, d\xi \right| \lesssim \min\left\{ 1, |t|^{-\frac{d-1}{2}} \right\}$$

related to the wave equation and the rate of decay $\frac{d-1}{2}$ is best possible. By scaling, it easily follows that $|\int e^{i(x \cdot \xi + t|\xi|)} \chi_0(2^{-j}|\xi|) \, d\xi| \lesssim \min\{2^{dj}, 2^{\frac{d+1}{2}j} |t|^{-\frac{d-1}{2}}\}$ for each $j \in \mathbb{Z}$. Thus, for any $\lambda < \frac{d-1}{2}$, a standard dyadic decomposition yields

$$\left| \int e^{i(x \cdot \xi + t|\xi|)} \frac{1}{|\xi|^{d-\lambda}} \, d\xi \right| \lesssim \sum_{2^j \leq |t|^{-1}} 2^{\lambda j} + |t|^{-\frac{d-1}{2}} \sum_{2^j > |t|^{-1}} 2^{(\lambda - \frac{d-1}{2})j} \lesssim |t|^{-\lambda}.$$

So, the estimate (3.1) may be regarded as the endpoint case of the above family of estimates. However, it appears to be difficult to establish Theorem 5 using these weighted estimates, and they seem to be useful only for the sub-optimal decay rate. The additional oscillation introduced by the factor $|\xi|^{i\kappa}$ allows us to recover the optimal decay rate $\frac{d-1}{2}$, and this is crucial for the approach taken in this paper. We remark that the extra κ factor in the left-hand side of (3.1) is also important, and it is not difficult to show that the estimate is not true without this factor (this becomes clear in the proofs of Lemma 1 and Proposition 3 below).

As preparation for the proof of Proposition 3, we establish the following elementary estimate.

Lemma 1. *There exists a constant $C < \infty$, independent of κ, ε and R, such that we have*

$$\left| \kappa \int_\varepsilon^R e^{\pm i A \rho} \rho^{-1 + i\kappa} \, d\rho \right| \leq C(1 + |\kappa|)^2.$$

In fact, what we shall need in the proof of Proposition 3 is the following estimate:

$$\left| \kappa \int_0^R \psi(\rho) e^{\pm i A \rho} \rho^{-1 + i\kappa} \, d\rho \right| \leq C(1 + |\kappa|)^2 (\|\psi'\|_1 + \|\psi\|_\infty), \tag{3.2}$$

where the constant $C < \infty$ is the same as in Lemma 1 (and hence independent of $R > 0$, $\kappa \in \mathbb{R}$, $A \geq 0$ and $\psi \in C_c^\infty(0, \infty)$). We note that (3.2) follows immediately from Lemma 1. Indeed, if we set

$$F(r) = \kappa \int_\varepsilon^r e^{\pm i A \rho} \rho^{-1 + i\kappa} \, d\rho,$$

where $\varepsilon > 0$ is a sufficiently small number so that $\psi(\varepsilon) = 0$, then, by integration by parts, we get

$$\left| \kappa \int_0^R \psi(\rho) e^{\pm i A \rho} \rho^{-1 + i\kappa} \, d\rho \right| \leq |\psi(R) F(R)| + \left| \int_0^R \psi'(\rho) F(\rho) \, d\rho \right|,$$

and thus (3.2) clearly follows from Lemma 1.

Proof (Proof of Lemma 1). Since $\frac{\mathrm{d}}{\mathrm{d}\rho}\rho^{i\kappa} = i\kappa\rho^{-1+i\kappa}$, we obviously have

$$\left| \kappa \int_{\varepsilon}^{R} \rho^{-1+i\kappa}\,\mathrm{d}\rho \right| \le 2,$$

which gives the desired estimate when $A = 0$.

Next, note that the case $A \ne 0$ follows from the case $A = 1$ by rescaling $\rho \to \rho/A$, and thus it is sufficient to show that

$$\left| \kappa \int_{\varepsilon}^{R} e^{\pm i\rho}\rho^{-1+i\kappa}\,\mathrm{d}\rho \right| \le C(1 + |\kappa|)^2$$

holds with C independent of $R, \varepsilon > 0$. We initially assume $\varepsilon \le 1 \le R$ and split the integral

$$\kappa \int_{\varepsilon}^{R} e^{\pm i\rho}\rho^{-1+i\kappa}\,\mathrm{d}\rho = \kappa \left(\int_{\varepsilon}^{1} + \int_{1}^{R} \right) e^{\pm i\rho}\rho^{-1+i\kappa}\,\mathrm{d}\rho := I + II.$$

By integration by parts, it is easy to see that $|II| \le (1 + |\kappa|)^2$. For I, if we write

$$I = \kappa \int_{\varepsilon}^{1} (e^{\pm i\rho} - 1)\rho^{-1+i\kappa}\,\mathrm{d}\rho + \kappa \int_{\varepsilon}^{1} \rho^{-1+i\kappa}\,\mathrm{d}\rho,$$

then it is clear that $|I| \le C(1 + |\kappa|)$, thus completing the argument when $\varepsilon \le 1 \le R$. If $\varepsilon \le R \le 1$, we only need the argument for I; and if $1 \le \varepsilon \le R$, the argument for II is enough. $\square$

Proof (Proof of Proposition 3). The trivial estimate

$$|\kappa\mathcal{W}_\kappa(x,t)| \lesssim |\kappa| \int |\xi|^{-\frac{d+1}{2}} \chi(|\xi|)\,\mathrm{d}\xi \lesssim |\kappa|$$

means that we may reduce to the case where $|t| \ge 1$. Furthermore, in the case $|t| \ge 1$, we argue that it suffices to consider the case where

$$2^{-1}|t| \le |x| \le 2|t|. \tag{3.3}$$

To see this, suppose either $|x| \ge 2|t|$ or $2|x| \le |t|$. Using the dyadic decomposition (2.2) and rescaling, we get

$$\mathcal{W}_\kappa(x,t) = \sum_{j \ge 1} 2^{-j(\frac{d-1}{2}+i\kappa)} \int e^{i\Phi_j(\xi)} |\xi|^{-\frac{d+1}{2}+i\kappa} \chi_0(|\xi|)\,\mathrm{d}\xi,$$

where

$$\Phi_j(\xi) = 2^{-j}x \cdot \xi + 2^{-j}t|\xi|.$$

In the case $|x| \ge 2|t|$, we have $|\nabla\Phi_j(\xi)| \gtrsim 2^{-j}|x|$ on the support of $\chi_0(|\cdot|)$, and it follows by repeated integration by parts that

$$\left| \int e^{i\Phi_j(\xi)} |\xi|^{-\frac{d+1}{2}+i\kappa} \chi_0(|\xi|)\,\mathrm{d}\xi \right| \lesssim_\kappa \min(1, (2^{-j}|x|)^{-M})$$

for any $M \ge 0$. From this, it follows that

$$|\mathcal{W}_\kappa(x,t)| \lesssim \sum_{\substack{j \ge 1 \\ 2^j \le |x|}} 2^{-j\frac{d-1}{2}} (2^{-j}|x|)^{-M} + \sum_{\substack{j \ge 1 \\ 2^j \ge |x|}} 2^{-j\frac{d-1}{2}} \lesssim |x|^{-\frac{d-1}{2}},$$

if we choose M sufficiently large, and hence

$$|\mathcal{W}_\kappa(x,t)| \lesssim |t|^{-\frac{d-1}{2}}$$

holds in the case $|x| \geq 2|t|$. It is easy to see the same estimate holds in the case $2|x| \leq |t|$ by a similar argument. Since

$$\left| \int e^{i\Phi_j(\xi)} |\xi|^{-\frac{d+1}{2}+i\kappa} \chi_0(|\xi|) \, d\xi \right| \lesssim_\kappa \min(1, (2^{-j}|t|)^{-M})$$

for any $M \geq 0$ if $2|x| \leq |t|$, repeating the argument for the case $|x| \geq 2|t|$ yields the desired estimate.

For the remainder of the proof, we assume that (3.3) holds. Since we have

$$\left| \int e^{i(x\cdot\xi+t|\xi|)} |\xi|^{-\frac{d+1}{2}+i\kappa} \chi(|t\xi|)\chi(|\xi|) \, d\xi \right| \lesssim \int |\xi|^{-\frac{d+1}{2}} \chi(|t\xi|) \, d\xi \lesssim |t|^{-\frac{d-1}{2}},$$

it suffices to consider

$$\widetilde{\mathcal{W}}_\kappa(x,t) := \int e^{i(x\cdot\xi+t|\xi|)} |\xi|^{-\frac{d+1}{2}+i\kappa} (1-\chi(|t\xi|))\chi(|\xi|) \, d\xi.$$

To estimate this oscillatory integral, we use spherical coordinates to write

$$\widetilde{\mathcal{W}}_\kappa(x,t) = \int \widehat{d\sigma}(\rho x) \, e^{it\rho} \rho^{\frac{d-3}{2}+i\kappa} (1-\chi(|t|\rho))\chi(\rho) \, d\rho,$$

where $d\sigma$ is the surface measure on the sphere $\{x : |x| = 1\}$. Since $\widehat{d\sigma}(x) = C_d |x|^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(|x|)$, using an asymptotic expansion for the Bessel function, we get

$$\widehat{d\sigma}(x) = C_\pm |x|^{-\frac{d-1}{2}} e^{\pm i|x|} + \sum_{j=1}^{N} C_{j,\pm} |x|^{-\frac{d-1}{2}-j} e^{\pm i|x|} + \mathcal{E}(|x|), \quad |x| \geq 1, \tag{3.4}$$

where $\left(\frac{d}{dr}\right)^k \mathcal{E}(r) = O(r^{-k-\frac{N+d-1}{2}})$. For more details regarding this, see [72, p. 338] and also [72, Proposition 3, p. 334]. Since we are assuming $|x| \sim |t| \gtrsim 1$, for the contribution from the leading term in this expansion, it is enough to show that

$$\left| \kappa \int e^{i(t\pm|x|)\rho} \rho^{-1+i\kappa} (1-\chi(|t|\rho))\chi(\rho) \, d\rho \right| \leq C(1+|\kappa|)^2.$$

However, this immediately follows from (3.2) since $\|((1-\chi(|t|\cdot))\chi)'\|_1 \lesssim 1$ uniformly in t. The other terms can be handled similarly but in an easier manner, so we omit the details. $\qquad \square$

3.2. The Klein–Gordon equation

In this subsection, we prove the weighted oscillatory integral estimates that are needed for the proof of Theorems 6 and 7. The associated surface $(\xi, \langle\xi\rangle)$ to the Klein–Gordon equation has nonvanishing Gaussian curvature everywhere, but its Gaussian curvature asymptotically vanishes at infinity. This gives rise to significant complication in the argument for obtaining the sharp decay rate in the oscillatory integral estimates.

For each $\kappa \in \mathbb{R}$, $\varepsilon > 0$ and $(x,t) \in \mathbb{R}^d \times \mathbb{R}$, we define the oscillatory integrals $\mathcal{J}_{\kappa,R}(x,t)$ and $\mathcal{K}_{\kappa,\varepsilon}(x,t)$ by

$$\mathcal{J}_{\kappa,\varepsilon}(x,t) = \int e^{i(x\cdot\xi+t\langle\xi\rangle)} \psi(\varepsilon\xi)(1+|\xi|^2)^{-\frac{d+1}{4}-i\kappa} \, d\xi$$

and

$$\mathcal{K}_{\kappa,\varepsilon}(x,t) = \int e^{i(x\cdot\xi+t\langle\xi\rangle)}\psi(\varepsilon\xi)(1+|\xi|^2)^{-\frac{d+2}{4}-i\kappa}\,d\xi.$$

Here, $\psi \in C_0^\infty(-1,1)$. We insert $\psi(\varepsilon|\xi|)$ to guarantee the existence of the integral. However, the presence of the additional factor $\psi(\varepsilon|\xi|)$ does not have any significance, and the overall argument below works as if there were no such factor. It is easy to see by integration by parts that, for $t \neq 0$, the limits $\lim_{\varepsilon\to 0}\mathcal{J}_{\kappa,\varepsilon}(x,t)$ and $\lim_{\varepsilon\to 0}\mathcal{K}_{\kappa,\varepsilon}(x,t)$ exist.

The following , respectively, give the optimal wave-like and Schrödinger-like dispersive estimates for the Klein-Gordon equation.

Proposition 4. *There exist constants $C < \infty$ and $N > 0$ such that*

$$|\kappa\mathcal{J}_{\kappa,\varepsilon}(x,t)| \leq C(1+|\kappa|)^N |t|^{-\frac{d-1}{2}} \tag{3.5}$$

for all $\kappa \in \mathbb{R}$, $\varepsilon > 0$ and $(x,t) \in \mathbb{R}^d \times \mathbb{R}$.

Proposition 5. *There exist constants $C < \infty$ and $N > 0$ such that*

$$|\mathcal{K}_{\kappa,\varepsilon}(x,t)| \leq C(1+|\kappa|)^N |t|^{-\frac{d}{2}} \tag{3.6}$$

for all $\kappa \in \mathbb{R}$, $\varepsilon > 0$ and $(x,t) \in \mathbb{R}^d \times \mathbb{R}$.

Before starting the proof, we recall the following (see [39, Appendix]), which is an easy consequence of the stationary phase method.

Lemma 2. *Let $\widetilde{\chi}$ be a smooth function supported in $\{\xi : 2^{-1} \leq |\xi| \leq 2\}$. For $0 < \rho \leq 1$, let us set*

$$I_\rho = \int e^{i(x\cdot\xi+t\sqrt{\rho^2+|\xi|^2})}\widetilde{\chi}(\xi)\,d\xi.$$

Then we have the estimate $|I_\rho| \lesssim \min(|t|^{-\frac{d-1}{2}}, \rho^{-1}|t|^{-\frac{d}{2}})$.

For convenience of the reader we briefly explain how to show this. Via a finite decomposition and rotation, we may assume the cutoff function $\widetilde{\chi}$ is supported in a small conic neighborhood of e_d such that $\xi_d \in [2^{-2}, 2^2]$. Thus we may write

$$\sqrt{\rho^2+|\xi|^2} = \xi_d\left(1+\frac{\rho^2+|\bar{\xi}|^2}{\xi_d^2}\right)^{\frac{1}{2}} = \xi_d + \frac{1}{2}\left(\frac{\rho^2}{\xi_d}+\frac{|\bar{\xi}|^2}{\xi_d}\right) + \text{error},$$

where $\xi = (\bar{\xi}, \xi_d)$. To get the estimate $|I_\rho| \lesssim |t|^{-\frac{d-1}{2}}$, we may simply ignore ξ_d and apply the stationary phase method in $\bar{\xi}$. For the estimate $|I_\rho| \lesssim \rho^{-1}|t|^{-\frac{d}{2}}$, note that one of the eigenvalues of the Hessian matrix of $\sqrt{\rho^2+|\xi|^2}$ is $\sim \rho^2$ while the other $d-1$ eigenvalues are ~ 1.

Proof (Proof of Proposition 4). As before, we break the integral dyadically. Using (2.2), we write

$$\mathcal{J}_{\kappa,\varepsilon} = J_\circ + \sum_{j=0}^\infty J_j := \int e^{i(x\cdot\xi+t\langle\xi\rangle)}\frac{\psi(\varepsilon|\xi|)\chi(|\xi|)}{(1+|\xi|^2)^{\frac{d+1}{4}+i\kappa}}\,d\xi$$

$$+ \sum_{j=0}^\infty \int e^{i(x\cdot\xi+t\langle\xi\rangle)}\frac{\psi(\varepsilon|\xi|)\chi_j(|\xi|)}{(1+|\xi|^2)^{\frac{d+1}{4}+i\kappa}}\,d\xi.$$

In what follows, we assume $2^j \leq 2/\varepsilon$ even if it is not explicitly mentioned. Clearly, $|J_\circ| \lesssim_\kappa \min(1, |t|^{-d/2})$ since $\det H\langle\cdot\rangle \neq 0$ on the support of $\chi(|\cdot|)$. Thus, it is sufficient to show that

$$\left| \sum_{j \geq 0} \kappa J_j \right| \lesssim_\kappa |t|^{-\frac{d-1}{2}}. \tag{3.7}$$

After rescaling, we see that

$$J_j = 2^{(d - \frac{d+1}{2} - 2\kappa i)j} \int e^{i(2^j x \cdot \xi + 2^j t \sqrt{2^{-2j} + |\xi|^2})} \widetilde{\chi}_j(|\xi|) \, d\xi, \tag{3.8}$$

where $\widetilde{\chi}_j$ is a smooth function supported in $[1/2, 2]$ that satisfies $\|\widetilde{\chi}_j\|_{C^N} \lesssim (1 + |\kappa|)^N$ for any N. As before, we may also assume that

$$2^{-1}|t| \leq |x| \leq 2|t|.$$

Otherwise, $|\nabla_\xi (2^j x \cdot \xi + 2^j t \sqrt{2^{-2j} + |\xi|^2})| \gtrsim 2^j \max(|x|, |t|)$. By integration by parts we see that, for any M,

$$|J_j| \lesssim_\kappa 2^{\frac{d-1}{2} j} \begin{cases} (2^j |x|)^{-M} & \text{if } |x| \geq 2|t|, \\ (2^j |t|)^{-M} & \text{if } |t| \geq 2|x|. \end{cases}$$

This gives $|\sum_{j \geq 0} J_j| \lesssim_\kappa \sum_{j \geq 0} 2^{\frac{d-1}{2} j} (2^j |t|)^{-M} \lesssim_\kappa |t|^{-\frac{d-1}{2}}$.

To show (3.7), we now consider the following cases separately:

$$\mathbf{A} : |t| \geq 1, \quad \mathbf{B} : |t| < 1.$$

From Lemma 2, we have

$$|J_j| \lesssim_\kappa 2^{\frac{d-1}{2} j} \min(2^{-\frac{d-2}{2} j} |t|^{-\frac{d}{2}}, (2^j |t|)^{-\frac{d-1}{2}}, 1). \tag{3.9}$$

Case A

Since $|t| \geq 1$, from (3.9), we observe that

$$\sum_{2^0 \leq 2^j \leq t} |J_j| \lesssim_\kappa \sum_{2^0 \leq 2^j \leq t} 2^{\frac{1}{2} j} |t|^{-\frac{d}{2}} \lesssim |t|^{-\frac{d-1}{2}}.$$

Let us set

$$\chi_{>t} := \sum_{|t| < 2^j} \chi_j(\cdot) \tag{3.10}$$

and set

$$J_{>t} = \kappa \int e^{i(x \cdot \xi + t \langle \xi \rangle)} (1 + |\xi|^2)^{-\frac{d+1}{4} - i\kappa} \chi_{>t}(|\xi|) \psi(\varepsilon|\xi|) \, d\xi.$$

Then to show (3.7), we are reduced to showing that

$$|J_{>t}| \lesssim |t|^{-\frac{d-1}{2}}.$$

As before, since $|x| \sim |t| \geq 1$, using polar coordinates, we may apply the asymptotic expansion (3.4). Taking into account the main contribution from the leading terms in (3.4), it is sufficient to consider

$$|x|^{\frac{1-d}{2}} \int e^{i(\pm|x|\rho + t \langle \rho \rangle)} (1 + \rho^2)^{-\frac{d+1}{4} - i\kappa} \rho^{\frac{d-1}{2}} \chi_{>t}(\rho) \psi(\varepsilon\rho) \, d\rho.$$

The other terms can be handled similarly, but they are easier. Since we are assuming $|x| \sim |t|$, the matter reduces to showing

$$\left| \int e^{i(\pm|x|\rho + t\langle\rho\rangle)}(1+\rho^2)^{-\frac{d+1}{4}-i\kappa}\rho^{\frac{d-1}{2}}\chi_{>t}(\rho)\psi(\varepsilon\rho)\,d\rho \right| \lesssim_\kappa 1. \qquad (3.11)$$

Since $|t| \geq 1$, from the mean value theorem we note that

$$\int_0^\infty |(1+\rho^2)^{-\frac{d+1}{4}-i\kappa} - (\rho^2)^{-\frac{d+1}{4}-i\kappa}|\chi_{>t}(\rho)\rho^{\frac{d-1}{2}}\,d\rho \lesssim_\kappa 1.$$

Let us set

$$\mathcal{J} = \kappa \int e^{i(\pm|x|\rho + t\langle\rho\rangle)}\rho^{-1-i\kappa}\chi_{>t}(\rho)\psi(\varepsilon\rho)\,d\rho.$$

Then (3.11) follows if we show that

$$|\mathcal{J}| \lesssim_\kappa 1.$$

Changing variables, we have

$$\mathcal{J} = \kappa|t|^{-i\kappa}\int e^{i(\pm|x||t|\rho + t\sqrt{1+t^2\rho^2})}\rho^{-1-i\kappa}\chi_{>1}(\rho)\psi(\varepsilon|t|\rho)\,d\rho.$$

Now let us set

$$F_{A,t}(r) = \kappa|t|^{-i\kappa}\int_0^r e^{iA\rho}\rho^{-1-i\kappa}\chi_{>1}(\rho)\psi(\varepsilon|t|\rho)\,d\rho, \qquad (3.12)$$

$$G(r) = e^{it(\sqrt{1+t^2r^2}-|t|r)}\chi_{>2^{-2}}(r).$$

Since $\chi_{>1}\chi_{>2^{-2}} = \chi_{>1}$, we note that

$$\mathcal{J} = \int \frac{d}{d\rho}F_{|t|(t\pm|x|),|t|}(\rho)G(\rho)\,d\rho = -\int F_{|t|(t\pm|x|),|t|}(\rho)G'(\rho)\,d\rho.$$

Since $\|(\chi_{>1}\psi(|t|\varepsilon\cdot))'\|_1 \lesssim 1$[3], by (3.2), we have $|F_{A,t}(r)| \lesssim_\kappa 1$. Thus, it is sufficient to check $\|G'\|_1 \lesssim 1$. This follows from

$$G'(r) = e^{it(\sqrt{1+t^2r^2}-|t|r)}\left(\chi_{>2^{-2}}(r)\frac{-it|t|}{\sqrt{1+t^2r^2}(|t|r+\sqrt{1+t^2r^2})} + \chi'_{>2^{-2}}(r)\right)$$

and the fact that $\chi'_{>2^{-2}}$ is supported in $[-1, 1]$.

Case B

In this case, we have $|t| < 1$. To begin with, from (3.9), we note that

$$\sum_{2^0 \leq 2^j \leq \frac{1}{|t|}} |J_j| \lesssim_\kappa \sum_{2^0 \leq 2^j \leq \frac{1}{|t|}} 2^{\frac{d-1}{2}j} \lesssim |t|^{-\frac{d-1}{2}}.$$

We set

$$J_{>t^{-1}} = \int e^{i(x\cdot\xi + t\langle\xi\rangle)}(1+|\xi|^2)^{-\frac{d+1}{4}-i\kappa}\chi_{>t^{-1}}(|\xi|)\psi(\varepsilon|\xi|)\,d\xi,$$

---

[3] Recall that $\chi'_{>1}$ is supported in $[-1, 1]$ because $1 - \chi = \chi_{>1}$.

where $\chi_{>t^{-1}}$ is given by (3.10). From the above observation, for (3.7), it is sufficient to show that

$$|J_{>t^{-1}}| \lesssim_\kappa |t|^{-\frac{d-1}{2}}.$$

Similar to before, since $|x| \sim |t|$, $|t||\xi| \sim |x||\xi| \gtrsim 1$ on the support of $\chi_{>t^{-1}}(\cdot)$. Thus, we may apply (3.4) after using polar coordinates. Considering the main contribution from the leading terms in (3.4), we need only to handle

$$|x|^{\frac{1-d}{2}} \int e^{i(\pm|x|\rho + t\langle\rho\rangle)} (1+\rho^2)^{-\frac{d+1}{4} - i\kappa} \rho^{\frac{d-1}{2}} \chi_{>t^{-1}}(\rho)\psi(\varepsilon\rho)\,\mathrm{d}\rho.$$

Thus the desired estimate (3.7) follows from

$$\left| \int e^{i(\pm|x|\rho + t\langle\rho\rangle)} (1+\rho^2)^{-\frac{d+1}{4} - i\kappa} \rho^{\frac{d-1}{2}} \chi_{>t^{-1}}(\rho)\psi(\varepsilon\rho)\,\mathrm{d}\rho \right| \lesssim_\kappa 1.$$

Let us set

$$\widetilde{\mathcal{J}} = \tau \int e^{i(\pm|x|\rho + t\langle\rho\rangle)} \rho^{-1 - i\kappa} \chi_{>t^{-1}}(\rho)\psi(\varepsilon\rho)\,\mathrm{d}\rho.$$

Proceeding along the lines of the case **A**, we are further reduced to showing that

$$|\widetilde{\mathcal{J}}| \lesssim_\kappa 1. \tag{3.13}$$

Changing variables $\rho \to \rho/|t|$, we see that

$$\widetilde{\mathcal{J}} = \kappa |t|^{i\kappa} \int e^{i(\pm\frac{|x|}{|t|}\rho + \mathrm{sgn}\, t\sqrt{t^2 + \rho^2})} \rho^{-1 - i\kappa} \chi_{>1}(\rho)\psi(\varepsilon\rho/|t|)\,\mathrm{d}\rho.$$

Let us set

$$H(r) = e^{i\,\mathrm{sgn}\, t(\sqrt{t^2 + r^2} - r)} \chi_{>2^{-2}}(r).$$

Then

$$\widetilde{\mathcal{J}} = \int F'_{\mathrm{sgn}\, t \pm \frac{|x|}{|t|}, |t|^{-1}}(\rho) H(\rho)\,\mathrm{d}\rho = -\int F_{\mathrm{sgn}\, t \pm \frac{|x|}{|t|}, |t|^{-1}}(\rho) H'(\rho)\,\mathrm{d}\rho,$$

where $F_{A,t}$ is given by (3.12). Since $\|(\chi_{>1}\psi(\varepsilon/|t|\cdot))'\|_1 \lesssim 1$, by (3.2) $|F_{A,|t|^{-1}}(r)| \lesssim_\kappa 1$. Finally, we note that

$$H'(r) = e^{i\,\mathrm{sgn}\, t(\sqrt{t^2 + r^2} - r)} \left(\chi_{>2^{-2}}(r) \frac{-i\,\mathrm{sgn}\, t\, t^2}{\sqrt{t^2 + r^2}(r + \sqrt{t^2 + r^2})} + \chi'_{>2^{-2}}(r) \right).$$

Since we are assuming $|t| < 1$, it follows that $\|H'\|_1 \lesssim 1$. Therefore we get (3.13). This completes the proof of Proposition 4.   □

Proof (Proof of Proposition 5). Compared with the proof of Proposition 4, that of Proposition 5 is much more involved since we show the damped oscillatory integral estimate to recover the best possible decay. We provide the proof by dividing it into several steps.

Step 1: Reduction to the case $|t| \gtrsim 1$ and $|x| \sim |t|$.

As before, we begin with a dyadic decomposition of the integral. Using (2.2), we have

$$\mathcal{K}_{\kappa, \varepsilon} = \mathcal{K}_\circ + \sum_{j \geq 0} \mathcal{K}_j,$$

where

$$\mathcal{K}_\circ(x,t) := \int e^{i(x\cdot\xi+t\langle\xi\rangle)} \frac{\chi(|\xi|)\psi(\varepsilon|\xi|)}{(1+|\xi|^2)^{\frac{d+2}{4}+i\kappa}} \, d\xi,$$

$$\mathcal{K}_j(x,t) := \int e^{i(x\cdot\xi+t\langle\xi\rangle)} \frac{\chi_j(|\xi|)\psi(\varepsilon|\xi|)}{(1+|\xi|^2)^{\frac{d+2}{4}+i\kappa}} \, d\xi, \qquad j \geq 0.$$

From the stationary phase method, it follows that $|\mathcal{K}_\circ(x,t)| \lesssim_\kappa \min(|t|^{-\frac{d}{2}}, 1)$. Thus, to show (3.6), it is enough to show

$$\left| \sum_{j\geq 0} \mathcal{K}_j(x,t) \right| \lesssim_\kappa |t|^{-\frac{d}{2}}. \tag{3.14}$$

After rescaling, we see that

$$\mathcal{K}_j(x,t) = 2^{(\frac{d-2}{2}-2i\kappa)j} \int e^{i(2^j x\cdot\xi+2^j t\sqrt{2^{-2j}+|\xi|^2})} \widetilde{\chi}_j(|\xi|) \, d\xi,$$

where $\widetilde{\chi}_j$ is a smooth function supported in $[1/2, 2]$ that satisfies $\|\widetilde{\chi}_j\|_{C^N} \lesssim (1+|\kappa|)^N$ for any N. Thus, from Lemma 2, we have the following estimate for $j \geq 1$:

$$|\mathcal{K}_j(x,t)| \lesssim_\kappa 2^{\frac{d-2}{2}j} \min(2^{-\frac{d-2}{2}j}|t|^{-\frac{d}{2}}, (2^j|t|)^{-\frac{d-1}{2}}, 1). \tag{3.15}$$

With the estimate (3.15), we justify that it suffices to prove (3.14) in the case where

$$|t| > 1 \qquad \text{and} \qquad 2^{-1}|t| \leq |x| \leq 2|t|. \tag{3.16}$$

Indeed, in the case $|t| \leq 1$, from (3.15), we have

$$\left| \sum_{j\geq 1} \mathcal{K}_j(x,t) \right| \lesssim_\kappa \sum_{j\geq 1} 2^{-\frac{1}{2}j}|t|^{-\frac{d-1}{2}} \lesssim |t|^{-\frac{d-1}{2}} \leq |t|^{-\frac{d}{2}}.$$

Now we assume $|t| > 1$ and $|x| \geq 2|t|$, in which case we have $|\nabla_\xi(2^j x\cdot\xi + 2^j t\sqrt{2^{-2l}+|\xi|^2}| \gtrsim 2^j|x|$ on the support of $\widetilde{\chi}_j(|\cdot|)$ and thus

$$|\mathcal{K}_j(x,t)| \lesssim_\kappa 2^{\frac{d-2}{2}j}(2^j|x|)^{-M} \leq 2^{\frac{d-2}{2}j}(2^j|t|)^{-M},$$

which follows by repeated integration by parts. Applying this with $M > \frac{d}{2}$, we obtain

$$\left| \sum_{j\geq 1} \mathcal{K}_j(x,t) \right| \lesssim_\kappa \sum_{j\geq 1} 2^{\frac{d-2}{2}j}(2^j|t|)^{-M} \lesssim |t|^{-M} \lesssim |t|^{-\frac{d}{2}}.$$

By similar arguments, we may handle the case $|t| > 1$ and $|t| \geq 2|x|$ where $|\nabla_\xi(2^j x\cdot\xi + 2^j t\sqrt{2^{-2l}+|\xi|^2}| \gtrsim 2^j|t|$. We omit the details.

Step 2: Reduction to a one-dimensional oscillatory integral.

For the remainder of the proof of Proposition 3, we assume that x and t satisfy (3.16). Since $\sum_{j\geq 0}\chi_j(\cdot) = 1 - \chi$, to show (3.14), it is sufficient to show that

$$|\widetilde{\mathcal{K}}(x,t)| \leq C(1+|\kappa|)^N |t|^{-\frac{d}{2}}, \tag{3.17}$$

where

$$\widetilde{\mathcal{K}}(x,t) = \int e^{i(x\cdot\xi+t\langle\xi\rangle)}(1-\chi(|\xi|)) \frac{\psi(\varepsilon|\xi|)d\xi}{(1+|\xi|^2)^{\frac{d+2}{4}+i\kappa}}.$$

By changing to spherical coordinates, we write this expression as

$$\widetilde{\mathcal{K}}(x,t) = \int \widehat{\mathrm{d}\sigma}(\rho x) e^{it\langle\rho\rangle}(1-\chi(\rho))\rho^{d-1}\frac{\psi(\varepsilon\rho)\mathrm{d}\rho}{(1+\rho^2)^{\frac{d+2}{4}+i\kappa}}.$$

Since $\rho \gtrsim 1$ on the support of the integrand and $|x| \gtrsim 1$, we may use the asymptotic expansion (3.4). In fact, as before, it is sufficient to consider the contributions from the leading terms, which take the form

$$|x|^{\frac{1-d}{2}}\int e^{i(\pm|x|\rho+t\langle\rho\rangle)}a(\rho)\,\mathrm{d}\rho,$$

where

$$a(\rho) = C_{\pm}(1-\chi(\rho))\rho^{\frac{d-1}{2}}\psi(\varepsilon\rho)(1+\rho^2)^{-\frac{d+2}{4}-i\kappa} = O(\rho^{-\frac{3}{2}})$$

for some constant $C_{\pm}$. We note that a satisfies, for $\ell \geq 0$,

$$\left|\left(\frac{\mathrm{d}}{\mathrm{d}\rho}\right)^{\ell} a(\rho)\right| \leq C_{\ell}(1+|\kappa|)^{N_{\ell}}\rho^{-\frac{3}{2}-\ell}. \tag{3.18}$$

Since we are considering the case $|x| \sim |t| \gtrsim 1$, to show the desired estimate (3.17), we are reduced to showing that

$$\left|\int e^{i(r\rho+t\langle\rho\rangle)}a(\rho)\,\mathrm{d}\rho\right| \leq C(1+|\kappa|)^N |t|^{-\frac{1}{2}}, \quad |r| \sim |t| \gtrsim 1 \tag{3.19}$$

for some N. If r and t have the same sign, then $|\frac{\mathrm{d}}{\mathrm{d}\rho}(r\rho+t\langle\rho\rangle)| \gtrsim |t|$ holds, and therefore (3.19) follows easily by integration by parts. Thus it is enough to consider (3.19) for the case $r < 0$ and $t > 0$, and the case $r > 0$ and $t < 0$. We provide the details of (3.19) when $r < 0$ and $t > 0$ since the case $r > 0$ and $t < 0$ will follow by essentially the same argument.

For $r, t \gtrsim 1$, we set $\widetilde{\chi}_{<t}(\rho) = \sum_{1 \leq 2^j < t}\chi_j(\cdot)$ and set

$$\mathfrak{R}(r,t) = \int e^{i(-r\rho+t\langle\rho\rangle)}\widetilde{\chi}_{<t}(\rho)a(\rho)\mathrm{d}\rho,$$

$$\mathfrak{R}_j(r,t) = \int e^{i(-r\rho+t\langle\rho\rangle)}\chi_j(\rho)a(\rho)\,\mathrm{d}\rho.$$

From (2.2), we have

$$\int e^{i(-r\rho+t\langle\rho\rangle)}a(\rho)\,\mathrm{d}\rho = \mathfrak{R}(r,t) + \sum_{2^j \geq t}\mathfrak{R}_j(r,t).$$

From (3.18), we have trivial estimates $|\mathfrak{R}_j(r,t)| \lesssim \int_{\rho\sim 2^j}|a(\rho)|\,\mathrm{d}\rho \lesssim 2^{-j/2}$. Hence

$$\sum_{2^j \geq t}|\mathfrak{R}_j(r,t)| \lesssim t^{-\frac{1}{2}}.$$

Therefore, to show (3.19) for the case $r < 0$ and $t > 0$, we need only to show that

$$|\mathfrak{R}(r,t)| \leq C(1+|\kappa|)^N t^{-\frac{1}{2}}, \quad r \sim t \gtrsim 1. \tag{3.20}$$

This will be taken up in Steps 3 and 4 below, corresponding to the cases $t \leq r$ and $t \geq r$. The first case is easier since the phase function $-r\rho + t\langle\rho\rangle$ does not have a stationary point. However, in the latter case the phase may have a stationary point, so we make use of additional dyadic decomposition away from the stationary point.

Step 3: Proof of (3.20) *when* $t \le r$.

From (2.2), we note that

$$\mathfrak{K}(r,t) = \sum_{1 \le 2^j < t} \mathfrak{K}_j(r,t). \tag{3.21}$$

Note that, for $\rho \sim 2^j$, we have $|\frac{\mathrm{d}}{\mathrm{d}\rho}(-r\rho + t\langle\rho\rangle)| \ge t(1 - \frac{\rho}{\langle\rho\rangle}) \gtrsim \frac{t}{2^{2j}}$ and from (3.18), it follows that

$$\left\| \frac{\mathrm{d}}{\mathrm{d}\rho}(\chi_j(\cdot)a) \right\|_1 \lesssim_\kappa 2^{-\frac{3}{2}j}, \quad \left\| \chi_j(\cdot)a \right\|_\infty \lesssim_\kappa 2^{-\frac{3}{2}j}. \tag{3.22}$$

Thus, by Proposition 2, we have

$$|\mathfrak{K}_j(r,t)| \lesssim_\kappa \frac{2^{j/2}}{t}. \tag{3.23}$$

From this, (3.20) immediately follows.

Step 4: Proof of (3.20) *when* $t > r$.

We set

$$\rho_* = \frac{r}{\sqrt{t^2 - r^2}},$$

which is the stationary point of the phase function $-r\rho + t\langle\rho\rangle$, and note that $\rho_* \ge 1/\sqrt{3}$ since $2r \ge t$. Also, we may write

$$\frac{\mathrm{d}}{\mathrm{d}\rho}(-r\rho + t\langle\rho\rangle) = -r + \frac{t\rho}{\langle\rho\rangle} = t \int_{\rho_*}^{\rho} \frac{\mathrm{d}s}{(1+s^2)^{\frac{3}{2}}}. \tag{3.24}$$

Now we distinguish the cases:

$$(i)\ \rho_* \ge 2^2 t \quad \text{and} \quad (ii)\ \rho_* \in (2^{-1}, 2^2 t).$$

Case (i). This case is easier to handle since the stationary point ρ_* does not appear in the region of integration. We estimate each of $\mathfrak{K}_j$ in (3.21). For each $\mathfrak{K}_j$, we may clearly assume $\rho \le 2t$. If $\rho \sim 2^j$ and $\rho \le 2t$, then

$$\left| \frac{\mathrm{d}}{\mathrm{d}\rho}(-r\rho + t\langle\rho\rangle) \right| \sim t \int_{\rho}^{\rho_*} \frac{\mathrm{d}s}{s^3} \gtrsim \frac{t}{\rho^2} \sim 2^{-2j} t.$$

Therefore, by Proposition 2 and (3.22), we again obtain (3.23), from which (3.20) follows.

Case (ii). The remainder of Step 4 is devoted to proof of (3.20) for Case (*ii*), and here we split

$$\mathfrak{K}(r,t) = \int_{\rho_*}^{\infty} e^{i(-r\rho+t\langle\rho\rangle)} \widetilde{\chi}_{<t}(\rho)a(\rho)\,\mathrm{d}\rho + \int_0^{\rho_*} e^{i(-r\rho+t\langle\rho\rangle)} \widetilde{\chi}_{<t}(\rho)a(\rho)\,\mathrm{d}\rho$$

and make decomposition away from the stationary point ρ_*. For the integral over (ρ_*, ∞), we make use of a dyadic decomposition to write

$$\int_{\rho_*}^{\infty} e^{i(-r\rho+t\langle\rho\rangle)} \widetilde{\chi}_{<t}(\rho)a(\rho)\,\mathrm{d}\rho = \sum_{2^j \ge \rho_*} \mathfrak{K}_j^{\rho_*} + \sum_{2^j < \rho_*} \mathfrak{K}_j^{\rho_*},$$

where

$$\mathfrak{R}_j^{\rho_*}(r,t) = \int_{\rho_*}^{\infty} e^{i(-r\rho + t\langle\rho\rangle)} \widetilde{\chi}_{<t}(\rho) a(\rho) \chi_j(\rho - \rho_*) \, d\rho.$$

We first consider the former sum $\sum_{2^j \geq \rho_*} \mathfrak{R}_j^{\rho_*}$, which is easier to handle. If $2^j \geq \rho_*$ and $\rho - \rho_* \sim 2^j$, from (3.24), it follows that

$$\left| \frac{d}{d\rho}(-r\rho + t\langle\rho\rangle) \right| \gtrsim t \int_{\rho}^{\rho^*} \frac{ds}{s^3} \gtrsim \frac{t2^j}{\rho\rho_*^2} \gtrsim t2^{-2j}.$$

Since $\rho \sim 2^j$ on the support of integrand, from (3.18), it is easy to see that $\|\widetilde{\chi}_{<t} \, a \, \chi_j(\cdot - \rho_*)\|_\infty \lesssim 2^{-3j/2}$ and $\|\frac{d}{d\rho}(\widetilde{\chi}_{<t} \, a \, \chi_j(\cdot - \rho_*))\|_1 \lesssim_\kappa 2^{-3j/2}$. Thus, by Proposition 2, we have

$$|\mathfrak{R}_j^{\rho_*}(r,t)| \lesssim 2^{j/2} t^{-1}.$$

Therefore

$$\sum_{2^j \geq \rho_*} |\mathfrak{R}_j^{\rho_*}(r,t)| \lesssim t^{-\frac{1}{2}}. \tag{3.25}$$

Here we also use $2^j \lesssim t$ (otherwise the integral is zero).

Now we consider the second sum $\sum_{2^j < \rho_*} \mathfrak{R}_j^{\rho_*}$. Since $2^j < \rho_*$ and $\rho - \rho_* \sim 2^j$, using (3.24), we have

$$\left| \frac{d}{d\rho}(-r\rho + t\langle\rho\rangle) \right| \sim \frac{t2^j}{\rho\rho_*^2} \gtrsim \frac{t2^j}{\rho_*^3}.$$

We also note that $\rho \sim \rho_*$ on the support of integrand. From (3.18), it follows that

$$\|\widetilde{\chi}_{<t} \, a \, \chi_j(\cdot - \rho_*)\|_\infty \lesssim \rho_*^{-3/2}, \quad \left\| \frac{d}{d\rho}(\widetilde{\chi}_{<t} \, a \, \chi_j(\cdot - \rho_*)) \right\|_1 \lesssim_\kappa \rho_*^{-3/2}. \tag{3.26}$$

Hence, Proposition 2 and the trivial estimate $|\mathfrak{R}_j^{\rho_*}(r,t)| \lesssim \rho_*^{-\frac{3}{2}} 2^j$ yield

$$|\mathfrak{R}_j^{\rho_*}(r,t)| \lesssim_\kappa \min(\rho_*^{\frac{3}{2}} 2^{-j} t^{-1}, \rho_*^{-\frac{3}{2}} 2^j).$$

Therefore, by splitting the summation further into the cases $2^j \geq \rho_*^{\frac{3}{2}} t^{-\frac{1}{2}}$ and $2^j < \rho_*^{\frac{3}{2}} t^{-\frac{1}{2}}$, we obtain

$$\sum_{2^j < \rho_*} |\mathfrak{R}_j^{\rho_*}(r,t)| \lesssim_\kappa t^{-\frac{1}{2}}.$$

Thus, combining this with (3.25), we get

$$\left| \int_{\rho_*}^{\infty} e^{i(-r\rho + t\langle\rho\rangle)} \widetilde{\chi}_{<t}(\rho) a(\rho) \, d\rho \right| \lesssim_\kappa t^{-\frac{1}{2}}.$$

To complete the proof, it remains to show

$$\left| \int_{0}^{\rho_*} e^{i(-r\rho + t\langle\rho\rangle)} \widetilde{\chi}_{<t}(\rho) a(\rho) \, d\rho \right| \lesssim_\kappa t^{-\frac{1}{2}}. \tag{3.27}$$

Let set $\widetilde{\chi}_{\rho_*}(\rho) = 1 - \sum_{2^j \leq 2^{-2}\rho_*} \chi_j(\rho_* - \rho)$. As before, we break the integral

$$\int_0^{\rho_*} e^{i(-r\rho + t\langle\rho\rangle)} \widetilde{\chi}_{<t}(\rho) a(\rho)\, d\rho = \sum_{2^j \leq 2^{-2}\rho_*} \Re_{\rho_*, j}(r, t) + \Re_{<\rho_*}(r, t),$$

where

$$\Re_{\rho_*, j}(r, t) = \int_0^{\rho_*} e^{i(-r\rho + t\langle\rho\rangle)} \widetilde{\chi}_{<t}(\rho) a(\rho) \chi_j(\rho - \rho_*)\, d\rho,$$

$$\Re_{<\rho_*}(r, t) = \int_0^{\rho_*} e^{i(-r\rho + t\langle\rho\rangle)} \widetilde{\chi}_{<t}(\rho) a(\rho) \widetilde{\chi}_{\rho_*}(\rho)\, d\rho.$$

For $2^j \leq 2^{-2}\rho_*$, $\rho \in [2^{-1}\rho_*, \rho_*]$ whenever ρ is contained in the support of the integrand of the integral $\Re_{\rho_*, j}(r, t)$. Thus the sum $\sum_{2^j \leq 2^{-2}\rho_*} \Re_{\rho_*, j}(r, t)$ can be handled as it was for $\sum_{2^j < \rho_*} \Re_j^{\rho_*}$. Indeed, note that $|\frac{d}{d\rho}(-r\rho + t\langle\rho\rangle)| \sim \frac{t2^j}{\rho^2\rho_*} \gtrsim \frac{t2^j}{\rho_*^3}$, and we also have (3.26). Thus, similarly to before, by Proposition 2 and the trivial estimate, it follows that $|\Re_{\rho_*, j}(r, t)| \lesssim_\kappa \min(\rho_*^{\frac{3}{2}} 2^{-j} t^{-1}, \rho_*^{-\frac{3}{2}} 2^j)$. This gives

$$\sum_{2^j \leq 2^{-2}\rho_*} |\Re_{\rho_*, j}(r, t)| \lesssim_\kappa |t|^{-\frac{1}{2}}.$$

Thus, to show (3.27), it remains to show that

$$|\Re_{<\rho_*}(r, t)| \lesssim_\kappa t^{-\frac{1}{2}}. \tag{3.28}$$

We make a dyadic decomposition away from the origin. Let us set

$$\mathcal{A}_\ell = \widetilde{\chi}_{<t}(\rho) a(\rho) \widetilde{\chi}_{\rho_*}(\rho) \chi_\ell(\rho).$$

We may write

$$\Re_{<\rho_*}(r, t) = \sum_{1 \lesssim 2^\ell \lesssim \rho_*} \int_0^{\rho_*} e^{i(-r\rho + t\langle\rho\rangle)} \mathcal{A}_\ell(\rho)\, d\rho. \tag{3.29}$$

We note that $\widetilde{\chi}_{<t} \widetilde{\chi}_{\rho_*} \chi_{(0,\rho_*)}$ is supported in the interval $[2^{-1}, (1 - 2^{-4})\rho_*]$. Since $\rho \sim 2^l$ and $\rho_* - \rho \sim \rho_*$ on the support of $\mathcal{A}_\ell$, by (3.24), we have

$$\left| \frac{d}{d\rho}(-r\rho + t\langle\rho\rangle) \right| \gtrsim \int_\rho^{\rho_*} \frac{ds}{s^3} \sim \frac{t}{\rho^2} \sim \frac{t}{2^{2\ell}},$$

provided ρ is contained in the support of $\mathcal{A}_\ell$. Also from (3.18), it follows that $\|\mathcal{A}_\ell\|_\infty \lesssim 2^{-3\ell/2}$ and $\|\frac{d}{d\rho}\mathcal{A}_\ell\|_1 \lesssim 2^{-3\ell/2}$. By making use of Proposition 2, we get

$$\left| \int_0^{\rho_*} e^{i(-r\rho + t\langle\rho\rangle)} \mathcal{A}_\ell(\rho)\, d\rho \right| \lesssim 2^{\frac{1}{2}\ell} t^{-1}.$$

Since $2^\ell \leq \rho_* \lesssim t$, by this and (3.29), we now obtain (3.28). This completes our proof of Step 4 and thus Proposition 5.   □

Remark 2. The estimates (3.5) and (3.6) continue to be valid with higher order of damping. Such estimates can be shown without difficulty by following the argument in the proofs of Proposition 4 and

Proposition 5, so we state the estimates without proof. For $\gamma > 0$, there exist constants $C < \infty$ and $N > 0$ such that

$$\left| \int e^{i(x \cdot \xi + t \langle \xi \rangle)} \psi(\varepsilon \xi)(1 + |\xi|^2)^{-\frac{d+1}{4} - \gamma - i\kappa} \, d\xi \right| \le C(1 + |\kappa|)^N |t|^{-\frac{d-1}{2}}, \tag{3.30}$$

$$\left| \int e^{i(x \cdot \xi + t \langle \xi \rangle)} \psi(\varepsilon \xi)(1 + |\xi|^2)^{-\frac{d+2}{4} - \gamma - i\kappa} \, d\xi \right| \le C(1 + |\kappa|)^N |t|^{-\frac{d}{2}} \tag{3.31}$$

for all $\kappa \in \mathbb{R}$, $\varepsilon > 0$ and $(x, t) \in \mathbb{R}^d \times \mathbb{R}$.

3.3. The fractional Schrödinger equation

We now provide the damped oscillatory integral estimates that are key to the proof of Theorem 4. For our purpose we only need to consider the oscillatory integrals with phases of the form $x \cdot \xi + t|\xi|^\alpha$, but our method here admits extension of $|\xi|^\alpha$ to more general symbols ϕ that are basically perturbations of a homogeneous function. To this end, we first introduce the notion of an *almost homogeneous phase* that is inspired by the condition introduced by Kenig–Ponce–Vega [49, Lemma 3.4].

Definition 1. Let $\alpha \in \mathbb{R} \setminus \{0, 1\}$. We say $\phi \in \mathfrak{R}(\alpha, B, \lambda, d_1, d_2)$ if $\psi \in C^{d+2}(\mathbb{R}^d \setminus \{0\})$ and there are positive constants d_1, d_2, λ and B such that, for all $\xi \in \mathbb{R}^d \setminus \{0\}$,

$$d_1|\xi|^{\alpha-1} \le |\nabla\phi(\xi)| \le d_2|\xi|^{\alpha-1}, \tag{3.32}$$

$$|\partial^\gamma \phi(\xi)| \le B|\xi|^{\alpha - |\gamma|}, \quad |\gamma| \le d + 2, \tag{3.33}$$

$$\lambda|\xi|^{\alpha-2} \le |v^t H\phi(\xi)v|, \quad \forall v \in \mathbb{S}^{d-1}. \tag{3.34}$$

We also say ϕ is *almost homogeneous of order α* if $\phi \in \mathfrak{R}(\alpha, B, \lambda, d_1, d_2)$ for some positive constants d_1, d_2, λ and B.

Kenig–Ponce–Vega [49] considered phase functions ϕ satisfying (3.32), (3.33) and

$$\lambda_1|\xi|^{d(\alpha-2)} \le |\det H\phi(\xi)| \le \lambda_2|\xi|^{d(\alpha-2)} \tag{3.35}$$

for some λ_1, λ_2 instead of (3.34). It is easy to see that (3.33) and (3.34) imply (3.35). Indeed, (3.33) implies that all the absolute values of the eigenvalues of $\det H\phi(\xi)$ are at most $d^2 B|\xi|^{\alpha-2}$, and from (3.34), we see the absolute values of all eigenvalues of $\det H\phi(\xi)$ are at least $\lambda|\xi|^{\alpha-2}$. Thus we get (3.35) with $\lambda_2 = (d^2 B)^d$ and $\lambda_1 = \lambda^d$. The converse is also not difficult to see. For this we use the following simple observation, which is also useful in what follows. This can be shown by direct computation, so we omit the proof.

Lemma 3. Let $\alpha \in \mathbb{R} \setminus \{0, 1\}$ and $\rho > 0$. Set

$$\phi_\rho^\alpha(\xi) = \rho^{-\alpha}\phi(\rho\xi).$$

Then if ϕ satisfies (3.32), so does ϕ_ρ^α. The same is true with the conditions (3.33), (3.34), (3.35). That is to say, for any $\rho > 0$, $\phi \in \mathfrak{R}(\alpha, B, \lambda, d_1, d_2)$ if and only if $\phi_\rho^\alpha \in \mathfrak{R}(\alpha, B, \lambda, d_1, d_2)$.

Proof of implication from (3.35) to (3.34).
Now we show (3.35) implies (3.34) under the condition (3.33). Let $\xi \in \mathbb{R}^d \setminus \{0\}$ and suppose $|\xi| \in (\lambda, 2\lambda]$ for some $\lambda > 0$, then the conditions (3.33), (3.34) and (3.35) are invariant under $\phi \to \phi_\lambda^\alpha$ by Lemma 3. Thus it is enough to show that, for $|\xi| \in (1, 2]$, the conditions (3.33) and (3.35) imply (3.34). By the condition (3.33), we see the eigenvalues $\mu_1, \mu_2, \ldots, \mu_d$ of the matrix $H\phi(\xi)$ satisfy $|\mu_i| \le b := d^2 B \max(2^{\alpha-2}, 1)$ for $i = 1, \ldots, d$. Since $\mu_1 \mu_2 \cdots \mu_d = \det H\phi(\xi)$ and $\lambda_1 \min(2^{\alpha-2}, 1) \le |\det H\phi(\xi)|$,

it follows that

$$b^{1-d}\lambda_1 \min(2^{\alpha-2}, 1) \le |\mu_i|.$$

Since $\min \mu_i \le v^t H\phi(\xi)v \le \max \mu_i$ for $v \in \mathbb{S}^{d-1}$, we see that (3.34) holds with $\lambda = b^{1-d}\lambda_1 \min(2^{\alpha-2}, 1)(\max(2^{\alpha-2}, 1))^{-1}$. This completes the proof of the implication. $\qquad\square$

For a given ϕ that is almost homogeneous of order $\alpha \in \mathbb{R} \setminus \{0, 1\}$, the definition of the relevant oscillatory integral depends on the sign of α. If $\alpha > 0$, for each $\kappa \in \mathbb{R}$ and $(x, t) \in \mathbb{R}^d \times \mathbb{R}$, we define the oscillatory integral

$$\mathcal{I}_{\varepsilon,\kappa}^{\phi,+}(x, t) = \int_{\mathbb{R}^d} e^{i(x\cdot\xi + t\phi(\xi))}\chi(\varepsilon\xi)|\det H\phi(\xi)|^{1/2+i\kappa} \, d\xi.$$

If $\alpha < 0$, for each $\kappa \in \mathbb{R}$ and $(x, t) \in \mathbb{R}^d \times \mathbb{R}$, we define the oscillatory integral

$$\mathcal{I}_{\varepsilon,\kappa}^{\phi,-}(x, t) = \int_{\mathbb{R}^d} e^{i(x\cdot\xi + t\phi(\xi))}\chi_\infty(\varepsilon^{-1}\xi)|\det H\phi(\xi)|^{1/2+i\kappa} \, d\xi,$$

where $\chi_\infty := 1 - \chi$. Note that the cutoff function χ has compact support, and χ_∞ is supported away from the origin. Observe also that

$$|\det H\phi(\xi)|^{1/2} \sim |\xi|^{\frac{\alpha-2}{2}d} \tag{3.36}$$

for $\alpha \ne 0, 1$. Thus, $\mathcal{I}_{\varepsilon,\kappa}^{\phi,+}, \mathcal{I}_{\varepsilon,\kappa}^{\phi,-}$ are well defined for the cases $\alpha > 0$, $\alpha < 0$, respectively.

Our main oscillatory integral estimate is as follows.

Proposition 6. *Let $\alpha \in \mathbb{R} \setminus \{0, 1\}$ and ϕ be almost homogeneous of order α. Then there exist $C < \infty$ and $N > 0$ such that*

$$|\mathcal{I}_{\varepsilon,\kappa}^{\phi,\pm}(x, t)| \le C(1 + |\kappa|)^N |t|^{-d/2}, \quad \pm\alpha > 0, \tag{3.37}$$

for all $\varepsilon > 0$, $\kappa \in \mathbb{R}$ and $(x, t) \in \mathbb{R}^d \times \mathbb{R}$.

Remark 3. As is to be clearly seen from the proof, for the estimate (3.37) with $+$ it is enough to have the conditions (3.32), (3.33) and (3.34) for $|\xi| \le 2\varepsilon^{-1}$. Similarly, for (3.37) with $-$, we only need to consider $|\xi| \ge 2\varepsilon$. Also, we remark that Proposition 6 has been already proved by Kenig, Ponce and Vega [49, Lemma 3.4] when $\alpha \ge 2$.

The proof of Proposition 6 relies heavily on stability of the oscillatory integral estimates that are obtained by the stationary phase method. For example, see [43, p. 220, Theorem 7.7.5] and [1, Theorem 1].

Lemma 4. *Let $\alpha \in \mathbb{R} \setminus \{0, 1\}$. Suppose $\phi \in \mathfrak{R}(\alpha, B, \lambda, d_1, d_2)$ for some positive constants λ, d_1, d_2 and ψ is a smooth function supported in the set $\{\xi : 2^{-1} \le |\xi| \le 2\}$. Then there exists a constant $C < \infty$ depending only on λ, B, d_1, d_2, such that*

$$\left| \int_{\mathbb{R}^d} e^{i(x\cdot\xi + t\phi(\xi))}\psi(\xi) \, d\xi \right| \le C\|\psi\|_{C^{d+1}} |t|^{-d/2}. \tag{3.38}$$

This may be seen as a simple consequence of [1, Theorem 1]. Since (3.34) gives uniform lower bounds for the eigenvalues of the Hessian matrix, the lemma also can be proved by making use of the standard argument that relies on the Morse Lemma.

Proof. Since ψ is supported in the set $\{\xi : 2^{-1} \le |\xi| \le 2\}$, from (3.32), it is easy to see that there are c_1, c_2, b_1, and $b_2 > 0$, depending only on d_1, d_2, such that $|\nabla_\xi(x \cdot \xi + t\phi(\xi))| \ge c_1|t|$ if $|t| > b_1|x|$ and $|\nabla_\xi(x \cdot \xi + t\phi(\xi))| \ge c_2|x|$ if $|x| > b_2|t|$. Thus, in either case, by integration by parts, we get (3.38)

with C depending only on c_1, c_2 and B. Thus we may assume $b_2^{-1}|x| \le |t| \le b_1|x|$. We may rewrite the estimate as

$$\left| \int_{\mathbb{R}^d} e^{it\Phi_{x,t}(\xi)} \psi(\xi) \, d\xi \right| \le C|t|^{-d/2}$$

with $\Phi_{x,t}(\xi) = t^{-1}x \cdot \xi + \phi(\xi)$. We note that $\Phi_{x,t}$ is uniformly bounded in C^{d+2} and $|\det(H\Phi_{x,t})|$ has a nonzero uniform lower bound. Once we have ensured these conditions, then we may employ the stationary phase method for the oscillatory integral (see [1, Theorem 1][4] or, alternatively, Theorem 7.7.5 in [43], and Theorem 3 and Example 4 in [1], for more explicit statements), to have the desired uniform bound. □

Proof (Proof of Proposition 6). First of all, we observe that it suffices to show the estimate when $\varepsilon = 1$. Let $\phi \in \Re(\alpha, B, \lambda, d_1, d_2)$ for some positive constants $\alpha, B, \lambda, d_1, d_2$. Since $\det H\phi_\rho^\alpha(\xi) = \rho^{d(2-\alpha)} \det H\phi(\rho\xi)$ for any $\rho > 0$, after rescaling $\xi \to \varepsilon^{\mp 1}\xi$, we have

$$\mathfrak{I}_{\varepsilon,\kappa}^{\phi,\pm}(x,t) = \varepsilon^{\pm[-\frac{\alpha d}{2}+id(2-\alpha)\kappa]} \mathfrak{I}_{1,\kappa}^{\phi_{\varepsilon^{\mp 1}}^\alpha,\pm}(\varepsilon^{\mp 1}x, \varepsilon^{\mp\alpha}t), \quad \pm\alpha > 0.$$

By Lemma 3, $\phi_{\varepsilon^{\mp 1}}^\alpha \in \Re(\alpha, B, \lambda, d_1, d_2)$. Thus, the desired estimate follows if we show

$$|\mathfrak{I}_{1,\kappa}^{\phi,\pm}(x,t)| \lesssim_\kappa |t|^{-d/2}, \tag{3.39}$$

whenever $\phi \in \Re(\alpha, B, \lambda, d_1, d_2)$. Since $|\mathfrak{I}_{1,\kappa}^{\phi,\pm}| \lesssim 1$ by (3.36), we may assume $|t| > 1$; otherwise the estimate is trivial. To show (3.39), we need to consider two cases, $\alpha > 0$ and $\alpha < 0$, separately.

The case $\alpha > 0$: estimate for $\mathfrak{I}_{1,\kappa}^{\phi,+}$.

We use the dyadic decomposition $\chi = \sum_{j=1}^\infty \chi_0(2^j \cdot)$ (recall (2.2)). By changing variables, we write

$$\mathfrak{I}_{1,\kappa}^{\phi,+} = \sum_{j=1}^\infty I_j := \sum_{j=1}^\infty C_{\kappa,j} 2^{-\frac{d\alpha}{2}j} \int e^{i(2^{-j}x \cdot \xi + 2^{-\alpha j}t\phi_j(\xi))} |\det H\phi_j(\xi)|^{1/2+i\kappa} \chi_0(|\xi|) \, d\xi,$$

where $\phi_j := \phi_{2^{-j}}^\alpha$ (recall the notation in Lemma 3), and $C_{\kappa,j}$ is a complex number with $|C_{\kappa,j}| = 1$. Let us set

$$a_1 = 2^{-1}d_1 \min(2^{1-\alpha}, 2^{\alpha-1}), \quad a_2 = 2d_2 \max(2^{1-\alpha}, 2^{\alpha-1}).$$

We consider the following three cases:

$$\mathbf{A}: \quad 2^{(\alpha-1)j}|x| \ge a_2|t|,$$
$$\mathbf{B}: \quad a_1|t| < 2^{(\alpha-1)j}|x| < a_2|t|,$$
$$\mathbf{C}: \quad 2^{(\alpha-1)j}|x| \le a_1|t|.$$

In case **B**, there are only finitely many j. By Lemma 3, $\phi_j \in \Re(\alpha, B, \lambda, d_1, d_2)$. Thus, applying Lemma 4 to each I_j, we see

$$\sum_{j \in \mathbf{B}} |I_j| \le C \sum_{j \in \mathbf{B}} 2^{-\frac{d\alpha}{2}j} |2^{-\alpha j}t|^{-d/2} \sim |t|^{-d/2}.$$

For case **C**, since ϕ_j satisfies (3.32), we have

$$|\nabla_\xi(2^{-j}x \cdot \xi + 2^{-\alpha j}t\phi_j(\xi))| \ge 2^{-\alpha j}d_1 \min(2^{1-\alpha}, 2^{\alpha-1})|t| - 2^{-j}|x| \ge a_1 2^{-\alpha j}|t|$$

---

[4]Here we need the condition $\phi \in C^{d+2}$.

on the support of χ_0. So, integration by parts gives $|I_j| \lesssim_\kappa 2^{-\frac{d\alpha}{2}j}\min(|2^{-\alpha j}t|^{-N}, 1)$ for $N \le d + 2$. Since $\alpha > 0$, taking $N > d/2$, we get

$$\sum_{j \in \mathbf{C}} |I_j| \lesssim_\kappa \sum_{j \in \mathbf{C}} 2^{-\frac{d\alpha}{2}j}\min(|2^{-\alpha j}t|^{-N}, 1) \lesssim |t|^{-d/2}.$$

We now consider case **A**. As before, we know from (3.32), Lemma 3 and assumption $j \in \mathbf{A}$ that

$$|\nabla_\xi(2^{-j}x \cdot \xi + 2^{-\alpha j}t\phi_j(\xi))| \ge 2^{-j}|x| - 2^{-\alpha j}d_2\max(2^{1-\alpha}, 2^{\alpha-1})|t| \ge 2^{-1}2^{-j}|x|.$$

Integration by parts yields $|I_j| \lesssim_\kappa 2^{-\frac{\alpha d}{2}j}\min(|2^{-j}x|^{-M}, 1)$ for any $M \le d + 2$. In particular, taking $M = d/2$, we get

$$|I_j| \lesssim_\kappa \min(2^{\frac{d}{2}(1-\alpha)j}|x|^{-\frac{d}{2}}, 2^{-\frac{\alpha d}{2}j}). \tag{3.40}$$

To sum each estimate, we need to separate two cases $0 < \alpha < 1$ and $\alpha > 1$. First, consider $0 < \alpha < 1$. In this case, $2^j \lesssim (|x|/|t|)^{\frac{1}{1-\alpha}}$, and using (3.40), we have

$$\sum_{j \in \mathbf{A}} |I_j| \lesssim_\kappa |x|^{-\frac{d}{2}} \sum_{1 \le 2^j \lesssim (|x|/|t|)^{\frac{1}{1-\alpha}}} 2^{\frac{d}{2}j(1-\alpha)} \sim |t|^{-\frac{d}{2}}.$$

For the case $\alpha > 1$, $2^j \gtrsim (|t|/|x|)^{1/(\alpha-1)}$. Thus, by (3.40), we have

$$\sum_{j \in \mathbf{A}} |I_j| \lesssim_\kappa \sum_{2^j \gtrsim (|t|/|x|)^{1/(\alpha-1)}}^{\infty} 2^{\frac{d}{2}j(1-\alpha)}|x|^{-\frac{d}{2}} \lesssim |t|^{-\frac{d}{2}}.$$

This completes the proof of (3.39) with + for the case $\alpha > 0$.

The case $\alpha < 0$: estimate for $\mathfrak{I}_{1,\kappa}^{\phi,-}$.

This can be handled in a similar manner, so we shall be brief. Using the dyadic decomposition $\chi_\infty = \sum_{j=0}^{\infty} \chi_j(\cdot)$ and changing variables, we write

$$\mathfrak{I}_{1,\kappa}^{\phi,-} = \sum_{j=0}^{\infty} \widetilde{I}_j := \sum_{j=0}^{\infty} \widetilde{C}_{\kappa,j} 2^{\frac{d\alpha}{2}j} \int e^{i(2^j x \cdot \xi + 2^{\alpha j}t\widetilde{\phi}_j(\xi))}|\det H\widetilde{\phi}_j(\xi)|^{1/2+i\kappa}\chi_0(|\xi|)\,d\xi,$$

where $\widetilde{\phi}_j = \phi_{2^j}^\alpha$ and $\widetilde{C}_{\kappa,j}$ is a complex number with $|\widetilde{C}_{\kappa,j}| = 1$. As before, we consider the following three cases:

$$\begin{aligned}
\widetilde{\mathbf{A}}: \quad & 2^{(1-\alpha)j}|x| \ge a_2|t|, \\
\widetilde{\mathbf{B}}: \quad & a_1|t| < 2^{(1-\alpha)j}|x| < a_2|t|, \\
\widetilde{\mathbf{C}}: \quad & 2^{(1-\alpha)j}|x| \le a_1|t|.
\end{aligned}$$

For case $\widetilde{\mathbf{B}}$, there are only finitely many j. By Lemma 3 and Lemma 4, it follows that

$$\sum_{j \in \widetilde{\mathbf{B}}} |\widetilde{I}_j| \le C \sum_{j \in \widetilde{\mathbf{B}}} 2^{\frac{d\alpha}{2}j}|2^{\alpha j}t|^{-d/2} \sim |t|^{-d/2}.$$

In case $\widetilde{\mathbf{C}}$, similar to before, we see $|\nabla_\xi(2^j x \cdot \xi + 2^{\alpha j}t\widetilde{\phi}_j(\xi))| \gtrsim 2^{\alpha j}|t|$. Hence, integration by parts gives $|\widetilde{I}_j| \lesssim 2^{\frac{d\alpha}{2}j}\min(|2^{\alpha j}t|^{-N}, 1)$ for $N \le d + 2$. Since $\alpha < 0$, again taking $N > \frac{d}{2}$ and splitting the

sum into the cases $2^j \le |t|^{-1/\alpha}$ and $2^j > |t|^{-1/\alpha}$, we see

$$\sum_{j \in \widetilde{\mathbf{C}}} |\widetilde{I}_j| \lesssim_\kappa \sum_{j=0}^{\infty} 2^{\frac{d\alpha}{2} j} \min\left(|2^{\alpha j} t|^{-N}, 1\right) \lesssim |t|^{-d/2}.$$

Finally, for case $\widetilde{\mathbf{A}}$, we have $|\nabla_\xi (2^j x \cdot \xi + 2^{\alpha j} t \widetilde{\phi}_j(\xi))| \gtrsim 2^j |x|$. Integration by parts yields $|\widetilde{I}_j| \lesssim 2^{\frac{d\alpha}{2} j} \min\left((2^j |x|)^{-N}, 1\right)$ for $N \le d + 2$. In particular, taking $N = d/2$, we see

$$\sum_{j \in \widetilde{\mathbf{A}}} |\widetilde{I}_j| \le C|x|^{-d/2} \sum_{2^j \gtrsim (|t|/|x|)^{1/(1-\alpha)}} 2^{\frac{d}{2}(\alpha-1)j} \lesssim |t|^{-d/2}.$$

Combining these estimates for the cases $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{C}}$ gives (3.37) with $-$. This completes the proof. $\qquad\square$

We conclude this section with some remarks on our results by comparing them with previously known results.

Remark 4. First, the estimate for the wave case (Proposition 3) is of a different nature from the standard context of damped oscillatory integral estimates since the determinant of the Hessian matrix is identically zero. As far as we are aware, no such result has previously appeared in the literature. The same also applies to Proposition 4. Secondly, concerning Proposition 5, note that if $|\xi|$ is large, then $|\nabla\langle\xi\rangle| \sim 1$ and

$$\det H\langle\xi\rangle = \langle\xi\rangle^{-3d} |\xi|^{2d-2} \sim |\xi|^{-d-2},$$

which means $\langle\xi\rangle$ is not almost homogeneous. Thus, the estimate (3.6) cannot be covered by Proposition 6, and we extend the result in Kenig–Ponce–Vega [49]. Also, (3.6) cannot be deduced from Carbery and Ziesler's work [12] since the result in [12] is local in its nature and the basic convexity assumption is not satisfied (precisely, $\phi(r) = \langle r \rangle$ is convex, but ϕ' is not convex).

4. Strichartz estimates for orthonormal families: the sharp admissible case

In this section, making use of the weighted oscillatory integral estimates in the previous section, we prove the sufficiency parts of Theorems 5, 6, 7 and 4 (in this order). The necessity part is to be discussed later in Section 6.

We begin by stating the following result, which formalizes the argument we use to establish the orthonormal Strichartz estimates.

Proposition 7. *Let $\sigma > 0$, and assume that (q, r) is sharp σ-admissible such that*[5]

$$\max\{1 + 2\sigma, 2\} < \widetilde{r} < 2 + 2\sigma. \tag{4.1}$$

Let S be a domain in the complex plane that contains the strip $\{z \in \mathbb{C} : -\widetilde{r}/2 \le \Re z \le 0\}$. Suppose that $(\Theta_z)_z$ is an analytic family of functions for $z \in S$[6] *that satisfies the following: for some constants A_0, A_1 and $N > 0$,*

$$\sup_{\xi \in \mathbb{R}^d} |\Theta_{i\kappa}(\xi)| \le (1 + |\kappa|)^N A_0, \tag{4.2}$$

$$\left| \int e^{i(x \cdot \xi + t\phi(\xi))} \Theta_{-\frac{\widetilde{r}}{2} + i\kappa}(\xi) \, d\xi \right| \le (1 + |\kappa|)^N A_1 |t|^{-\sigma}. \tag{4.3}$$

---

[5]Equivalently, $\frac{2(\sigma+1)}{\sigma} < r < \infty$ if $0 < \sigma \le \frac{1}{2}$, or $\frac{2(\sigma+1)}{\sigma} < r < \frac{2(2\sigma+1)}{2\sigma-1}$ if $\sigma > \frac{1}{2}$.
[6]This means the map $z \mapsto \Theta_z(\xi)$ is an analytic function on S for each $\xi \in \mathbb{R}^d$.

Additionally, if Θ_{-1} is nonnegative, then

$$\left\| \sum_j \nu_j |U_\phi \sqrt{\Theta_{-1}(D)} f_j|^2 \right\|_{L_t^{\frac{q}{2},\beta} L_x^{\frac{r}{2}}} \lesssim A_0^{\frac{2}{r}} A_1^{1-\frac{2}{r}} \|\nu\|_{\ell^\beta}$$

holds for all families of orthonormal functions $(f_j)_j$ in L^2 and $\beta = \frac{2r}{r+2}$.

The assumption that Θ_{-1} is nonnegative is not essential. As long as $\sqrt{\Theta_{-1}(D)}$ can be properly defined, Proposition 7 is valid.

Proof. Consider the analytic family operators $(T_z)_z$, which are given by

$$\mathcal{F}(T_z G)(\xi, \tau) = \frac{(\tau - \phi(\xi))_+^z}{\Gamma(1+z)} A_0^{-1-\frac{2}{r}z} A_1^{\frac{2}{r}z} \Theta_z(\xi) \widehat{G}(\xi, \tau).$$

Note that

$$T_{-1} = \frac{1}{2\pi} A_0^{-1+\frac{2}{r}} A_1^{-\frac{2}{r}} U_\phi \Theta_{-1}(D) U_\phi^*.$$

Therefore, by Proposition 1, it suffices to prove

$$\|W_1 T_{-1} W_2\|_{\mathcal{C}^{\tilde{r}}} \lesssim \|W_1\|_{(\tilde{q},2\tilde{r}),\tilde{r}} \|W_2\|_{(\tilde{q},2\tilde{r}),\tilde{r}}.$$

Since (q, r) is sharp σ-admissible, we note that $\frac{\tilde{r}}{2\tilde{q}} = \frac{\tilde{r}-2\sigma}{4}$. By analytic interpolation, this follows from the estimates

$$\|W_1 T_{i\kappa} W_2\|_{\mathcal{C}^\infty} \leq C_\kappa \|W_1\|_{\infty,\infty} \|W_2\|_{\infty,\infty} \tag{4.4}$$

and

$$\|W_1 T_{-\frac{\tilde{r}}{2}+i\kappa} W_2\|_{\mathcal{C}^2} \leq C_\kappa \|W_1\|_{(\frac{4}{\tilde{r}-2\sigma},4),2} \|W_2\|_{(\frac{4}{\tilde{r}-2\sigma},4),2} \tag{4.5}$$

for constants C_κ, which grow at most exponentially with κ. Since (4.4) is equivalent to the L^2 boundedness of $T_{i\kappa}$, the estimate in (4.4) is an easy consequence of Plancherel's theorem and the assumption (4.2).

For (4.5), let $z = -\frac{\tilde{r}}{2} + i\kappa$, in which case the kernel of T_z is given by

$$\frac{C}{\Gamma(1 - \frac{\tilde{r}}{2} + i\kappa)} \int \tau_+^z e^{i(t-s)\tau} \left(\int e^{i((x-y)\cdot\xi + (t-s)\phi(\xi))} \Theta_{-\frac{\tilde{r}}{2}+i\kappa}(\xi) \, d\xi \right) d\tau.$$

Thus, by the assumption (4.3), we see[7] that the kernel $K_{-\frac{\tilde{r}}{2}+i\kappa}$ of the operator $W_1 T_{-\frac{\tilde{r}}{2}+i\kappa} W_2$ satisfies the bound

$$|K_{-\frac{\tilde{r}}{2}+i\kappa}(x, t, y, s)| \leq C_\kappa |W_1(x, t)| |t - s|^{-\sigma - 1 + \frac{\tilde{r}}{2}} |W_2(y, s)|$$

for some constant C_κ, which grows at most exponentially with κ. Using (2.1) and the assumption (4.1), we obtain

$$\|W_1 T_{-\frac{\tilde{r}}{2}+i\kappa} W_2\|_{\mathcal{C}^2}^2 \leq C_\kappa \iiiint |W_1(x, t)|^2 |t - s|^{-2\sigma - 2 + \tilde{r}} |W_2(y, s)|^2 \, dx\,dy\,ds\,dt$$

$$\leq C_\kappa \|W_1^2\|_{(u,2),1} \|W_2^2\|_{(u,2),1},$$

where $\frac{2}{u} = \tilde{r} - 2\sigma$. The estimate in (4.5) follows. $\qquad\square$

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[7] Here we are using the fact that $\mathcal{F}(\frac{t_+^z}{\Gamma(z+1)})(\tau) = ie^{iz\pi/2}(\tau + i0)^{-z-1} = i(e^{iz\pi/2}\tau_+^{-z-1} - e^{-iz\pi/2}\tau_-^{-z-1})$, where the latter equality is true if $z \notin \mathbb{Z}$ (see [33, p. 172]).

4.1. The wave equation

Since we now have Proposition 7, to show the sufficiency part of Theorem 5, we need to choose an appropriate analytic family $\Theta_z(\xi)$. However it is already more or less clear from the perspective of Proposition 3.

Proof (Proof of Theorem 5 (sufficiency part)). It suffices to prove

$$\left\| \sum_j \nu_j |e^{it\sqrt{-\Delta}} f_j|^2 \right\|_{(\frac{q}{2},\beta),\frac{r}{2}} \lesssim \|\nu\|_{\ell^\beta} \tag{4.6}$$

for (q,r), which are sharp $\frac{d-1}{2}$-admissible, $\beta = \frac{2r}{r+2}$, and

$$\frac{2(d+1)}{d-1} < r < \frac{2d}{d-2}, \tag{4.7}$$

where $(f_j)_j$ is an orthonormal family in $\dot{H}^s$ and $s = \frac{d+1}{2}(\frac{1}{2} - \frac{1}{r})$. Indeed, for such (q,r), we have $\frac{q}{2} > \beta$, and thus the strong type estimate (1.12) follows by embedding between Lorentz spaces. Since this estimate is trivially valid when $(q,r,\beta) = (\infty,2,1)$, by interpolation we may extend the range to the full range $2 \leq r < \frac{2d}{d-2}$.

Let us set

$$g_j = |D|^{\frac{d+1}{2}(\frac{1}{2}-\frac{1}{r})} f_j.$$

Clearly, $(g_j)_j$ is an orthonormal family in L^2. To prove (4.6) under the condition (4.7), we use Proposition 7 with

$$\Theta_z(\xi) = \frac{\widetilde{r} + 2z}{\widetilde{r} - 2} |\xi|^{\frac{d+1}{\widetilde{r}}z} \chi^2(|\xi|).$$

Then (4.2) holds, and from Proposition 3, we have (4.3) with $\sigma = \frac{d-1}{2}$. Also, note that $d < \widetilde{r} < d+1$ from (4.7). Since $\Theta_{-1}(\xi) = |\xi|^{-(d+1)(\frac{1}{2}-\frac{1}{r})} \chi^2(|\xi|)$, it follows from Proposition 7 that we have the estimate

$$\left\| \sum_j \nu_j \left| e^{it\sqrt{-\Delta}} \chi(|D|)|D|^{-s} g_j \right|^2 \right\|_{(\frac{q}{2},\beta),\frac{r}{2}} \lesssim \|\nu\|_{\ell^\beta}$$

with $s = \frac{d+1}{2}(\frac{1}{2} - \frac{1}{r})$ and $\beta = \frac{2r}{r+2}$. This immediately yields

$$\left\| \sum_j \nu_j |e^{it\sqrt{-\Delta}} \chi(|D|) f_j|^2 \right\|_{(\frac{q}{2},\beta),\frac{r}{2}} \lesssim \|\nu\|_{\ell^\beta}$$

for all families of orthonormal functions $(f_j)_j$ in $\dot{H}^s$, with $s = \frac{d+1}{2}(\frac{1}{2} - \frac{1}{r})$ and $\beta = \frac{2r}{r+2}$. Note that the sharp $\frac{d-1}{2}$-admissible estimate (4.6) with $s = \frac{d+1}{2}(\frac{1}{2} - \frac{1}{r})$ is invariant under rescaling. Thus, it is easy to see that rescaling the above estimate gives

$$\left\| \sum_j \nu_j |e^{it\sqrt{-\Delta}} \chi(\varepsilon|D|)^{1/2} f_j|^2 \right\|_{(\frac{q}{2},\beta),\frac{r}{2}} \lesssim \|\nu\|_{\ell^\beta}$$

uniformly in $\varepsilon > 0$, and letting ε tend to zero, we obtain (4.6). □

4.2. The Klein–Gordon equation

As mentioned in the introduction, we separately handle the Schrödinger-like case and the wave-like case. As before, the proof is rather straightforward once we have the right analytic family, so we shall be brief.

Proof (Proof of Theorem 7 (sufficiency part)). We only handle the critical case $s = \frac{d+1}{2}(\frac{1}{2} - \frac{1}{r})$ since the other case $s > \frac{d+1}{2}(\frac{1}{2} - \frac{1}{r})$ can be shown in exactly same manner by making use of the estimate (3.30) instead of (3.5).

In a similar manner to the proof of Theorem 5, it suffices to prove the estimate

$$\left\| \sum_j v_j |e^{it\sqrt{1-\Delta}} \chi(\varepsilon|D|) f_j|^2 \right\|_{(\frac{q}{2},\beta),\frac{r}{2}} \lesssim \|v\|_{\ell^\beta} \tag{4.8}$$

uniformly in $\varepsilon > 0$. Here, (q,r) is sharp $\frac{d-1}{2}$-admissible with r satisfying (4.7), $\beta = \frac{2r}{r+2}$, and an orthonormal family $(f_j)_j$ in H^s, $s = \frac{d+1}{2}(\frac{1}{2} - \frac{1}{r})$. To this end, we use Proposition 7 with

$$\Theta_z(\xi) = \frac{\widetilde{r} + 2z}{\widetilde{r} - 2} \langle\xi\rangle^{\frac{d+1}{\widetilde{r}}z} \chi^2(\varepsilon|\xi|),$$

and we see (4.2) holds with A_0 independent of ε; and from the estimate (3.5) in Proposition 4, we obtain (4.3) with $\sigma = \frac{d-1}{2}$ and A_1 independent of ε. Since $\Theta_{-1}(\xi) = \langle\xi\rangle^{-2s} \chi^2(\varepsilon|\xi|)$ and $(\langle D\rangle^{-s} f_j)_j$ forms an orthonormal family in L^2, the desired uniform estimate (4.8) follows from Proposition 7. □

Proof (Proof of Theorem 6 (sufficiency part)). As before, we only handle the critical case $s = \frac{d+2}{2}(\frac{1}{2} - \frac{1}{r})$ since the other case $s > \frac{d+2}{2}(\frac{1}{2} - \frac{1}{r})$ can be shown by following the argument below if we use the estimate (3.30) instead of (3.5).

It is enough to show that (4.8) holds uniformly in $\varepsilon > 0$ for sharp $\frac{d}{2}$-admissible (q,r) with $2 \le r < \frac{2(d+1)}{d-1}$, $\beta = \frac{2r}{r+2}$, and an orthonormal family $(f_j)_j$ in H^s, $s = \frac{d+2}{2}(\frac{1}{2} - \frac{1}{r})$. We use Proposition 7 with

$$\Theta_z(\xi) = \langle\xi\rangle^{\frac{d+2}{\widetilde{r}}z} \chi^2(\varepsilon|\xi|).$$

Then the assumption (4.2) trivially holds, and from Proposition 5, we have (4.3) with $\sigma = \frac{d}{2}$, which is uniform in ε. Since $\Theta_{-1}(\xi) = \langle\xi\rangle^{-2s} \chi^2(\varepsilon|\xi|)$ and $(\langle D\rangle^{-s} f_j)_j$ is an orthonormal family in L^2, we deduce from Proposition 7 that (4.8) holds for all families of orthonormal functions $(f_j)_j$ in H^s, $\beta = \frac{2r}{r+2}$ and where (q,r) is sharp $\frac{d}{2}$-admissible with

$$\frac{2(d+2)}{d} < r < \frac{2(d+1)}{d-1}. \tag{4.9}$$

Since this estimate holds trivially when $(q,r,\beta) = (\infty, 2, 1)$, by interpolation we may extend the range to $2 \le r < \frac{2(d+1)}{d-1}$, and this completes the proof. □

Remark 5. In the proofs of Theorem 7 and Theorem 6, the noncritical cases $s > \frac{d+1}{2}(\frac{1}{2} - \frac{1}{r})$ and $s > \frac{d+2}{2}(\frac{1}{2} - \frac{1}{r})$ can also be deduced from the critical cases by making use of the inequality

$$\left\| \left(\sum_j |\langle D\rangle^{-\beta} g_j|^2 \right)^{\frac{1}{2}} \right\|_p \lesssim \left\| \left(\sum_j |g_j|^2 \right)^{\frac{1}{2}} \right\|_p,$$

which is valid for $\beta \ge 0$ and $1 \le p < \infty$ with any $(g_j)_j$ not necessarily orthogonal. This inequality may be shown from the trivial inequality $\|\langle D\rangle^{-\beta} f\|_p \lesssim \|f\|_p$, $1 \le p \le \infty$, and randomization (Kintchin's inequality).

4.3. The fractional Schrödinger equation

Clearly, our phase function $\phi_\alpha(\xi) = |\xi|^\alpha$, $\alpha \in \mathbb{R} \setminus \{0,1\}$ is almost homogeneous of order α and $|\det H\phi_\alpha(\xi)| = C_{d,\alpha}|\xi|^{d(\alpha-2)}$ for some constant $C_{d,\alpha}$. Hence, Theorem 4 may be deduced from the following theorem.

Theorem 8. *Let $d \geq 1$, $\alpha \in \mathbb{R} \setminus \{0, 1\}$ and ϕ be almost homogeneous of order α. Suppose (q, r) is sharp $\frac{d}{2}$-admissible. Then the following hold for all families of orthonormal functions $(f_j)_j$ in L^2.*

(i) *If $2 \leq r < \frac{2(d+1)}{d-1}$ and $\beta = \frac{2r}{r+2}$, then we have the estimate*

$$\left\| \sum_j \nu_j \left| U_\phi (|\det H\phi(D)|^{\frac{r-2}{4r}} f_j) \right|^2 \right\|_{L_t^{\frac{q}{2}} L_x^{\frac{r}{2}}} \lesssim \|\nu\|_{\ell^\beta}. \tag{4.10}$$

(ii) *If $d = 2$ and $6 \leq r < \infty$, or if $d \geq 3$ and $\frac{2(d+1)}{d-1} \leq r \leq \frac{2d}{d-2}$, then (4.10) holds with $\beta < \frac{q}{2}$.*

Proof (Proof of Theorem 8). It is enough to prove (i). Indeed, the standard argument combined with the Littlewood–Paley inequality[8] gives the estimate

$$\left\| U_\phi (|\det H\phi(D)|^{\frac{r-2}{4r}} f) \right\|_{L_t^q L_x^r} \lesssim \|f\|_2,$$

for all $\frac{d}{2}$-admissible (q, r). For example, see [48, p. 978]. The estimate is trivially equivalent to (4.10) with $\beta = 1$. In particular, for $d \geq 3$, interpolation between the estimate in (i) and (4.10) with $(q, r, \beta) = (2, \frac{2d}{d-2}, 1)$ proves (ii). When $d = 2$, a similar argument works, except that we must interpolate between (i) and (4.10) with $\beta = 1$ and (q, r) sharp 1-admissible with $(\frac{1}{r}, \frac{1}{q})$ arbitrarily close to $(0, \frac{1}{2})$.

To show (i), first let us consider $\alpha > 0$, in which case it suffices to prove the estimate

$$\left\| \sum_j \nu_j \left| U_\phi (|\det H\phi(D)|^{\frac{r-2}{4r}} \chi(\varepsilon|D|) f_j) \right|^2 \right\|_{L_t^{\frac{q}{2}} L_x^{\frac{r}{2}}} \lesssim \|\nu\|_{\ell^\beta} \tag{4.11}$$

uniformly in $\varepsilon > 0$. Here, $(f_j)_j$ is an orthonormal family in L^2, $\beta = \frac{2r}{r+2}$, and (q, r) is sharp $\frac{d}{2}$-admissible satisfying (4.9). Indeed, once this is established, we take the limit $\varepsilon \to 0$ and then interpolate the resulting bound with the case $(q, r, \beta) = (\infty, 2, 1)$ to obtain the desired estimates for the range $2 \leq r \leq \frac{2d}{d-2}$.

To prove (4.11), we consider

$$\Theta_z(\xi) = |\det H\phi(\xi)|^{-z/\tilde{r}} \chi^2(\varepsilon|\xi|).$$

Obviously, (4.2) holds, and we use Proposition 6 to verify (4.3) with $\sigma = \frac{d}{2}$. Since $\Theta_{-1}(\xi) = |\det H\phi(\xi)|^{\frac{r-2}{2r}} \chi^2(\varepsilon|\xi|)$, we obtain (4.11) from Proposition 7.

The case $\alpha < 0$ can be proved in a very similar manner. It suffices to prove the uniform bound (4.11) with χ replaced by χ_∞, upon which we take the limit $\varepsilon \to \infty$. To this end, we apply Proposition 7 to

$$\Theta_z(\xi) = |\det H\phi(\xi)|^{-z/\tilde{r}} \chi_\infty^2(\varepsilon|\xi|)$$

and again use Proposition 6 to verify (4.3). The remainder is identical to the previous case, so we omit the details. $\qquad \square$

Remark 6. Since the proofs of Theorems 4–7 all rely on Proposition 7, it is clear that the estimates in part (i) of Theorems 5 and 7 are true with the Lorentz norm $\| \cdot \|_{(\frac{q}{2}, \frac{2r}{r+2}), \frac{r}{2}}$ on the left-hand side for sharp $\frac{d-1}{2}$–admissible (q, r) satisfying (4.7), and similarly the same is also true for the estimates in part (i) of Theorems 4 and 6 if (q, r) is $\frac{d}{2}$–admissible and satisfies (4.9).

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[8]After a Littlewood–Paley decomposition, one may use a rescaling argument with Lemma 3 to get uniform bounds for each dyadic piece.

5. Strichartz estimates for orthonormal families: the non-sharp admissible case

In this section, we present extensions of Theorems 4–7 to the corresponding non-sharp admissible cases. In this case, emphasis lies on proving the estimates with initial data of the sharp regularity. If the dispersion relation is homogeneous, the optimal regularity is naturally determined by the homogeneous degree of the dispersion relation. In the case of the classical Strichartz estimates, the sharp regularity estimates are well known for the non-sharp admissible case. However, in contrast to the classical Strichartz estimates, generalizing to the estimates for orthonormal families with the optimal summability exponent β is no longer trivial. For a discussion on why more elementary arguments, such as those based on Littlewood–Paley type considerations, do not seem to yield the desired estimates, we refer the reader to [5]. However, for the Schrödinger equation, this issue was resolved in [5] with an argument that made use of improved estimates in the scale of Lorentz spaces (see Remark 6). The basic strategy devised in [5] also works for the propagators under consideration here to recover the non-sharp admissible bounds. However, here we provide a somewhat more straightforward proof based on Lieb's version of the Sobolev inequality for families of orthonormal functions [56].

In what follows, the estimates we have already obtained in the sharp case play an important role in establishing the non-sharp case, and to a certain degree, this 'deduction' can be captured in an abstract framework. Thus, prior to the statements for each particular equation, we present some results that hold in a level of generality.

To facilitate our presentation, we introduce some notations. For $\sigma \geq \frac{1}{2}$, we introduce the points A_σ and B_σ in $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$ given by

$$A_\sigma = \left(\frac{2\sigma - 1}{2(2\sigma + 1)}, \frac{\sigma}{2\sigma + 1} \right), \quad B_\sigma = \left(\frac{\sigma}{2(\sigma + 1)}, \frac{\sigma}{2(\sigma + 1)} \right).$$

For $q, r \geq 2$, define $\beta_\sigma(q, r) \in [1, \infty]$ by

$$\frac{\sigma}{\beta_\sigma(q, r)} = \frac{1}{q} + \frac{2\sigma}{r}.$$

We also introduce the points

$$C = \left(\frac{1}{2}, 0 \right), \qquad D = \left(0, \frac{1}{2} \right)$$

and, for $\sigma \geq 1$, the point E_σ by

$$E_\sigma = \left(\frac{\sigma - 1}{2\sigma}, \frac{1}{2} \right).$$

See Figure 1. Additionally, if $P_1, P_2, \ldots, P_n$ are points in $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$, by $\mathrm{int}(P_1 P_2 \cdots P_n)$, we denote the interior of the convex hull of the set $\{P_1, P_2, \ldots, P_n\}$.

Proposition 8. *Let $\sigma \geq \frac{1}{2}$, $s(q, r) = \frac{a}{q} + d(\frac{1}{2} - \frac{1}{r})$ for some $a \in \mathbb{R}$, and $\Psi(\xi)$ be $|\xi|$ or $\langle \xi \rangle$. Assume that for sharp σ-admissible (q, r) satisfying (4.1) and, for all families of orthonormal functions $(f_j)_j$ in L^2, the estimate*

$$\left\| \sum_j \nu_j |U_\phi \Psi(D)^{-s(q,r)} f_j|^2 \right\|_{\frac{q}{2}, \frac{r}{2}} \lesssim \|\nu\|_{\ell^\beta} \tag{5.1}$$

holds with $\beta = \beta_\sigma(q, r)$. Then for all (q, r) that are non-sharp σ-admissible with $q > \frac{2\sigma - 1}{2\sigma} r$, the estimate

$$\left\| \sum_j \nu_j |U_\phi \Psi(D)^{-s(q,r)} f_j|^2 \right\|_{\frac{q}{2}, \frac{r}{2}} \lesssim \|\nu\|_{\ell^\beta} \tag{5.2}$$

holds for all families of orthonormal functions $(f_j)_j$ in L^2 with $\beta = \beta_\sigma(q, r)$.
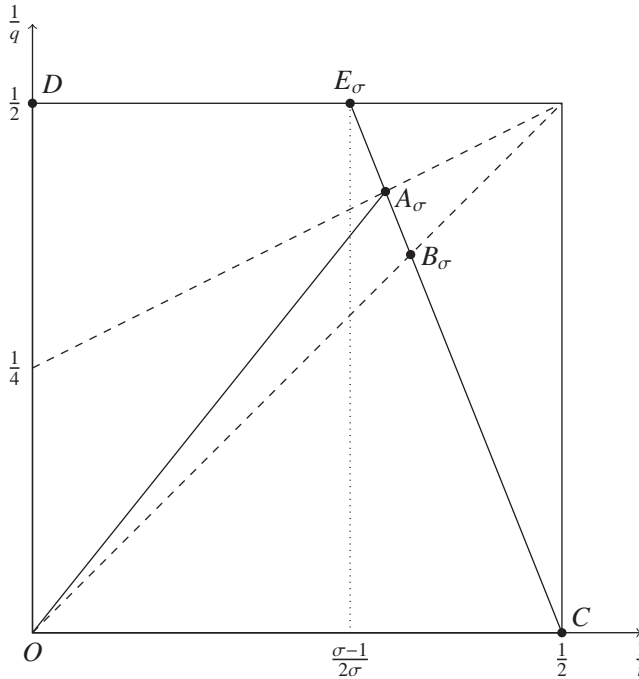
Figure 1. *The points $A_\sigma, B_\sigma, C, D, E_\sigma$ in the case $\sigma > 1$.*

Remark 7. (1) Suppose $\sigma > 1$; and, in addition to the assumptions in Proposition 8, we assume that the (single-function) Strichartz estimate

$$\|U_\phi \Psi(D)^{-s(2,r)} f\|_{2,r} \lesssim \|f\|_2$$

holds for $\frac{\sigma-1}{\sigma} < r < \infty$; that is, $(\frac{1}{r}, \frac{1}{2})$ lies on the line segment (D, E_σ). Then for such r, clearly (5.2) holds with $(q, \beta) = (2, 1)$. We also note that $\beta_\sigma(q, r) = \frac{q}{2}$ whenever $(\frac{1}{r}, \frac{1}{q})$ belongs to the line segment (O, A_σ). Thus, by complex interpolation between estimates (5.2) with $(\frac{1}{r}, \frac{1}{q})$ on (D, E_σ) and points in the region $\text{int}(OA_\sigma C)$ arbitrarily close to the line segment (O, A_σ), we deduce that (5.2) holds if $(\frac{1}{r}, \frac{1}{q})$ belongs to $\text{int}(ODE_\sigma A_\sigma)$ and $\beta < \frac{q}{2}$.

(2) If (q, r) is sharp σ-admissible, then $\beta_\sigma(q, r) = \frac{2r}{r+2}$. As we have seen, $\beta = \frac{2r}{r+2}$ is the sharp summability exponent appearing on the right-hand side of the estimates in Theorems 4 and 6 for $(\frac{1}{r}, \frac{1}{q})$ lying on the line segment $(A_{\frac{d}{2}}, C)$.

Proof (Proof of Proposition 8). The key estimate is the following version of the Sobolev inequality for orthonormal functions, due to Lieb [56]. For all $1 < p < \infty$,

$$\left\| \sum_j \nu_j |\Psi(D)^{-\frac{d}{2p'}} f_j|^2 \right\|_{L^p(\mathbb{R}^d)} \lesssim \|\nu\|_{\ell^{p,1}} \tag{5.3}$$

holds for all families of orthonormal functions $(f_j)_j$ in $L^2(\mathbb{R}^d)$ and sequences ν in $\ell^{p,1}$. Notice that for each fixed $t \in \mathbb{R}$, $(U_\phi f_j(t, \cdot))_j$ remains to be an orthonormal family in $L^2(\mathbb{R}^d)$. Also, note that $s(\infty, r) = \frac{d}{2(r/2)'}$. So, (5.3) implies

$$\left\| \sum_j \nu_j |U_\phi \Psi(D)^{-s(\infty,r)} f_j|^2 \right\|_{\infty, \frac{r}{2}} \lesssim \|\nu\|_{\ell^{\frac{r}{2},1}}. \tag{5.4}$$

In view of $\beta_\sigma(\infty, r) = \frac{r}{2}$, complex interpolation between (5.4) and (5.1) implies

$$\left\| \sum_j v_j |U_\phi \Psi(D)^{-s(q,r)} f_j|^2 \right\|_{\frac{q}{2}, \frac{r}{2}} \lesssim \|v\|_{\ell^{\beta,1}} \tag{5.5}$$

for $(\frac{1}{r}, \frac{1}{q})$ belonging to int$(OA_\sigma C)$, with $\beta = \beta_\sigma(q,r)$; we shall, in fact, only need to make use of this estimate for $(\frac{1}{r}, \frac{1}{q})$ in int$(OA_\sigma B_\sigma)$.

The estimate (5.5) already provides the sharp estimates for various equations if we accept the weaker Lorentz space norm on the right-hand side; however, we need to strengthen (5.5) to the desired strong-type estimate (5.2), and this can be done by using real interpolation. We fix $(\frac{1}{r}, \frac{1}{q})$ belonging to int$(OA_\sigma B_\sigma)$ and choose two distinct points $(q_0, r_0), (q_1, r_0)$ such that $(\frac{1}{r_j}, \frac{1}{q_j})$ belongs to int$(OA_\sigma B_\sigma)$ and $s(q,r) = s(q_j, r_j)$ for $j = 0, 1$. By real interpolation, this[9] yields

$$\left\| \sum_j v_j |U_\phi \Psi(D)^{-s(q,r)} f_j|^2 \right\|_{\frac{q}{2}, (\frac{r}{2}, \frac{q}{2})} \lesssim \|v\|_{\ell^{\beta, \frac{q}{2}}},$$

where $\beta = \beta_\sigma(q,r)$. Now, notice that

$$\frac{q}{2} > \beta_\sigma(q,r) \tag{5.6}$$

holds whenever $(\frac{1}{r}, \frac{1}{q})$ belongs to int$(OA_\sigma B_\sigma)$. Since we have both (5.6) and $q \leq r$ when $(\frac{1}{r}, \frac{1}{q})$ belongs to int$(OA_\sigma B_\sigma)$, we obtain (5.2) whenever $(\frac{1}{r}, \frac{1}{q})$ lies in int$(OA_\sigma B_\sigma)$, from the embeddings $L_t^{\frac{q}{2}} L_x^{\frac{r}{2}, \frac{q}{2}} \subseteq L_t^{\frac{q}{2}} L_x^{\frac{r}{2}}$ and $\ell^\beta \subseteq \ell^{\beta, \frac{q}{2}}$.

Finally, we observe that (5.2) trivially holds when $(q, r, \beta) = (\infty, 2, 1)$. Indeed, from the trivial estimate

$$\|U_\phi f\|_{\infty, 2} \lesssim \|f\|_2$$

and since $s(\infty, 2) = 0$, we see that (5.2) follows by the triangle inequality. By interpolation, we obtain the desired estimate (5.2) whenever $(\frac{1}{r}, \frac{1}{q})$ belongs to int$(OA_\sigma C)$. □

Once we have Proposition 8, obtaining the desired estimates for non-sharp admissible (q, r) is rather straightforward. We prove the sufficiency parts of the theorems stated below, and the necessity parts will be shown in Section 6.

Theorem 9 (The wave equation; non-sharp $\frac{d-1}{2}$-admissible). *Let $d \geq 2$, and suppose (q, r) is non-sharp $\frac{d-1}{2}$-admissible.*

(i) *If $(\frac{1}{r}, \frac{1}{q})$ belongs to* int$(OA_{\frac{d-1}{2}} C)$, *then (1.12) holds for all families of orthonormal functions $(f_j)_j$ in $\dot{H}^s$, with $s = \frac{d}{2} - \frac{d}{r} - \frac{1}{q}$, and $\beta = \beta_{\frac{d-1}{2}}(q,r)$.*

(ii) *If $d \geq 4$ and $(\frac{1}{r}, \frac{1}{q})$ belongs to* int$(ODA_{\frac{d-1}{2}} E_{\frac{d-1}{2}})$, *then (1.12) holds for all families of orthonormal functions $(f_j)_j$ in $\dot{H}^s$, with $s = \frac{d}{2} - \frac{d}{r} - \frac{1}{q}$, and $\beta < \frac{q}{2}$. This is sharp in the sense that the estimate fails if $\beta > \frac{q}{2}$.*

Proof (Proof of Theorem 9 (sufficiency part)). From the classical Strichartz estimates (1.6) and (1) in Remark 7, the claimed estimates in (ii) follow from those estimates in (i) by interpolating them with trivial estimates. To prove the claimed estimates in (i), we apply Proposition 8 with $\Psi(\xi) = |\xi|$ and $s(q,r) = \frac{d}{2} - \frac{d}{r} - \frac{1}{q}$. Thus, it is sufficient to verify (5.1). This follows from Theorem 5. □

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[9]Here, we are using the fact that $(L^{q_0}(L^{r_0}), L^{q_1}(L^{r_1}))_{\theta, q} = L^q(L^{r,q})$ holds whenever $q_0, q_1, r_0, r_1 \in [1, \infty)$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$, $\theta \in (0, 1)$ (see [58] and [27]).

Following an argument similar to that used to prove Theorem 9, we prove the following theorems concerning the Klein–Gordon equation in the non-sharp σ-admissible cases, where $\sigma = \frac{d}{2}, \frac{d-1}{2}$. Also, we later discuss further estimates that are available for the Klein–Gordon equation.

Theorem 10 (The Klein–Gordon equation; non-sharp $\frac{d-1}{2}$-admissible). *Let $d \geq 2$, and suppose (q, r) is non-sharp $\frac{d-1}{2}$-admissible.*

(i) *If $(\frac{1}{r}, \frac{1}{q})$ belongs to $\mathrm{int}(OA_{\frac{d-1}{2}}C)$, then (1.13) holds for all families of orthonormal functions $(f_j)_j$ in H^s, with $s \geq \frac{d}{2} - \frac{d}{r} - \frac{1}{q}$, and $\beta = \beta_{\frac{d-1}{2}}(q, r)$.*

(ii) *If $d \geq 4$ and $(\frac{1}{r}, \frac{1}{q})$ belongs to $\mathrm{int}(ODA_{\frac{d-1}{2}}E_{\frac{d-1}{2}})$, then (1.13) holds for all families of orthonormal functions $(f_j)_j$ in H^s, with $s \geq \frac{d}{2} - \frac{d}{r} - \frac{1}{q}$, and $\beta < \frac{q}{2}$. This is sharp in the sense that the estimate fails if $\beta > \frac{q}{2}$.*

Proof (Proof of Theorem 10 (sufficiency part)). As we have noticed before, it suffices to prove the estimates in (i) since (ii) follows by interpolation between the estimates in (i) and the trivial estimates. We only consider the critical case $s(q, r) = \frac{d}{2} - \frac{d}{r} - \frac{1}{q}$ since the other case can be shown by the same argument with a little modification, or using the inequality in Remark 5. Similar to the proof of Theorem 9, we apply Proposition 8 with $\Psi(\xi) = \langle \xi \rangle$ and $s(q, r) = \frac{d}{2} - \frac{d}{r} - \frac{1}{q}$. For the estimate (5.1), we employ Theorem 7. □

Theorem 11 (The Klein–Gordon equation; non-sharp $\frac{d}{2}$-admissible). *Let $d \geq 1$, and suppose (q, r) is non-sharp $\frac{d}{2}$-admissible.*

(i) *If $(\frac{1}{r}, \frac{1}{q})$ belongs to $\mathrm{int}(OA_{\frac{d}{2}}C)$, then (1.13) holds for all families of orthonormal functions $(f_j)_j$ in H^s, with $s \geq \frac{d}{2} - \frac{d}{r} - \frac{d-2}{dq}$, and $\beta = \beta_{\frac{d}{2}}(q, r)$. This is sharp in the sense that the estimate fails if $\beta > \beta_{\frac{d}{2}}(q, r)$.*

(ii) *If $d \geq 3$ and $(\frac{1}{r}, \frac{1}{q})$ belongs to $\mathrm{int}(ODA_{\frac{d}{2}}E_{\frac{d}{2}})$, then (1.13) holds for all families of orthonormal functions $(f_j)_j$ in H^s, with $s \geq \frac{d}{2} - \frac{d}{r} - \frac{d-2}{dq}$, and $\beta < \frac{q}{2}$. This is sharp in the sense that the estimate fails if $\beta > \frac{q}{2}$.*

Proof (Proof of Theorem 11 (sufficiency part)). Similar to the proof of Theorem 10 (sufficiency part), we only consider the critical case $s(q, r) = \frac{d}{2} - \frac{d}{r} - \frac{d-2}{dq}$. It suffices also to prove the estimates in (i) and to this end we apply Proposition 8 with $\Psi(\xi) = \langle \xi \rangle$ and $s(q, r) = \frac{d}{2} - \frac{d}{r} - \frac{d-2}{dq}$. For the estimate (5.1), we use Theorem 6. □

For simplicity of the exposition, we have presented our main results for the Klein–Gordon equation (Theorems 6, 7, 10 and 11) in the case where (q, r) is σ-admissible for $\sigma = \frac{d}{2}, \frac{d-1}{2}$. Complex interpolation between these estimates gives the orthonormal Strichartz estimates for the Klein–Gordon equation corresponding to the sharp σ-admissible cases with $\sigma \in (\frac{d-1}{2}, \frac{d}{2})$. However, we can obtain these estimates in a unified way, with $\sigma = \frac{d-1}{2} + \rho$ for $\rho \in [0, \frac{1}{2}]$ ($d \geq 2$), and $\rho \in (0, \frac{1}{2}]$ ($d = 1$). For example, since the dispersive estimate with $O(|t|^{-\sigma})$ follows if we interpolate the estimates (3.5) ((3.30)) and (3.6) ((3.31), respectively), then simple modification to the proof of Theorems 6 and 7 yields the following.

Theorem 12 (The Klein–Gordon equation; sharp σ-admissible, $d = 1$). *Let $d = 1$, $\sigma \in (0, \frac{1}{2}]$, and suppose (q, r) is sharp σ-admissible. Then (1.13) holds for all families of orthonormal functions $(f_j)_j$ in H^s, with $s \geq (\sigma + 1)(\frac{1}{2} - \frac{1}{r})$, and $\beta = \frac{2r}{r+2}$.*

Theorem 13 (The Klein–Gordon equation; sharp σ-admissible, $d \geq 2$). *Let $d \geq 2$, $\rho \in [0, \frac{1}{2}]$, and suppose (q, r) is sharp $(\frac{d-1}{2} + \rho)$-admissible.*

(i) *If $2 \leq r < \frac{2(d+2\rho)}{d-2+2\rho}$, then (1.13) holds for all families of orthonormal functions $(f_j)_j$ in H^s, with $s \geq (\frac{d+1}{2} + \rho)(\frac{1}{2} - \frac{1}{r})$, and $\beta = \frac{2r}{r+2}$.*

(ii) When $d = 2$, suppose $\frac{2(1+\rho)}{\rho} \le r < \infty$ for $\rho \in (0, \frac{1}{2}]$. When $d = 3$, suppose $6 \le r < \infty$ for $\rho = 0$, and $\frac{2(d+2\rho)}{d-2+2\rho} \le r \le \frac{2(d-1+2\rho)}{d-3+2\rho}$ for $\rho \in (0, \frac{1}{2}]$. When $d \ge 4$, suppose $\frac{2(d+2\rho)}{d-2+2\rho} \le r \le \frac{2(d-1+2\rho)}{d-3+2\rho}$ for $\rho \in [0, \frac{1}{2}]$. Then (1.13) holds for all families of orthonormal functions $(f_j)_j$ in H^s, with $s \ge (\frac{d+1}{2} + \rho)(\frac{1}{2} - \frac{1}{r})$ and $\beta < \frac{q}{2}$. This estimate is sharp in the sense that the estimate fails for $\beta > \frac{q}{2}$.

By following the approach taken to prove our main results in the non-sharp admissible case, it is possible to extend Theorems 12 and 13 in a similar manner. In Remark 8, we offer some further remarks regarding the sharpness of the exponent β in the case of the Klein–Gordon equation.

Theorem 14 (The fractional Schrödinger equation; non-sharp $\frac{d}{2}$-admissible). *Let $\alpha \in \mathbb{R} \setminus \{0, 1\}$ and $d \ge 1$. Suppose (q, r) is non-sharp $\frac{d}{2}$-admissible with $\frac{d}{r} + \frac{\alpha}{q} < d$.*

(i) *If $(\frac{1}{r}, \frac{1}{q})$ belongs to $\mathrm{int}(OA_{\frac{d}{2}}C)$, then (1.15) holds for all families of orthonormal functions $(f_j)_j$ in $\dot{H}^s$, with $s = \frac{d}{2} - \frac{d}{r} - \frac{\alpha}{q}$, and $\beta = \beta_{\frac{d}{2}}(q, r)$. This is sharp in the sense that the estimate fails if $\beta > \beta_{\frac{d}{2}}(q, r)$.*

(ii) *If $d \ge 3$ and $(\frac{1}{r}, \frac{1}{q})$ belongs to $\mathrm{int}(ODA_{\frac{d}{2}}E_{\frac{d}{2}})$, then (1.15) holds for all families of orthonormal functions $(f_j)_j$ in $\dot{H}^s$, with $s = \frac{d}{2} - \frac{d}{r} - \frac{\alpha}{q}$, and $\beta < \frac{q}{2}$. This is sharp in the sense that the estimate fails if $\beta > \frac{q}{2}$.*

Proof (Proof of Theorem 14 (sufficiency part)). As in the proof of Theorems 9–11, the claimed estimates in (ii) follow from those in (i). To prove the claimed estimates in (i), we apply Proposition 8 with $\Psi(\xi) = |\xi|$ and $s(q, r) = \frac{d}{2} - \frac{d}{r} - \frac{\alpha}{q}$. For the estimate (5.1), we use Theorem 4. □

6. Necessary conditions

Let $\widetilde{\chi}_0 \in C_c^\infty(2^{-2}, 2^2)$ and $\widetilde{\chi}_0 \gtrsim 1$ on $(2^{-1}, 2)$. For a given $(q, r) \in [2, \infty) \times [2, \infty)$, consider the frequency localized estimate

$$\left\| \sum_j \nu_j |U_\phi \widetilde{\chi}_0(|D|) f_j|^2 \right\|_{\frac{q}{2}, \frac{r}{2}} \lesssim \|\nu\|_{\ell^\beta}. \tag{6.1}$$

We show this estimate implies necessary conditions under fairly mild conditions on the dispersion relation ϕ. The following are based on a slight generalization of the construction given in [5].

Proposition 9. *Suppose (6.1) holds for all orthonormal family $(f_j)_j$ and ϕ is continuously differentiable away from the origin. Then we have*

$$\beta \le \beta_{\frac{d}{2}}(q, r). \tag{6.2}$$

Additionally if ϕ is nonnegative and radial with $\phi(\xi) = \phi_0(|\xi|)$ and ϕ_0 is continuously differentiable away from the origin and increasing, then

$$\beta \le \frac{q}{2}. \tag{6.3}$$

From this, the necessity claims in Theorems 4–7 and Theorems 9–14 all follow. Indeed, the orthonormal Strichartz estimates in these theorems clearly imply the corresponding frequency localized estimate of the form (6.1) with suitable choices of $\widetilde{\chi}_0$. The exponents q, r and β should satisfy (6.2) and (6.3).

It seems reasonable to expect that our estimates in (i) of Theorems 5, 7, 9 and 10 (corresponding to the line segment $(A_{\frac{d-1}{2}}, C)$ in the sharp $\frac{d-1}{2}$-admissible case) are also sharp with respect to the range of allowable β. This would follow if we could have $\beta \le \beta_{\frac{d-1}{2}}(q, r)$. Unfortunately, we are not able to reach this conclusion yet.

Proof (Proof of Proposition 9). To show (6.2), we use the family of vectors

$$(v_j)_{j \in J} := \{\xi \in R^{-1}\mathbb{Z}^d : 2^{-1} \leq |\xi| \leq 2\},$$

where $\#J \sim R^d$. If we let

$$\widehat{f_j}(\xi) = cR^{d/2}\chi(2R|\xi - v_j|)$$

for each $j \in J$, then $(f_j)_j$ becomes orthonormal system in $L^2(\mathbb{R}^d)$ for an appropriate choice of the constant c. Note that

$$|U_\phi \widetilde{\chi_0}(|D|) f_j(x)| \gtrsim R^{-d/2} 1_{\{|t| \lesssim R^2, |x+t\nabla\phi(v_j)| \lesssim R\}}.$$

Also, we have $|\nabla\phi(v_j)| \lesssim 1$ since $|v_j| \sim 1$, and this gives the uniform lower bound $1_{\{|t| \lesssim R^2, |x+t\nabla\phi(v_j)| \lesssim R\}} \gtrsim 1_{\{|t|, |x| \lesssim R\}}$. Therefore, if we assume (6.1) holds, then

$$R^{\frac{2}{q}+\frac{2d}{r}} \lesssim R^{\frac{d}{\beta}}.$$

By taking $R \to \infty$, we see that $\beta \leq \beta_{\frac{d}{2}}(q,r)$.

To get the second condition (6.3), let us consider

$$f_j = ce^{-ij\phi(D)}g,$$

where

$$\widehat{g}(\xi) = 1_{[\phi_0^{-1}(\ell\pi), \phi_0^{-1}((\ell+2)\pi)]}(|\xi|)|\xi|^{-\frac{d-1}{2}}\phi_0'(|\xi|)^{1/2},$$

c is a constant to be chosen momentarily, and we here choose $\ell \in \mathbb{Z}$ so that the set $\{\phi_0^{-1}(\ell\pi) \leq |\xi| \leq \phi_0^{-1}((\ell+2)\pi)\} \cap \{2^{-1} \leq |\xi| \leq 2\}$ has nonzero measure. Then by changing to spherical coordinates, it is easy to check that $(f_j)_j$ is an orthonormal family by choosing an appropriate constant c.

Notice that

$$\left\| \sum_j v_j |U_\phi \widetilde{\chi_0}(|D|) f_j|^2 \right\|_{\frac{q}{2}, \frac{r}{2}}^{\frac{q}{2}} \geq \sum_{n \in \mathbb{Z}} \int_n^{n+\varepsilon} v_n^{\frac{q}{2}} \|e^{i(t-n)\phi(D)}g\|_r^q \, dt \gtrsim \|v\|_{\ell^{\frac{q}{2}}}^{\frac{q}{2}},$$

by choosing $\varepsilon > 0$ small enough so that $\|e^{is\phi(D)}g\|_r \sim \|g\|_r$ uniformly in $s \in (0, \varepsilon)$. Therefore, if we assume (6.1) holds, then we see $\|v\|_{\frac{q}{2}} \lesssim \|v\|_\beta$, which shows (6.3). □

Remark 8. In cases where the Sobolev exponent s is determined by scaling considerations in terms of q and r, the summability exponent β is considered as a function of these exponents. In other cases such as the Klein-Gordon equation

$$\left\| \sum_j v_j |e^{it\sqrt{1-\Delta}} f_j|^2 \right\|_{\frac{q}{2}, \frac{r}{2}} \lesssim \|v\|_{\ell^\beta}$$

for families of orthonormal functions $(f_j)_j$ in H^s, it is reasonable to expect that the sharp value of β depends on q, r and s. To highlight this, for simplicity, we consider the case $q = r = \frac{2(d+1)}{d-1}$. Regarding (q, r) as a non-sharp $\frac{d}{2}$-admissible pair, it follows from Theorem 11 that we have the estimate

$$\left\| \sum_j v_j |e^{it\sqrt{1-\Delta}} f_j|^2 \right\|_{\frac{2(d+1)}{d-1}} \lesssim \|v\|_{\ell^{\beta_1}} \tag{6.4}$$

for any family of orthonormal functions $(f_j)_j$ in H^{s_1}, where

$$(\beta_1, s_1) = \left(\frac{d}{d-1}, \frac{d^2 + 3d - 2}{2d(d+1)} \right).$$

On the other hand, regarding (q, r) as a sharp $\frac{d-1}{2}$-admissible pair, from Theorem 3, we have

$$\left\| \sum_j \nu_j |e^{it\sqrt{1-\Delta}} f_j|^2 \right\|_{\frac{2(d+1)}{d-1}} \lesssim \|\nu\|_{\ell^{\beta_2}} \tag{6.5}$$

for any family of orthonormal functions $(f_j)_j$ in H^{s_2}, where

$$(\beta_2, s_2) = \left(\frac{d+1}{d}, \frac{1}{2} \right).$$

Easy computations show $\beta_1 > \beta_2$ and $s_1 > s_2$. So, from the viewpoint of the regularity, (6.5) is better than (6.4), but if we want further gain in the summability exponent β, then (6.4) is better. It seems to be an interesting problem to identify the sharp value of β for allowable (q, r, s), but we do not pursue this here.

7. Applications

In this section, we present several applications of our results toward local well-posedness of infinite systems of Hartree type, weighted velocity averaging estimates for kinetic transport equations and refined versions of classical Strichartz estimates. As we touched on in the Introduction, that Strichartz estimates for orthonormal families of initial data enjoy such a variety of connections has already been observed by other authors, and we clarify this at appropriate points in the remainder of this section. Our purpose here is to obtain improvements in various respects over the existing literature in this direction based on our progress on Strichartz estimates for orthonormal families obtained in this paper. Although, broadly speaking, the methods overlap with existing work, we believe there is value in taking care to include some detail, for clarity of the exposition, and since it allows us to draw certain connections between the applications themselves.

To begin with, it will be helpful to introduce an operator-theoretic version of the Strichartz estimates for orthonormal families, which is useful for the forthcoming applications. For this, we introduce the notion of the density function ρ_γ (also written $\rho[\gamma]$ where appropriate) of certain classes of compact and self-adjoint operators γ on $L^2(\mathbb{R}^d)$. Formally, $\rho_\gamma(x) = \gamma(x, x)$, where (with the usual abuse of notation) $\gamma(x, y)$ denotes the integral kernel of the operator γ. In terms of the spectral decomposition $\gamma = \sum_j \nu_j \Pi_{f_j}$, where $(f_j)_j$ is an orthonormal family in $L^2(\mathbb{R}^d)$ and $\Pi_{f_j} := \langle \cdot, f_j \rangle f_j$ is the orthogonal projection onto the span of f_j, we formally obtain

$$\rho_\gamma(x) = \sum_j \nu_j |f_j(x)|^2.$$

It turns out that the estimate

$$\left\| \sum_j \nu_j |U_\phi \Psi(D)^{-s} f_j|^2 \right\|_{\frac{q}{2}, \frac{r}{2}} \leq C_* \|\nu\|_{\ell^\beta}, \qquad (f_j)_j \text{ orthonormal in } L^2(\mathbb{R}^d) \tag{7.1}$$

ensures that the density function $\rho[\Psi(D)^{-s}\gamma(t)\Psi(D)^{-s}]$ is well defined in $L_t^{\frac{q}{2}} L_x^{\frac{r}{2}}$, where $\gamma(t) = e^{it\phi(D)}\gamma_0 e^{-it\phi(D)}$, and satisfies

$$\|\rho[\Psi(D)^{-s}\gamma(t)\Psi(D)^{-s}]\|_{\frac{q}{2}, \frac{r}{2}} \leq C_* \|\gamma_0\|_{\mathcal{C}^\beta(L^2)} \tag{7.2}$$

whenever $\gamma_0 \in \mathcal{C}^\beta(L^2)$. Here $q, r \geq 2$, $\beta \geq 1$, $s \in \mathbb{R}$, and $\Psi(\xi) = |\xi|$ or $\Psi(\xi) = \langle \xi \rangle$. We refer the reader to [29] and [30] for further details.

As a further word on notation, in the remainder of this section we usually abbreviate $\mathcal{C}^\beta(L^2)$ to $\mathcal{C}^\beta$. Furthermore, $\mathcal{C}^{\beta,s}$ denotes the Sobolev-type Schatten space with norm

$$\|\gamma\|_{\mathcal{C}^{\beta,s}} = \|\Psi(D)^s \gamma \Psi(D)^s\|_{\mathcal{C}^\beta(L^2)}.$$

Also, the homogeneous and inhomogeneous Besov spaces $\dot{B}^s_{q,r}$ and $B^s_{q,r}$ will arise in this section; their norms are given by

$$\|f\|_{\dot{B}^s_{q,r}} = \left(\sum_{j \in \mathbb{Z}} (2^{js} \|P_j f\|_q)^r \right)^{\frac{1}{r}},$$

$$\|f\|_{B^s_{q,r}} = \|P_{\leq 0} f\|_q + \left(\sum_{j \geq 1} (2^{js} \|P_j f\|_q)^r \right)^{\frac{1}{r}}$$

for finite r, with the obvious modification in the case $r = \infty$. Here, $(P_j)_{j \in \mathbb{Z}}$ is a smooth Littlewood–Paley family of operators associated with dyadic annuli given by

$$\widehat{P_j f}(\xi) = \chi_0(2^{-j}|\xi|) \widehat{f}(\xi), \tag{7.3}$$

where χ_0 is a non-negative and smooth function supported on $[\frac{1}{2}, 2]$ such that $\sum_{j \in \mathbb{Z}} \chi_0(2^{-j}|\xi|) = 1$ for $\xi \neq 0$. Also, $\widehat{P_{\leq 0} f}(\xi) = \chi(\xi) \widehat{f}(\xi)$, where χ is a non-negative and smooth function supported on $[-\frac{1}{2}, \frac{1}{2}]$.

7.1. Local well-posedness for Hartree-type infinite systems

Estimates of the form (7.1) are a powerful tool in the analysis of infinite systems. For concreteness, we begin by discussing the infinite system of fractional Schrödinger equations on $\mathbb{R}^{1+d}$ with Hartree-type nonlinearity

$$\begin{cases} i\partial_t u_j = (-\Delta)^{\alpha/2} u_j + (w * \rho) u_j, & j \in \mathbb{N} \\ u_j(0, \cdot) = f_j, \end{cases} \tag{7.4}$$

where $\rho = \sum_{k=1}^\infty |u_k|^2$ is the density function and w is the interaction potential. We focus on the case where the initial data $(f_j)_j$ forms an orthonormal family in $L^2(\mathbb{R}^d)$, in which case we necessarily have $\alpha \geq 2$ in order for (7.1) to hold when $s = 0$. Recent developments on the rigorous mathematical theory of the above system in the case $\alpha = 2$ have made significant use of estimate (7.1) (or variants that incorporate additional smoothing effects). We refer the reader to [19], [20], [30], [54], [55] and [69] for further background and details on these applications and to other recent developments in the theory of such systems in [41] (on finite-time blowup criteria) and [34] (on ground states). We also remark that in the case of a single particle, the system (7.4) reduces to

$$i\partial_t u = (-\Delta)^{\alpha/2} u + (w * |u|^2) u,$$

and this has been the subject of numerous papers; see, for example, [22], [23], [35] and the references therein.

In the standard way, in order to treat the infinite system (7.4), we consider the operator formalism

$$\begin{cases} i\partial_t \gamma = [(-\Delta)^{\alpha/2} + w * \rho_\gamma, \gamma], & (t, x) \in \mathbb{R}^{1+d}, \ d \geq 1, \\ \gamma(t, \cdot) = \gamma_0. \end{cases} \tag{7.5}$$

Here, $\gamma = \gamma(t)$ is a bounded and self-adjoint operator on $L^2(\mathbb{R}^d)$, ρ_γ is the associated density function and $[\cdot, \cdot]$ denotes the commutator. Regarding this equation, we have the following local-in-time existence theorem.

Theorem 15. *Let $d \geq 1$, $\alpha > 2$, $r \in [2, r_\alpha]$ and $\frac{\alpha}{q} = d(\frac{1}{2} - \frac{1}{r})$, where $r_\alpha = \frac{2d}{d-\alpha}$ for $\alpha < d$, and $r_\alpha = \infty$ for $\alpha \geq d$. For $d = 1$, we assume $\beta \in [1, \beta_{\frac{1}{2}}(q, r)]$, and for $d \geq 2$, we assume β satisfies*

$$\beta \in \begin{cases} [1, \beta_{\frac{d}{2}}(q, r)] & \text{if } r < \frac{2(d+\alpha-1)}{d-1}, \\ [1, \frac{q}{2}) & \text{if } r \geq \frac{2(d+\alpha-1)}{d-1}. \end{cases}$$

Then for any $w \in L^{(r/2)'}$ and $\gamma_0 \in \mathcal{C}^\beta$, there exist $T = T(\|\gamma_0\|_{\mathcal{C}^\beta}, \|w\|_{L^{(r/2)'}}) > 0$ and a unique solution $\gamma \in C_t^0([0,T]; \mathcal{C}^\beta)$ to (7.5) on $[0,T] \times \mathbb{R}^d$ whose density ρ_γ belongs to $L_t^{q/2} L_x^{r/2}([0,T] \times \mathbb{R}^d)$.

Our proof of Theorem 15 follows the contraction mapping arguments in [30] and [54] for the Duhamel formulation of (7.5), into which we incorporate our new estimates of the form (7.1) from Theorem 14. The argument is robust, and similar applications are possible for wider classes of equations of the type

$$\begin{cases} i\partial_t \gamma = [\phi(D) + w * \rho_\gamma, \gamma], & (t,x) \in \mathbb{R}^{1+d}, \ d \geq 1, \\ \gamma(t, \cdot) = \gamma_0, \end{cases} \tag{7.6}$$

and thus we present an abstract statement in the forthcoming Proposition 10 based on the assumption that the associated propagator U_ϕ satisfies the estimate (7.1). In addition, for control of the nonlinearity, we shall also assume the operator norm estimates

$$\|\Psi(D)^{\pm s}(w * \rho)\Psi(D)^{\mp s}\|_{\mathcal{C}^\infty} \leq C_{**}\|w\|_{\mathcal{X}_{r,s}}\|\rho\|_{\frac{r}{2}}, \tag{7.7}$$

where $\mathcal{X}_{r,s}$ is an appropriate function space. It is of course immediate from Hölder's inequality that in the case $s = 0$ (needed for Theorem 15), we may take the Lebesgue space $\mathcal{X}_{r,0} = L^{(r/2)'}$. Some remarks on the case of general s will follow later.

Proposition 10. *Suppose (7.1) and (7.7) hold. Then for any $w \in \mathcal{X}_{r,s}$ and $\gamma_0 \in \mathcal{C}^{\beta,s}$, there exist $T = T(\|\gamma_0\|_{\mathcal{C}^{\beta,s}}, \|w\|_{\mathcal{X}_{r,s}}) > 0$ and a unique solution $\gamma \in C_t^0([0,T]; \mathcal{C}^{\beta,s})$ to (7.6) on $[0,T] \times \mathbb{R}^d$ whose density ρ_γ satisfies $\rho_\gamma \in L_t^{q/2} L_x^{r/2}([0,T] \times \mathbb{R}^d)$.*

By the Duhamel principle, the solution of the inhomogeneous equation

$$\begin{cases} i\partial_t \gamma = [\phi(D), \gamma] + R(t), & (t,x) \in \mathbb{R}^{1+d}, \ d \geq 1 \\ \gamma(t, \cdot) = \gamma_0 \end{cases}$$

is formally given by

$$\gamma(t) = e^{-it\phi(D)}\gamma_0 e^{it\phi(D)} - i \int_0^t e^{-i(t-t')\phi(D)} R(t') e^{i(t-t')\phi(D)} \, dt',$$

and thus the following proposition provides control over the inhomogeneous component of the Duhamel formula for (7.6).

Proposition 11.

(i) *If (7.1) holds, then for any $V \in L_t^{(q/2)'} L_x^{(r/2)'}$, we have*

$$\left\| \int_{\mathbb{R}} e^{it\phi(D)} \Psi(D)^{-s} V(t, \cdot) \Psi(D)^{-s} e^{-it\phi(D)} \, dt \right\|_{\mathcal{C}^{\beta'}} \leq C_* \|V\|_{(\frac{q}{2})', (\frac{r}{2})'}. \tag{7.8}$$

(ii) *(Inhomogeneous estimate) If* (7.1) *holds, then*

$$\|\rho[\Psi(D)^{-s}\gamma_R(t)\Psi(D)^{-s}]\|_{\frac{q}{2},\frac{r}{2}} \leq 4C_* \left\| \int_{\mathbb{R}} e^{it\phi(D)} |R(t)| e^{-it\phi(D)} \, dt \right\|_{\mathfrak{C}^\beta}. \tag{7.9}$$

Here, for each $t \in \mathbb{R}$, $R(t) : L^2 \to L^2$ is a self-adjoint operator and

$$\gamma_R(t) = \int_0^t e^{-i(t-t')\phi(D)} R(t') e^{i(t-t')\phi(D)} \, dt'.$$

Proof. For both (i) and (ii), we follow the argument in [29, Section 2.3], and so we will be brief with details.

For (i), by duality it suffices to show

$$\left| \mathrm{Tr}\left(\gamma_0 \int_{\mathbb{R}} e^{it\phi(D)} \Psi(D)^{-s} V(t,\cdot) \Psi(D)^{-s} e^{-it\phi(D)} \, dt \right) \right| \leq C_* \|V\|_{(\frac{q}{2})',(\frac{r}{2})'} \tag{7.10}$$

whenever $\|\gamma_0\|_{\mathfrak{C}^\beta} = 1$. Using the cyclic property of trace, it follows that the left-hand side of (7.10) coincides with

$$\left| \int_{\mathbb{R}} \int_{\mathbb{R}^d} \rho[\Psi(D)^{-s}\gamma(t)\Psi(D)^{-s}](x) V(t,x) \, dx dt \right|,$$

where $\gamma(t) = e^{-it\phi(D)} \gamma_0 e^{it\phi(D)}$, and thus (7.10) follows from Hölder's inequality and (7.2).

For (ii), we again proceed using duality. First, for non-negative functions $V \in L_t^{(q/2)'} L_x^{(r/2)'}$, we write

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \rho[\Psi(D)^{-s}\gamma_R(t)\Psi(D)^{-s}](x) V(t,x) \, dx dt = \int_{\mathbb{R}} \int_0^t \mathrm{Tr}(A(t)B(t')) \, dt' dt.$$

Here the operators $A(t)$ and $B(t)$ are given by

$$A(t) = e^{it\phi(D)} \Psi(D)^{-s} V(t,\cdot) \Psi(D)^{-s} e^{-it\phi(D)}, \quad B(t) = e^{it\phi(D)} R(t) e^{-it\phi(D)},$$

and since they are self-adjoint, it follows that $|\mathrm{Tr}(A(t)B(t'))| \leq \mathrm{Tr}(|A(t)||B(t')|)$. Therefore, by Hölder's inequality for the Schatten spaces along with Part (i) of this proposition, we obtain

$$\left| \int_{\mathbb{R}} \int_{\mathbb{R}^d} \rho[\Psi(D)^{-s}\gamma_R(t)\Psi(D)^{-s}](x) V(t,x) \, dx dt \right|$$

$$\leq C_* \|V\|_{(\frac{q}{2})',(\frac{r}{2})'} \left\| \int_{\mathbb{R}} e^{it'\phi(D)} |R(t')| e^{-it'\phi(D)} \, dt' \right\|_{\mathfrak{C}^\beta}.$$

The same inequality holds for general $V \in L_t^{(q/2)'} L_x^{(r/2)'}$ with constant $4C_*$ by appropriately decomposing V, and hence (7.9) follows from duality. $\qquad\square$

Proof (Proof of Proposition 10). The argument below follows the proof of [30, Theorem 14] (see also [54]). To begin, we set $M := \|\gamma_0\|_{\mathfrak{C}^{\beta,s}}$ and define the map Λ by

$$\Lambda(\gamma, \rho) = (\Lambda_1(\gamma, \rho), \rho[\Lambda_1(\gamma, \rho)]),$$

where

$$\Lambda_1(\gamma, \rho)(t) = e^{-it\phi(D)} \gamma_0 e^{it\phi(D)} - i \int_0^t e^{-i(t-t')\phi(D)} [w * \rho(t'), \gamma(t')] e^{i(t-t')\phi(D)} \, dt'.$$

We consider these mappings on the space

$$X_T = \{(\gamma, \rho) \in C_t^0([0,T]; \mathcal{C}^{\beta,s}) \times L_t^{q/2} L_x^{r/2}([0,T] \times \mathbb{R}^d) : \|(\gamma, \rho)\|_{X_T} \leq C^* M\}$$

equipped with the norm

$$\|(\gamma, \rho)\|_{X_T} := \|\gamma\|_{C_t^0([0,T]; \mathcal{C}^{\beta,s})} + \|\rho\|_{L_t^{q/2} L_x^{r/2}([0,T] \times \mathbb{R}^d)}.$$

Here, $T = T(M, \|w\|_{\mathcal{X}_{r,s}}) \leq 1$ is to be chosen momentarily and C^* is a constant such that $C^* > \max(10, 10C_*)$. With this setup, it suffices to show that there is some $(\gamma, \rho) \in X_T$ that is a fixed point of Λ, and thus our goal is to show that Λ is a contraction on the space X_T if T is small enough.

The proof that Λ maps X_T to itself rests on the estimates

$$\|\Lambda_1(\gamma, \rho)\|_{C_t^0([0,T]; \mathcal{C}^{\beta,s})} \leq M + 2C_{**}\|w\|_{\mathcal{X}_{r,s}} T^{1/(q/2)'}(C^* M)^2 \tag{7.11}$$

and

$$\|\rho[\Lambda_1(\gamma, \rho)]\|_{L_t^{q/2} L_x^{r/2}([0,T] \times \mathbb{R}^d)} \leq C_*\big(M + 8C_{**}\|w\|_{\mathcal{X}_{r,s}} T^{1/(q/2)'}(C^* M)^2\big) \tag{7.12}$$

for any $(\gamma, \rho) \in X_T$. Once these claims are proved, then it is clear that a sufficiently small choice of $T = T(M, \|w\|_{\mathcal{X}_{r,s}})$ shows $\Lambda : X_T \to X_T$. A similar argument shows that Λ is also a contraction, and we omit the details.

Since Schatten norms are unitarily invariant, it follows that $\|e^{-it\phi(D)} \gamma_0 e^{it\phi(D)}\|_{\mathcal{C}^{\beta,s}} = \|\gamma_0\|_{\mathcal{C}^{\beta,s}} = M$, and thus for (7.11), it suffices to establish

$$\int_0^T \mathcal{N} \, dt' \leq 2C_{**}\|w\|_{\mathcal{X}_{r,s}} T^{1/(q/2)'}(C^* M)^2, \tag{7.13}$$

where

$$\mathcal{N} = \|e^{-i(t-t')\phi(D)}[w * \rho(t'), \gamma(t')] e^{i(t-t')\phi(D)}\|_{\mathcal{C}^{\beta,s}}.$$

For the integrand, we again use the unitary invariance of the Schatten norms along with Hölder's inequality to obtain

$$\mathcal{N} \leq \big(\|\Psi(D)^s (w * \rho(t')) \Psi(D)^{-s}\|_{\mathcal{C}^\infty} + \|\Psi(D)^{-s}(w * \rho(t')) \Psi(D)^s\|_{\mathcal{C}^\infty}\big) \|\gamma(t')\|_{\mathcal{C}^{\beta,s}}$$

and thus, using (7.7), we get

$$\int_0^T \mathcal{N} \, dt' \leq 2C_{**}\|\gamma\|_{C_t^0([0,T]; \mathcal{C}^{\beta,s})} \|w\|_{\mathcal{X}_{r,s}} \int_0^T \|\rho(t')\|_{\frac{r}{2}} \, dt'.$$

Hence (7.13) follows from Hölder's inequality.

For (7.12), we first use (7.2) to see that

$$\|\rho[e^{-it\phi(D)} \gamma_0 e^{it\phi(D)}]\|_{L_t^{q/2} L_x^{r/2}([0,T] \times \mathbb{R}^d)} \leq C_* M.$$

From this, along with (7.9) and unitary invariance of Schatten norms, we obtain

$$\|\rho[\Lambda_1(\gamma, \rho)]\|_{L_t^{q/2} L_x^{r/2}([0,T] \times \mathbb{R}^d)}$$
$$\leq C_* M + 4C_* \int_0^T \|\Psi(D)^s [w * \rho(t'), \gamma(t')] \Psi(D)^s\|_{\mathcal{C}^\beta} \, dt,$$

and estimating as in the proof of (7.11) yields (7.12). $\square$

We conclude with some remarks about the pseudo-relativistic case $\phi(\xi) = \langle\xi\rangle$, in which case the single-equation system of Hartree type has been studied in numerous papers; see, for example, [22], [32], [53] and further references in these papers. In this case, we have estimates of the form (7.1) with $\Psi(\xi) = \langle\xi\rangle$ and certain $s > 0$, and (7.7) holds with the Besov space $\mathcal{X}_{r,s} = B^{s+\delta}_{(r/2)',\infty}$ for arbitrary $\delta > 0$. This latter assertion follows from the estimate

$$\|fg\|_{H^s} \leq C_{s,\delta}\|f\|_{B^{|s|+\delta}_{\infty,\infty}}\|g\|_{H^s}, \tag{7.14}$$

which is valid for all $s \in \mathbb{R}$ and $\delta > 0$ (see, for example, [77, p. 205]). We also remark that the analogous estimate to (7.14) on the torus was used in [62] to establish local well-posedness of infinite systems of Hartree equations in the periodic setting.

7.2. Velocity averages for kinetic transport equations

Associated with the dispersion relation ϕ, consider the kinetic transport equation

$$\begin{cases} \partial_t F(t,x,\xi) + \nabla\phi(\xi) \cdot \nabla_x F(t,x,\xi) = 0, & (t,x,\xi) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d, \\ F(0,x,\xi) = f(x,\xi). \end{cases} \tag{7.15}$$

It is easy to see that the solution is given by the explicit formula

$$F(t,x,\xi) = f(x - t\nabla\phi(\xi),\xi).$$

The standard kinetic transport equation $\partial_t F + \xi \cdot \nabla_x F = 0$ corresponds to the dispersion relation $\phi(\xi) = \frac{1}{2}|\xi|^2$, and the Strichartz estimates in this context

$$\|F\|_{L^a_t L^b_x L^c_\xi(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} \lesssim \|f\|_{L^\beta_{x,\xi}(\mathbb{R}^d\times\mathbb{R}^d)}$$

were first studied by Castella and Perthame [16]. From the explicit form of the solution, it is obvious that it suffices to consider the case $c = 1$ or, equivalently, the velocity averaging estimate[10]

$$\|\varrho f\|_{L^{\frac{q}{2}}_t L^{\frac{r}{2}}_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|f\|_{L^\beta_{x,\xi}(\mathbb{R}^d\times\mathbb{R}^d)}, \tag{7.16}$$

where ϱ is the velocity averaging operator given by

$$\varrho f(t,x) = \int_{\mathbb{R}^d} f(x - t\xi,\xi) \, \mathrm{d}\xi.$$

Combining the contribution in [16], along with an observation by Keel and Tao in [48], all non-endpoint estimates (7.16) were settled; specifically, the estimates were shown to be true for $r \in [2, \frac{2(d+1)}{d-1})$ and false for $r \in (\frac{2(d+1)}{d-1}, \infty]$, where (q,r) is $\frac{d}{2}$-sharp admissible and $\beta = \frac{2r}{r+1}$. The failure of the endpoint case $r = \frac{2(d+1)}{d-1}$ is due to Guo and Peng [37] (see also the work of Ovcharov [64]) when $d = 1$, and [4] for all $d \geq 2$. Some recent advances in the theory of nonlinear kinetic systems relied on Strichartz estimates for the standard kinetic transport equation; we refer the reader to work of Bourneveas, Calvez, Gutiérrez and Perthame [9] on global weak solutions for the Othmer–Dunbar–Alt kinetic model of chemotaxis, and also Arsénio [2] and He–Jiang [40] for contributions to the theory of the Boltzmann equation. In [40], it is suggested that weighted Strichartz estimates may be beneficial for the study of certain kinds of potentials, and in this section we establish the following weighted extensions of the estimates in [16] and [48] as applications of our main results in this paper.

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[10]As will become clear, the relabelling $(a,b) \to (\frac{q}{2},\frac{r}{2})$ is convenient for facilitating the connection to (7.1).

Theorem 16. *Suppose $d \geq 1$. Let $(\frac{1}{r}, \frac{1}{q})$ belong to* $\mathrm{int}(OA_{\frac{d}{2}}C) \cup [C, A_{\frac{d}{2}})$*, and define* $\beta = \beta_{\frac{d}{2}}(q, r)$*.*

(i) *Suppose $\alpha \in \mathbb{R} \setminus \{0, 1\}$ and let $s = \frac{d}{2} - \frac{d}{r} - \frac{\alpha}{q}$. If $\frac{d}{r} + \frac{\alpha}{q} < d$. Then*

$$\left\| \int_{\mathbb{R}^d} f(x - t|\xi|^{\alpha-2}\xi, \xi)|\xi|^{-2s} \, \mathrm{d}\xi \right\|_{L_t^{\frac{q}{2}} L_x^{\frac{r}{2}}} \lesssim \|f\|_{L_{x,\xi}^{\beta}}$$

for all $f \in L_{x,\xi}^{\beta}$.

(ii) *Let $s = \frac{d}{2} - \frac{d}{r} - \frac{d-2}{dq}$. Then*

$$\left\| \int_{\mathbb{R}^d} f(x - t\frac{\xi}{\langle \xi \rangle}, \xi)\langle \xi \rangle^{-2s} \, \mathrm{d}\xi \right\|_{L_t^{\frac{q}{2}} L_x^{\frac{r}{2}}} \lesssim \|f\|_{L_{x,\xi}^{\beta}}$$

for all $f \in L_{x,\xi}^{\beta}$.

Remark 9. (*i*) The case $\alpha = 2$ in (*ii*) was established in [5]. Also, in the case $\alpha > 1$, the estimates were obtained in [7] in the restricted range corresponding to $(\frac{1}{r}, \frac{1}{q})$ in $\mathrm{int}(OFC)$, where $F = (\frac{d-1}{4d}, \frac{1}{4})$; the approach in [7] was based on a Fourier-analytic argument and is completely different to the approach taken here.

(*ii*) By duality, the velocity averaging estimate

$$\left\| \int_{\mathbb{R}^d} f(x - t\nabla\phi(\xi), \xi)\Psi(\xi)^{-2s} \, \mathrm{d}\xi \right\|_{L_t^{\frac{q}{2}} L_x^{\frac{r}{2}}} \lesssim \|f\|_{L_{x,\xi}^{\beta}} \tag{7.17}$$

is equivalent to the *X*-ray transform type estimate

$$\left\| \int_{\mathbb{R}} V(t, x + t\nabla\phi(\xi))\Psi(\xi)^{-2s} \, \mathrm{d}t \right\|_{L_{x,\xi}^{\beta'}} \lesssim \|V\|_{L_t^{(q/2)'} L_x^{(r/2)'}}. \tag{7.18}$$

Since $e^{it\phi(D)} x e^{-it\phi(D)} = x + t\nabla\phi(D)$, formally at least we have

$$e^{it\phi(D)} V(t, x) e^{-it\phi(D)} = V(t, x + t\nabla\phi(D)),$$

and thus (7.18) has a striking resemblance to (7.8). In fact, the latter implies the former via a semi-classical limiting argument, and this is the mechanism through which we establish Theorem 16 as an application of our main results in this paper. The following proposition formalizes this link, albeit at the level of the density functions ρ_γ and ϱ, rather than their dual counterparts in (7.8) and (7.18).

Proposition 12. *Suppose (7.1) holds for some $q, r \geq 2$, $s \in \mathbb{R}$, $\beta = \beta_{\frac{d}{2}}(q, r)$ and ϕ is even. Then (7.17) holds.*

In the case of the Schrödinger equation $\phi(\xi) = |\xi|^2$ when (q, r) is sharp $\frac{d}{2}$-admissible, Proposition 12 can be found in [69, Lemma 9]. A more general statement for the Schrödinger equation can be found in [5, Proposition 5.1]. Our proof of Proposition 12 follows the same lines of argument, and thus we simply provide a sketch; for further details, we refer the reader to [69] and [5].

Proof (Proof of Proposition 12). Fix any $f \in \mathcal{S}(\mathbb{R}^d)$ and, for $h > 0$, define the quantization γ_h by

$$\gamma_h \psi(x) = \int_{\mathbb{R}^{2d}} f(\tfrac{h}{2}(x + x'), \xi) e^{i(x-x')\cdot\xi} \psi(x') \, \mathrm{d}\xi \mathrm{d}x', \quad \psi \in \mathcal{S}(\mathbb{R}^d).$$

Let us use $K_h(x, x')$ to denote the integral kernel of γ_h. Then we have

$$\widehat{K_h}(\xi, \xi') = \left(\tfrac{2\pi}{h}\right)^d \mathcal{F}_x[f(\cdot, \tfrac{\xi - \xi'}{2})]\left(\tfrac{\xi + \xi'}{h}\right),$$

and from the definition of the density function and changes of variable, we have

$$\rho[\Psi(D)^{-s}\gamma_h(t/h)\Psi(D)^{-s}](x/h)$$
$$= \int_{\mathbb{R}^{2d}} e^{-i\frac{t}{2h}(\phi(h\xi - \xi') - \phi(\xi'))} \Psi(h\xi - \xi')^{-s} \Psi(\xi')^{-s} \mathcal{F}_x[f(\cdot, \tfrac{h\xi - 2\xi'}{2})](\xi) e^{ix\cdot\xi} \, d\xi d\xi'.$$

Hence, it is clear that

$$\lim_{h\to 0} \rho[\Psi(D)^{-s}\gamma_h(t/h)\Psi(D)^{-s}](x/h)$$
$$= \int_{\mathbb{R}^{2d}} e^{-i\frac{t}{2}\nabla\phi(\xi')\cdot\xi} \Psi(\xi')^{-2s} \mathcal{F}_x[f(\cdot, -\xi')](\xi) e^{ix\cdot\xi} \, d\xi d\xi'$$
$$= \int_{\mathbb{R}^d} f(x - \tfrac{t}{2}\nabla\phi(\xi'), -\xi') \Psi(\xi')^{-2s} \, d\xi'.$$

This and (7.2) imply

$$\left\| \int_{\mathbb{R}^d} f(x - t\nabla\phi(\xi), \xi) \Psi(\xi)^{-2s} \, d\xi \right\|_{\frac{q}{2}, \frac{r}{2}} \lesssim \liminf_{h\to 0} \left\| \rho[\Psi(D)^{-s}\gamma_h(t/h)\Psi(D)^{-s}](\cdot/h) \right\|_{\frac{q}{2}, \frac{r}{2}}$$
$$\lesssim \liminf_{h\to 0} h^{\frac{2}{q} + \frac{2d}{r}} \|\gamma_h\|_{\mathcal{C}^\beta}.$$

For the right-hand side, we have $\|\gamma_h\|_{\mathcal{C}^\beta} \lesssim h^{-\frac{d}{\beta}} \|f\|_{L^\beta_{x,v}}$ for $h > 0$ sufficiently small, and the result follows since $\beta = \beta_{\frac{d}{2}}(q, r)$. $\qquad\square$

Proof (Proof of Theorem 16). Combine Theorems 4, 6, 11 and 14 with Proposition 12. $\qquad\square$

7.3. Refined Strichartz estimates

Dispersive PDE theory has been significantly enriched by the exploitation of refined versions of the classical single-function Strichartz estimates, including their decisive role in the construction of profile decompositions and understanding of concentration phenomena; see, for example, [3], [8], [13], [17], [60], [61] and [68]. In particular, we emphasize the key role of the papers by Bourgain [8] and Moyua–Vargas–Vega [60], [61] in many of these developments.

An example of a refined Strichartz estimate takes the form

$$\|U_\phi f\|_{L^q_t L^r_x} \lesssim \|f\|_{\mathcal{X}}^{1-\theta} \|f\|_{\mathcal{H}}^\theta \tag{7.19}$$

for some $\theta \in (0, 1)$ and some norm $\|\cdot\|_{\mathcal{X}}$ that satisfies $\|f\|_{\mathcal{X}} \lesssim \|f\|_{\mathcal{H}}$. The role of the norm $\|\cdot\|_{\mathcal{X}}$ is to identify the extent to which the frequency support of the initial data concentrates on certain subsets of $\mathbb{R}^d$. Frank and Sabin [30] observed that estimates of the form

$$\left\| \sum_j v_j |U_\phi f_j|^2 \right\|_{L^{\frac{q}{2}}_t L^{\frac{r}{2}}_x} \lesssim \|v\|_{\ell^\beta} \tag{7.20}$$

for families of orthonormal functions $(f_j)_j$ in $\mathcal{H}$, lead to

$$\|U_\phi f\|_{L_t^q L_x^r} \lesssim \left(\sum_{j \in \mathbb{Z}} \|P_j f\|_{\mathcal{H}}^{2\beta} \right)^{\frac{1}{2\beta}}, \tag{7.21}$$

as long as $r < \infty$. In the case $\mathcal{H} = \dot{H}^s(\mathbb{R}^d)$, the right-hand side of (7.21) is the homogeneous Besov norm $\|f\|_{\dot{B}_{2,2\beta}^s}$, and if $\beta > 1$, we immediately obtain (7.19) with $\theta = \frac{1}{\beta}$ and

$$\|f\|_{\mathcal{X}} = \sup_{k \in \mathbb{Z}} \|P_k f\|_{\dot{H}^s}.$$

In this regard, Besov space refinements of the classical Strichartz estimates are highly desirable. Prior to the recent developments on estimates of the form (7.20), typically refined Strichartz estimates were established using results or ideas from bilinear Fourier restriction theory, in many cases relying on sophisticated bilinear estimates (see, for example, [3] and [68], which relied upon bilinear estimates in [75] and [76], respectively). The observation of Frank–Sabin that Besov-space refinements are derivable from (7.20) opens a fresh line of attack that is likely to be fruitful in future developments.

Although in certain contexts it is necessary to obtain yet further refinements of (7.21) that detect concentration on smaller sets (such as dyadic cubes), as discussed in more detail below in the context of the fractional Schrödinger equation, the Besov space refinements (7.21) themselves suffice for certain applications to mass concentration phenomena. Beforehand, we briefly recall the argument given by Frank and Sabin for generating (7.21) from (7.20). In doing so, we observe that whenever $(P_j)_{j \in \mathbb{Z}}$ is a family of operators acting on a suitable subspace of $\mathcal{H}$ satisfying the following properties (1)–(4), we obtain (7.21) from (7.20).

1. *Decomposition of the identity:* $f = \sum_{j \in \mathbb{Z}} P_j f$.
2. *Commutativity with the solution operator:* $U_\phi P_j f = P_j U_\phi f$ for $j \in \mathbb{Z}$.[11]
3. *Littlewood–Paley inequality:* $\|f\|_{L^r(\mathbb{R}^d)} \lesssim \|(\sum_{j \in \mathbb{Z}} |P_j f|^2)^{\frac{1}{2}}\|_{L^r(\mathbb{R}^d)}$.
4. *Almost orthogonality:* $N \in \mathbb{N}$ exists such that $P_k^* P_j = 0$ whenever $|j - k| \geq N$.

Indeed, as a result of the first three properties, we have

$$\|U_\phi f\|_{L_t^q L_x^r} \lesssim \left\| \sum_{j \in \mathbb{Z}} |U_\phi P_j f|^2 \right\|_{L_t^{\frac{q}{2}} L_x^{\frac{r}{2}}}^{\frac{1}{2}}.$$

From the almost orthogonality property, we may decompose $\mathbb{Z} = \cup_{k=1}^N J_k$ into disjoint index sets such that, for each $k \in \{1, \ldots, N\}$, the family $(P_j f)_{j \in J_k}$ is orthogonal in $\mathcal{H}$. Hence, by the triangle inequality in $L_t^{\frac{q}{2}} L_x^{\frac{r}{2}}$ followed by (7.20), we obtain

$$\|U_\phi f\|_{L_t^q L_x^r} \lesssim \left(\sum_{k=1}^N \left(\sum_{j \in J_k} \|P_j f\|_{\mathcal{H}}^{2\beta} \right)^{\frac{1}{\beta}} \right)^{\frac{1}{2}}$$

and thus, by the equivalence of norms on finite-dimensional spaces, (7.21) follows.

We also remark that if, in addition, $\sum_{j \in \mathbb{Z}} \|P_j f\|_{\mathcal{H}}^2 \lesssim \|f\|_{\mathcal{H}}^2$ (which, for example, trivially follows if we add the assumption that $P_k^* P_j$ is positive semi-definite for each $j, k \in \mathbb{Z}$), then (7.21) implies the classical Strichartz estimate

$$\|U_\phi f\|_{L_t^q L_x^r} \lesssim \|f\|_{\mathcal{H}}$$

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[11]Strictly speaking, the right-hand side should be interpreted as P_j acting on the function $x \mapsto U_\phi f(x, t)$ for fixed $t \in \mathbb{R}$.

via the trivial embedding $\ell^{2\beta} \subseteq \ell^2$. In this regard, (7.21) represents a refined version of the classical Strichartz estimate for $\beta > 1$.

Clearly, if $(P_j)_{j\in\mathbb{Z}}$ is the Littlewood–Paley family given by (7.3), then the properties (1)–(4) hold (with $N = 2$), and $P_k^* P_j$ is positive semi-definite for each j, k.

As our first application in this direction, for appropriate (q, r, β, s) described in Theorem 17 below, we may use Theorems 4 and 14 to deduce (7.21), and consequently

$$\|e^{it(-\Delta)^{\alpha/2}} f\|_{L_t^q L_x^r} \lesssim \|f\|_{\dot{B}_{2,2\beta}^s} \tag{7.22}$$

for all f in the homogeneous Besov space $\dot{B}_{2,2\beta}^s$.

Theorem 17. *Let $\alpha \in \mathbb{R} \setminus \{0, 1\}$. Suppose $q, r \geq 2$ satisfy $\frac{d}{r} + \frac{\alpha}{q} < d$, and define $s = \frac{d}{2} - \frac{d}{r} - \frac{\alpha}{q}$.*

(i) *Let $d \geq 2$. If $(\frac{1}{r}, \frac{1}{q})$ belongs to $\mathrm{int}(OA_{\frac{d}{2}}C) \cup [C, A_{\frac{d}{2}})$, then (7.22) holds with $\beta \leq \beta_{\frac{d}{2}}(q, r)$. If $(\frac{1}{r}, \frac{1}{q})$ belongs to $\mathrm{int}(ODE_{\frac{d}{2}} A_{\frac{d}{2}}) \cup (O, A_{\frac{d}{2}}] \cup [A_{\frac{d}{2}}, E_{\frac{d}{2}})$, then (7.22) holds with $\beta < \frac{d}{2}$.*

(ii) *Let $d = 1$. If $(\frac{1}{r}, \frac{1}{q})$ belongs to $\mathrm{int}(OA_{\frac{1}{2}}C) \cup [C, A_{\frac{1}{2}})$, then (7.22) holds with $\beta \leq \beta_{\frac{1}{2}}(q, r)$.*

Estimates of the form (7.22) have appeared several times in the literature for the fractional Schrödinger equation. As an example, in work on the mass concentration phenomena for L^2-critical nonlinear fractional Schrödinger equations in [18], a key component of the proof (see equation (2.8) in [18]) is establishing (7.22) in the case $s = 0$ for some $\beta > 1$. The argument in [18] proceeds by establishing (7.22) with $(q, r, \beta, s) = (4, 4, 2, \frac{d-\alpha}{4})$ (using bilinear arguments) and interpolation with standard Strichartz estimates (i.e. $\beta = 1$). As a result, (7.22) is obtained in the case of non-sharp $\frac{d}{2}$-admissible (q, r) with $s = 0$ and certain $\beta \in (1, 2)$. Observe that the technical condition $\frac{d}{r} + \frac{\alpha}{q} < d$ in Theorem 17 is completely redundant for such (q, r), and applying our result may lead to improvement in the range of β for (7.22). For example, for $(\frac{1}{r}, \frac{1}{q})$ on $(O, A_{\frac{d}{2}})$ with $s = 0$, Theorem 17 gives (7.22) with β arbitrarily close to $\frac{d-1+\alpha}{d}$, and this exponent clearly grows unboundedly with α.

The estimate (7.22) also appeared in [10] in their study of the existence of extremizers for Fourier extension estimates associated with plane curves of the form $(\xi, |\xi|^\alpha)$, where $\alpha > 1$. In particular, it was shown that (7.22) holds when $(d, q, r, \beta, s) = (1, 6, 6, \frac{3}{2}, \frac{2-\alpha}{6})$; since $\beta_{\frac{1}{2}}(6, 6) = \frac{3}{2}$, we completely recover this result from Theorem 17 *(ii)*.

For the wave equation, we obtain

$$\|e^{it\sqrt{-\Delta}} f\|_{L_t^q L_x^r} \lesssim \|f\|_{\dot{B}_{2,2\beta}^s} \tag{7.23}$$

as a consequence of Theorems 5 and 9 combined with (7.21), where (q, r, β, s) are prescribed in the following.

Theorem 18. *Suppose $q, r \geq 2$, and define $s = \frac{d}{2} - \frac{d}{r} - \frac{1}{q}$.*

(i) *Let $d \geq 3$. If $(\frac{1}{r}, \frac{1}{q})$ belongs to $\mathrm{int}(OA_{\frac{d-1}{2}}C) \cup [C, A_{\frac{d-1}{2}})$, then (7.23) holds with $\beta \leq \beta_{\frac{d-1}{2}}(q, r)$. If $(\frac{1}{r}, \frac{1}{q})$ belongs to $\mathrm{int}(ODE_{\frac{d-1}{2}} A_{\frac{d-1}{2}}) \cup (O, A_{\frac{d-1}{2}}] \cup [A_{\frac{d-1}{2}}, E_{\frac{d-1}{2}})$, then (7.22) holds with $\beta < \frac{q}{2}$.*

(ii) *Let $d = 2$. If $(\frac{1}{r}, \frac{1}{q})$ belongs to $\mathrm{int}(OA_{\frac{1}{2}}C) \cup [C, A_{\frac{1}{2}})$, then (7.23) holds with $\beta \leq \beta_{\frac{1}{2}}(q, r)$.*

Although the estimates in Theorem 18 for $(\frac{1}{r}, \frac{1}{q})$ belonging to $[C, B_{\frac{d-1}{2}}]$ may be obtained from Theorem 2 (due to Frank–Sabin), for such q and r, the estimates (7.23) appeared earlier with coincidental or superior thresholds for β and via different techniques. For example, Quilodrán [67] obtained (7.23) in the case $(d, q, r, \beta, s) = (2, 6, 6, \frac{3}{2}, \frac{1}{2})$ in his work on sharp Fourier extension estimates for the two-dimensional cone at the Stein–Tomas exponent; in this instance, the threshold for β is the same as in Theorem 18. In a significant development, Ramos [68] independently obtained the same estimate as in

[67] when $d = 2$, and[12] (7.23) with $(q, r, \beta, s) = (\frac{2(d+1)}{d-1}, \frac{2(d+1)}{d-1}, \frac{d+1}{d-1}, \frac{1}{2})$ for $d \geq 3$ using deep bilinear Fourier restriction theory; this is superior to the threshold $\beta = \frac{d+1}{d}$ given by Theorem 18 in this special case. We remark that by the aforementioned result of Ramos and a trivial interpolation, one obtains (7.23) with $(\frac{1}{r}, \frac{1}{q})$ on $[(\frac{d-1}{2(d+1)}, \frac{d-1}{2(d+1)}), E_{\frac{d-1}{2}}]$, $s = \frac{d+1}{4}(\frac{1}{2} - \frac{1}{r})$ and $\beta = \frac{q}{2}$; interestingly, from Theorem 18, we are essentially able to recover this result on $[A_{\frac{d-1}{2}}, E_{\frac{d-1}{2}}]$ since our theorem yields (7.23) with $\beta < \frac{q}{2}$ in such a case.

Finally, we note that Theorems 6, 7 and 10–13 all lead to refinements of the Strichartz estimates for the Klein–Gordon equation

$$\|e^{it\sqrt{1-\Delta}}f\|_{L_t^q L_x^r} \lesssim \|f\|_{B_{2,2\beta}^s},$$

where $\|f\|_{\dot{B}_{2,2\beta}^s}$ is the inhomogeneous Besov norm; the details of the statements are left to the interested reader. We also refer the reader to earlier work on refined Strichartz estimates for the Klein–Gordon propagator in [11], [14], [15] and [50].

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