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GROUPS WITH THE SAME CHARACTER DEGREES AS SPORADIC ALMOST SIMPLE GROUPS

SEYED HASSAN ALAVI, ASHRAF DANESHKHAH✉ and ALI JAFARI

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Abstract

Let G be a finite group and $\mathsf{cd}(G)$ denote the set of complex irreducible character degrees of G. We prove that if G is a finite group and H is an almost simple group whose socle is a sporadic simple group H_0 and such that $\mathsf{cd}(G) = \mathsf{cd}(H)$, then $G' \cong H_0$ and there exists an abelian subgroup A of G such that G/A is isomorphic to H. In view of Huppert's conjecture, we also provide some examples to show that G is not necessarily a direct product of A and H, so that we cannot extend the conjecture to almost simple groups.

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1. Introduction

Let G be a finite group, and let $\mathsf{Irr}(G)$ be the set of complex irreducible characters of G. Denote the set of these character degrees of G by $\mathsf{cd}(G) = \{\chi(1) \mid \chi \in \mathsf{Irr}(G)\}$. When the context allows, the set of irreducible character degrees will be referred to simply as the set of character degrees. There is growing interest in the structural information which can be determined from the character degree set of G, although it is well known that the character degree set of G cannot completely determine the structure of G. For example, the nonisomorphic groups D_8 and Q_8 not only have the same set of character degrees, but also share the same character table. The character degree set cannot be used to distinguish between solvable and nilpotent groups. For example, if G is either Q_8 or S_3, then $\mathsf{cd}(G) = \{1, 2\}$.

In the late 1990s, Huppert [8] posed a conjecture which, if true, would sharpen the connection between the character degree set of a nonabelian simple group and the structure of the group.

Conjecture 1.1 (Huppert). Let G be a finite group, and let H be a finite nonabelian simple group such that the sets of character degrees of G and H are the same. Then $G \cong H \times A$, where A is an abelian group.

The conjecture asserts that the nonabelian simple groups are essentially characterised by their character degree set. In addition to verifying this conjecture for many of the simple groups of Lie type, it is also verified for all sporadic simple groups [1, 3, 12]. Note that this conjecture does not extend to solvable groups (for example, Q_8 and D_8), nor to almost simple groups. In fact, there are four groups G of order 240 whose character degrees are the same as $\mathsf{Aut}(A_5) = S_5$. These groups are $SL_2(5) \cdot \mathbb{Z}_2$ (nonsplit), $SL_2(5) : \mathbb{Z}_2$ (split), $A_5 : \mathbb{Z}_4$ (split) and $S_5 \times \mathbb{Z}_2$. If we further assume that $G' = A_5$, we still have two possibilities for G, namely, $A_5 : \mathbb{Z}_4$ and $S_5 \times \mathbb{Z}_2$. Indeed, the groups $A_5 : \mathbb{Z}_{2^n}$, for $n \geq 1$, have the same character degree set as S_5. Although we cannot establish Huppert's conjecture for almost simple groups, we can prove the following result for finite groups whose character degrees are the same as those of almost simple groups with socle a sporadic simple group.

Theorem 1.2. *Let G be a finite group and let H be an almost simple group whose socle H_0 is one of the sporadic simple groups. If $\mathsf{cd}(G) = \mathsf{cd}(H)$, then $G' \cong H_0$ and $G/Z(G)$ is isomorphic to H.*

In order to prove Theorem 1.2, we follow the steps introduced in [8]. In the notation of the theorem, we show that

(1) if G'/M is a chief factor of G, then G'/M is isomorphic to H_0;
(2) if $\theta \in \mathsf{Irr}(M)$ with $\theta(1) = 1$, then $I_{G'}(\theta) = G'$ and so $M = M'$;
(3) $M = 1$ and $G' \cong H_0$; and
(4) $G/Z(G)$ is isomorphic to H.

In Propositions 3.3–3.6, we will verify steps (1)–(4) and the proof of Theorem 1.2 follows immediately from these statements.

Remark 1.3. Recall that Theorem 1.2, for the case where $H = H_0$ is a sporadic simple group, has already been settled (see [1, 3, 9, 12]). Moreover, if H is the automorphism group of one of the Mathieu groups, then Theorem 1.2 is also proved by the authors [2]. Therefore, we only need to consider the remaining cases where $H = \mathsf{Aut}(H_0)$ with H_0 one of J_2, HS, J_3, McL, He, Suz, $O'N$, Fi_{22}, HN and Fi_{24}'.

2. Preliminaries

Throughout this paper, all groups are finite. A group H is said to be an almost simple group with socle H_0 if $H_0 \leqslant H \leqslant \mathsf{Aut}(H_0)$, where H_0 is a nonabelian simple group. If $N \trianglelefteq G$ and $\theta \in \mathsf{Irr}(N)$, then the inertia group $I_G(\theta)$ of θ in G is defined by $I_G(\theta) = \{g \in G \mid \theta^g = \theta\}$. If the character $\chi = \sum_{i=1}^{k} e_i\chi_i$, where each χ_i is an irreducible character of G and e_i is a nonnegative integer, then those χ_i with $e_i > 0$ are called the *irreducible constituents* of χ. The set of all irreducible constituents of θ^G is denoted by $\mathsf{Irr}(G \mid \theta)$. All further notation and definitions are standard as in [7, 10]. For computation, we use GAP [6].

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LEMMA 2.1 [7, Theorems 19.5 and 21.3]. *Suppose that $N \trianglelefteq G$ and $\chi \in \mathsf{Irr}(G)$.*

(a) *If $\chi_N = \theta_1 + \theta_2 + \cdots + \theta_k$ with $\theta_i \in \mathsf{Irr}(N)$, then k divides $|G/N|$. In particular, if $\chi(1)$ is prime to $|G/N|$, then $\chi_N \in \mathsf{Irr}(N)$.*
(b) *(Gallagher's Theorem) If $\chi_N \in \mathsf{Irr}(N)$, then $\chi\psi \in \mathsf{Irr}(G)$ for all $\psi \in \mathsf{Irr}(G/N)$.*

LEMMA 2.2 [7, Theorems 19.6 and 21.2]. *Suppose that $N \trianglelefteq G$ and $\theta \in \mathsf{Irr}(N)$. Let $I = I_G(\theta)$.*

(a) *If $\theta^I = \sum_{i=1}^{k} \phi_i$ with $\phi_i \in \mathsf{Irr}(I)$, then $\phi_i^G \in \mathsf{Irr}(G)$ and $\phi_i(1)|G : I| \in \mathsf{cd}(G)$.*
(b) *If θ extends to $\psi \in \mathsf{Irr}(I)$, then $(\psi\tau)^G \in \mathsf{Irr}(G)$ for all $\tau \in \mathsf{Irr}(I/N)$. In particular, $\theta(1)\tau(1)|G : I| \in \mathsf{cd}(G)$.*
(c) *If $\rho \in \mathsf{Irr}(I)$ such that $\rho_N = e\theta$, then $\rho = \theta_0\tau_0$, where θ_0 is a character of an irreducible projective representation of I of degree $\theta(1)$ and τ_0 is a character of an irreducible projective representation of I/N of degree e.*

A character $\chi \in \mathsf{Irr}(G)$ is said to be *isolated* in G if $\chi(1)$ is divisible by no proper nontrivial character degree of G and if no proper multiple of $\chi(1)$ is a character degree of G. In this situation, we also say that $\chi(1)$ is an *isolated degree* of G. We define a *proper power* degree of G to be a character degree of G of the form f^a for integers f and a, with $a > 1$.

LEMMA 2.3 [12, Lemma 3]. *Let G/N be a solvable factor group of G minimal with respect to being nonabelian. Then two cases can occur.*

(a) *G/N is an r-group for some prime r. In this case, G has a proper prime power degree.*
(b) *G/N is a Frobenius group with an elementary abelian Frobenius kernel F/N. Then $f := |G : F| \in \mathsf{cd}(G)$ and $|F/N| = r^a$ for some prime r, and a is the smallest integer such that $r^a \equiv 1 \bmod f$.*
    (1) *If $\chi \in \mathsf{Irr}(G)$ such that no proper multiple of $\chi(1)$ is in $\mathsf{cd}(G)$, then either f divides $\chi(1)$, or r^a divides $\chi(1)^2$.*
    (2) *If $\chi \in \mathsf{Irr}(G)$ is isolated, then $f = \chi(1)$ or r^a divides $\chi(1)^2$.*

LEMMA 2.4 [4, Theorems 2–4]. *If S is a nonabelian simple group, then there exists a nontrivial irreducible character θ of S that extends to $\mathsf{Aut}(S)$. Moreover:*

(a) *if S is an alternating group of degree at least seven, then S has two characters of consecutive degrees $n(n-3)/2$ and $(n-1)(n-2)/2$ that both extend to $\mathsf{Aut}(S)$;*
(b) *if S is a simple group of Lie type, then the Steinberg character of S of degree $|S|_p$ extends to $\mathsf{Aut}(S)$; and*
(c) *if S is a sporadic simple group or the Tits group, then S has two nontrivial irreducible characters of coprime degrees which both extend to $\mathsf{Aut}(S)$.*

LEMMA 2.5 [4, Lemma 5]. *Let N be a minimal normal subgroup of G so that $N \cong S^k$, where S is a nonabelian simple group. If $\theta \in \mathsf{Irr}(S)$ extends to $\mathsf{Aut}(S)$, then $\theta^k \in \mathsf{Irr}(N)$ extends to G.*

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Lemma 2.6 [8, Lemma 6]. *Suppose that $M \trianglelefteq G' = G''$ and, for every $\lambda \in \mathsf{Irr}(M)$ with $\lambda(1) = 1$, $\lambda^g = \lambda$ for all $g \in G'$. Then $M' = [M, G']$ and $|M/M'|$ divides the order of the Schur multiplier of G'/M.*

Lemma 2.7 [11, Theorem D]. *Let N be a normal subgroup of a finite group G and let $\varphi \in \mathsf{Irr}(N)$ be G-invariant. Assume that $\chi(1)/\varphi(1)$ is odd, for all $\chi(1) \in \mathsf{Irr}(G \mid \varphi)$. Then G/N is solvable.*

3. Proof of the main result

In this section, we prove Theorem 1.2 for an almost simple group H whose socle is one of the sporadic simple groups H_0 listed in Remark 1.3. For convenience, we first mention some properties of H and H_0 which can be drawn from ATLAS [5].

Lemma 3.1. *Suppose that H_0 is one of the sporadic simple groups shown in the first column of Table 1 and $H = \mathsf{Aut}(H_0)$. Then:*

(a) *the outer automorphism group $\mathsf{Out}(H_0)$ of H_0 is isomorphic to $\mathbb{Z}_2$, and the Schur multiplier $\mathsf{M}(H_0)$ of H_0 is as shown in Table 1;*
(b) *H has neither consecutive, nor proper power degrees; and*
(c) *if K is a maximal subgroup of H_0 whose index in H_0 divides some degrees $\chi(1)$ of H, then K is given in Table 1 and, for each K, $\chi(1)/|H_0 : K|$ divides $t(K)$, where $t(K)$ is as in Table 1.*

Proof. Parts (a) and (b) follow from ATLAS [5], and (c) is a simple calculation. □

Proposition 3.2. *Let S be a sporadic simple group or the Tits group ${}^2F_4(2)'$ whose character degrees divide some degrees of an almost simple group H with socle a sporadic simple group H_0. Then either S is isomorphic to H_0 or (H, S) is as in Table 2.*

Proof. The proof follows from ATLAS [5]. □

Proposition 3.3. *Let G be a finite group and let H be an almost simple group whose socle is a sporadic simple group H_0. If $\mathsf{cd}(G) = \mathsf{cd}(H)$, then the chief factor G'/M of G is isomorphic to H_0.*

Proof. We first apply Remark 1.3, so that we may assume that $H = \mathsf{Aut}(H_0)$, where H_0 is one of the sporadic groups J_2, J_3, McL, HS, He, HN, Fi_{22}, Fi_{24}', $O'N$ and Suz.

We now prove that $G' = G''$. Assume the contrary. Then there is a normal subgroup N of G, where N is a maximal such that G/N is a nonabelian solvable group. Since G has no prime power degree, by Lemma 2.3, G/N is a Frobenius group with kernel F/N of order r^a. In this case, $1 < f = |G : F| \in \mathsf{cd}(G)$.

Suppose that H_0 is not J_2 and Suz. Then G has three isolated coprime degrees, as in Table 3. But Lemma 2.3(b)(2) implies that f must be equal to these degrees, which is impossible.

Suppose $H_0 = Suz$. Let $r = 2$. For $1 \leqslant a \leqslant 4$, $r^a - 1 = 2^a - 1$ is less than the smallest nontrivial degree, 143, of G. By Lemma 2.3(b)(2), f must divide both degrees

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TABLE 1. Properties of some sporadic simple groups H_0 and their automorphism groups.

H_0	$\mathsf{Aut}(H_0)$	$M(H_0)$	K	$t(K)$
J_2	$J_2:2$	$\mathbb{Z}_2$	$U_3(3)$	3
HS	$HS:2$	$\mathbb{Z}_2$	M_{22}	2^5
			$U_3(5):2$	6 or 8
J_3	$J_3:2$	$\mathbb{Z}_3$	-	-
McL	$McL:2$	$\mathbb{Z}_3$	$U_4(3)$	$5\cdot 7$ or $2^2\cdot 3\cdot 5$
He	$He:2$	1	$S_4(4):2$	1
Suz	$Suz:2$	$\mathbb{Z}_6$	$G_2(4)$	$3^2\cdot 13$ or $3\cdot 5\cdot 7$
			$U_5(2)$	5
$O'N$	$O'N:2$	$\mathbb{Z}_3$	-	-
Fi_{22}	$Fi_{22}:2$	$\mathbb{Z}_6$	$2\cdot U_6(2)$	$2^2\cdot 3\cdot 5\cdot 11$, $2^4\cdot 5\cdot 7$, $3\cdot 5\cdot 11$ or 3^5
			$O_8^+(2):S_3$	6
			$2^{10}:M_{22}$	6
HN	$HN:2$	1	-	-
Fi_{24}'	$Fi_{24}':2$	$\mathbb{Z}_3$	$2\cdot Fi_{23}$	$2^4\cdot 5^2\cdot 7\cdot 17\cdot 23$, $2\cdot 3^3\cdot 7\cdot 11\cdot 13\cdot 17$, $2^2\cdot 3\cdot 11\cdot 13\cdot 17\cdot 23$, $2^3\cdot 3\cdot 7\cdot 11\cdot 13\cdot 23$, $2^4\cdot 3\cdot 13\cdot 17\cdot 23$, $2^2\cdot 7\cdot 11\cdot 17\cdot 23$, $11\cdot 13\cdot 17\cdot 23$

The symbol '-' means that there is no subgroup K satisfying these conditions in Lemma 3.1(c).

$75\,075 = 3\cdot 5^2\cdot 7\cdot 11\cdot 13$ and $5940 = 2^2\cdot 3^3\cdot 5\cdot 11$, and so f divides $3\cdot 5\cdot 11$. But no divisor of $3\cdot 5\cdot 11$ is a degree of G. Therefore $r\neq 2$. Now we apply Lemma 2.3(b)(2) to the isolated degrees $66\,560 = 2^{10}\cdot 5\cdot 13$ and $133\,056 = 2^6\cdot 3^3\cdot 7\cdot 11$. Since $r\neq 2$, it follows that f must be equal to both of these degrees, which is impossible.

Suppose, finally, that $H_0 = J_2$. We make the same argument as in the case of Suz. If $r = 5$, then, by Lemma 2.3(b)(2), f must divide both $2^5\cdot 3^2$ and $2\cdot 3^3\cdot 7$, and so f divides $2\cdot 3^2$, but G has no degree which divides $2\cdot 3^2$. Thus $r\neq 5$. Now we apply Lemma 2.3(b)(2) to the isolated degrees $2^5\cdot 5$ and $5^2\cdot 7$. Again, f must be equal to both of these degrees, which is impossible.

We conclude that $G' = G''$. Let G'/M be a chief factor of G. As G' is perfect, G'/M is nonabelian and so G'/M is isomorphic to S^k for some nonabelian simple group S and some integer $k\geq 1$.

We first show that $k = 1$. Assume the contrary. Then, by Lemma 2.4, S possesses a nontrivial irreducible character θ extendible to $\mathsf{Aut}(S)$, and so Lemma 2.5 implies that $\theta^k\in\mathsf{Irr}(G'/M)$ extends to G/M, that is, G has a proper power degree contradicting Lemma 3.1(b). Therefore $k = 1$ and $G'/M\cong S$.

Suppose S is an alternating group of degree $n\geq 7$. By Lemma 2.4(a), S has nontrivial irreducible characters θ_1 and θ_2 with $\theta_1(1) = n(n-3)/2$ and $\theta_2(1) = \theta_1(1) + 1 = (n-1)(n-2)/2$ and both θ_1 and θ_2 extend to $\mathsf{Aut}(S)$. Thus G possesses two consecutive nontrivial character degrees, which contradicts Lemma 3.1(b).

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TABLE 2. Sporadic simple groups S (and the Tits group) and almost simple groups H for Proposition 3.2.

H	S
$M_{12}, M_{12}:2$	M_{11}, M_{12}
M_{23}	M_{11}, M_{23}
M_{24}	M_{11}, M_{24}
J_4	$M_{11}, M_{12}, M_{22}, J_4$
$HS, HS:2$	M_{11}, M_{22}, HS
$McL, McL:2$	M_{11}, McL
$Suz, Suz:2$	$M_{11}, M_{12}, M_{22}, J_2, Suz, {}^2F_4(2)'$
Co_3	$M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, Co_3$
Co_2	$M_{11}, M_{12}, M_{22}, M_{23}, M_{24}\ J_2, Co_2$
Co_1	$M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, McL, J_2, HS, Co_1, Co_3, {}^2F_4(2)'$
$Fi_{22}, Fi_{22}:2$	$M_{11}, M_{12}, M_{22}, J_2, Fi_{22}$
Fi_{23}	$M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, HS, J_2, Fi_{23}, {}^2F_4(2)'$
$Fi'_{24}, Fi'_{24}:2$	$M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, He, J_2, Fi'_{24}, {}^2F_4(2)'$
Th	$J_2, Th, {}^2F_4(2)'$
Ru	$J_2, Ru, {}^2F_4(2)'$
Ly	M_{11}, M_{12}, J_2, Ly
$HN, HN:2$	$M_{11}, M_{12}, M_{22}, J_1, J_2, HS, HN$
$O'N, O'N:2$	$M_{11}, M_{12}, M_{22}, O'N$
B	$M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_2, J_3, HS, McL, Suz, Fi_{22}, Co_3, Co_2, Th, B, {}^2F_4(2)'$
M	$M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_2, J_3, HS, McL, Suz, Fi_{22}, Co_3, Co_2, He, O'N, Ru, M, {}^2F_4(2)'$

TABLE 3. Some isolated degrees of automorphism groups of sporadic simple groups.

H	$\chi_1(1)$	$\chi_2(1)$	$\chi_3(1)$
$HS:2$	$825 = 3\cdot 5^2\cdot 11$	$1792 = 2^8\cdot 7$	$2520 = 2^3\cdot 3^2\cdot 5\cdot 7$
$J_3:2$	$170 = 2\cdot 5\cdot 17$	$324 = 2^2\cdot 3^4$	$1215 = 3^5\cdot 5$
$McL:2$	$1750 = 2\cdot 5^3\cdot 7$	$4500 = 2^2\cdot 3^2\cdot 5^3$	$5103 = 3^6\cdot 7$
$He:2$	$1920 = 2^7\cdot 3\cdot 5$	$2058 = 2\cdot 3\cdot 7^3$	$20825 = 5^2\cdot 7^2\cdot 17$
$O'N:2$	$10944 = 2^6\cdot 3^2\cdot 19$	$26752 = 2^7\cdot 11\cdot 19$	$116963 = 7^3\cdot 11\cdot 31$
$Fi_{22}:2$	$360855 = 3^8\cdot 5\cdot 11$	$577368 = 2^3\cdot 3^8\cdot 11$	$1164800 = 2^9\cdot 5^2\cdot 7\cdot 13$
$HN:2$	$1575936 = 2^{10}\cdot 3^4\cdot 19$	$2784375 = 3^4\cdot 5^5\cdot 11$	$3200000 = 2^{10}\cdot 5^5$
$Fi'_{24}:2$	$159402880 = 2^7\cdot 5\cdot 7^2\cdot 13\cdot 17\cdot 23$	$5775278080 = 2^{14}\cdot 5\cdot 11\cdot 13\cdot 17\cdot 29$	$156321775827 = 3^{14}\cdot 7^2\cdot 23\cdot 29$

If $S \neq {}^2F_4(2)'$ is a simple group of Lie type in characteristic p, then the Steinberg character of S of degree $|S|_p$ extends to $\mathsf{Aut}(S)$ so that G possesses a nontrivial prime power degree, which contradicts Lemma 3.1(b).

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TABLE 4. Particular degrees of some sporadic simple groups S and the Tits group.

S	M_{11}	M_{12}	M_{22}	M_{23}	M_{24}	J_1	J_2	HS	He	${}^2F_4(2)'$
Degree	10	54	21	22	23	76	36	22	1275	27

Finally, if S is a sporadic simple group or the Tits group ${}^2F_4(2)'$, then the irreducible character degrees of S divide some degrees of H, and so, by Proposition 3.2, $S \cong H_0$ or (H, S) is as in Table 2. In the later case, for a given H as in the first row of Table 2, assume that S is not isomorphic to H_0. Then we apply Lemma 2.4(c). For each possible S, as shown in the first row of Table 4, G possesses an irreducible character of the degree shown in the second row of Table 4. This leads us to a contradiction. Therefore $S \cong H_0$, and hence G'/M is isomorphic to H_0. □

PROPOSITION 3.4. *Let G be a finite group with $\mathsf{cd}(G) = \mathsf{cd}(H)$, where H is an almost simple group whose socle is a sporadic simple group H_0. Suppose that the chief factor G'/M is isomorphic to H_0. If $\theta \in \mathsf{Irr}(M)$ with $\theta(1) = 1$, then $I_{G'}(\theta) = G'$.*

PROOF. By Remark 1.3, we may assume that $H = \mathsf{Aut}(H_0)$, where H_0 is one of the sporadic groups J_2, J_3, McL, HS, He, HN, Fi_{22}, Fi'_{24}, $O'N$ and Suz.

Suppose that $I = I_{G'}(\theta) < G'$. Let $\theta^I = \sum_{i=1}^{k} \phi_i$, where $\phi_i \in \mathsf{Irr}(I)$ for $i = 1, 2, \ldots, k$. Assume that U/M is a maximal subgroup of $G'/M \cong H_0$ containing I/M and set $t := |U : I|$. It follows, from Lemma 2.2(a), that $\phi_i(1)|G' : I| \in \mathsf{cd}(G')$, and so $t\phi_i(1)|G' : U|$ divides some degrees of G. Then $|G' : U|$ must divide some character degrees of G. By Lemma 3.1(c), for each H_0 in the first column of Table 1, U/M must be one of the subgroups K listed in the fifth column of Table 1 and $t\phi_i(1)|G' : U|$ must divide the positive integers $t(K)$ in the sixth column of Table 1.

If H_0 is J_3, $O'N$ or HN, then, by Lemma 3.1(b), there is no such subgroup U/M, and so $I_{G'}(\theta) = G'$ in these cases. We now discuss each remaining case separately.

Case 1. $H_0 = J_2$. By Lemma 3.1(b), $U/M \cong U_3(3)$ and $t\phi_i(1)$ divides three, for all i. Since $U_3(3)$ has no subgroup of index three [5, page 14], it follows that $t = 1$, that is, $I/M = U/M \cong U_3(3)$. Since $U_3(3)$ has trivial Schur multiplier, it follows, from [10, Theorem 11.7], that θ extends to $\theta_0 \in \mathsf{Irr}(I)$ and so, by Lemma 2.2(b), $(\theta_0\tau)^{G'} \in \mathsf{Irr}(G')$, for all $\tau \in \mathsf{Irr}(I/M)$. For $\tau(1) = 27 \in \mathsf{cd}(U_3(3))$, we find that $3 \cdot 27 \cdot \theta_0(1)$ divides some degrees of G, which is a contradiction. Therefore θ is G'-invariant.

Case 2. $H_0 = HS$. By Lemma 3.1(b), one of the following holds.

(i) $U/M \cong M_{22}$ and $t\phi_i(1)$ divides 2^5, for all i.

As $U/M \cong M_{22}$ does not have any subgroup of index 2^m for $m = 1, \ldots, 5$, by [5, pages 80–81], $t = 1$ and $I/M = U/M \cong M_{22}$ and $\phi_i(1)$ divides 2^5. Assume, first, that $e_j = 1$ for some j. Then θ extends to $\varphi_j \in \mathsf{Irr}(I)$. By Lemma 2.1(b), $\tau\varphi_j$ is an irreducible constituent of θ^I for every $\tau \in \mathsf{Irr}(I/M)$, so $\tau(1)\varphi_j(1) = \tau(1)$ divides 2^5. Now we choose $\tau \in \mathsf{Irr}(I/M) = \mathsf{Irr}(M_{22})$ with $\tau(1) = 21$ and find that this degree does not divide 2^5, which is a contradiction. Therefore each $e_i > 1$ and each e_i is the degree of

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a nontrivial proper irreducible projective representation of M_{22}. As $\phi_i(1) = e_i\theta(1) = e_i$, each e_i divides 2^5. But, according to [5, pages 39–41], there is no such projective degree.

(ii) $U/M \cong U_3(5) : 2$ and $t\phi_i(1)$ divides six or eight, for all i.

Let $M \leqslant W \leqslant U$ such that $W/M \cong U_3(5)$. Then $W \trianglelefteq U$. Assume that $W \nleqslant I$. Since $t = |U : I| = |U : WI| \cdot |WI : I|$ and $|WI : I| = |W : WI|$, the index of some maximal subgroup of $W/M \cong U_3(5)$ divides t and so divides six or eight, which is a contradiction, by [5, pages 34–35]. Thus $W \leqslant I \leqslant U$. Let $M \leqslant V \leqslant W$ such that $V/M \cong M_{10}$. Since θ is V-invariant and the Schur multiplier of V/M is trivial, θ extends to $\theta_0 \in \mathrm{Irr}(V)$. By Lemma 2.1(b), $\tau\theta_0$ is an irreducible constituent of θ^V for every $\tau \in \mathrm{Irr}(V/M)$. Choose $\tau \in \mathrm{Irr}(V/M)$ with $\tau(1) = 16$ and let $\gamma = \tau\theta_0 \in \mathrm{Irr}(V \mid \theta)$. If $\chi \in \mathrm{Irr}(I)$ is an irreducible constituent of γ^I, then $\chi(1) \geqslant \gamma(1)$, by Frobenius reciprocity [10, Lemma 5.2], and also $\chi(1)$ divides six or eight, which implies that $16 = \gamma(1) \leqslant \chi(1) \leqslant 8$, which is contradiction.

Case 3. $H_0 = McL$. By Lemma 3.1(b), $U/M \cong U_4(3)$ and $t\phi_i(1)$ divides 35 or 60, for all i. By inspecting the list of maximal subgroups of $U/M \cong U_4(3)$ in [5, pages 52–59], no index of a maximal subgroup of $U_4(3)$ divides 35 or 60, and so $t = 1$. Thus $I/M = U/M \cong U_4(3)$ and $\phi_i(1)$ divides 35 or 60, for all i. First, assume that $e_j = 1$, for some j. Then θ extends to $\varphi_j \in \mathrm{Irr}(I)$. It follows, from Lemma 2.1(b), that $\tau\varphi_j$ is an irreducible constituent of θ^I for every $\tau \in \mathrm{Irr}(I/M)$, so $\tau(1)\varphi_j(1) = \tau(1)$ divides 35 or 60. Now let $\tau \in \mathrm{Irr}(I/M) = \mathrm{Irr}(U_4(3))$ with $\tau(1) = 21$. This degree does not divide 35 or 60, which is a contradiction. Therefore, for each i, $e_i > 1$ and e_i is the degree of a nontrivial proper irreducible projective representation of $U_4(3)$. As $\phi_i(1) = e_i\theta(1) = e_i$, each e_i divides 35 or 60, and it follows, from [5, pages 53–59], that $e_i \in \{6, 15, 20, 35\}$. Let $M \leqslant V \leqslant U$ such that $V/M \cong U_3(3)$. Since θ is V-invariant and the Schur multiplier of V/M is trivial, θ extends to $\theta_0 \in \mathrm{Irr}(V)$. It follows, from Lemma 2.1(b), that $\tau\theta_0$ is an irreducible constituent of θ^V for every $\tau \in \mathrm{Irr}(V/M)$. Take $\tau \in \mathrm{Irr}(V/M)$ with $\tau(1) = 32$ and let $\gamma = \tau\theta_0 \in Irr(V \mid \theta)$. If $\chi \in \mathrm{Irr}(I)$ is an irreducible constituent of γ^I, then $\chi(1) \geq \gamma(1) = 32$, by Frobenius reciprocity [10, Lemma 5.2]. This shows that $e_i = 35$, for all i, that is, $\varphi_i(1)/\theta(1)$ divides 35, and so Lemma 2.7 implies that $I/M \cong U_4(3)$ is solvable, which is a contradiction.

Case 4. $H_0 = He$. By Lemma 3.1(b), $U/M \cong S_4(4) : 2$ and $t = 1$ or, equivalently, $I/M = U/M \cong S_4(4) : 2$. Moreover, $\phi_i(1) = 1$, for all i. Then θ extends to $\phi_i \in \mathrm{Irr}(I)$, and so, by Lemma 2.2(b), $2058\tau(1)$ divides some degrees of G. For $\tau(1) = 510$, this gives a contradiction. Therefore $I_{G'}(\theta) = G'$.

Case 5. $H_0 = Suz$. By Lemma 3.1(b), one of the following holds.

(i) $U/M \cong G_2(4)$ and $t\phi_i(1)$ divides $3^2 \cdot 13$ or $3 \cdot 5 \cdot 7$, for all i.

By inspecting the list of maximal subgroups of $G_2(4)$ in [5, pages 97–99], no index of a maximal subgroup of $G_2(4)$ divides $3^2 \cdot 13$ or $3 \cdot 5 \cdot 7$, so $t = 1$ and $I/M = U/M \cong G_2(4)$. Note that $\phi_i(1)/\theta(1)$ divides $3^2 \cdot 13$ or $3 \cdot 5 \cdot 7$, for all i. If $\phi_i(1)/\theta(1) > 1$,

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for all i, then we apply Lemma 2.7 to conclude that I/M is solvable, which is a contradiction. Therefore, $\varphi_i(1) = \theta(1) = 1$, in which case, θ extends to φ_i, for some i. It follows, from Lemma 2.1(b), that $\tau\varphi_i$ is an irreducible constituent of θ^I for every $\tau \in \mathrm{Irr}(I/M)$, so $\tau(1)\varphi_i(1) = \tau(1)$ divides $3^2 \cdot 13$ or $3 \cdot 5 \cdot 7$. We can choose $\tau \in \mathrm{Irr}(I/M) = \mathrm{Irr}(G_2(4))$ with $\tau(1) = 65$, and this degree does not divide $3^2 \cdot 13$ or $3 \cdot 5 \cdot 7$, which is a contradiction.

(ii) $U/M \cong U_5(2)$ and $t\phi_i(1)$ divides 5, for all i.

As $U/M \cong U_5(2)$ does not have any subgroup of index five, by [5, pages 72–73], $t = 1$ and $I/M = U/M \cong U_5(2)$. Thus $\phi_i(1)/\theta(1)$ divides five, for all i. Since $U_5(2)$ has trivial Schur multiplier, it follows that θ extends to $\theta_0 \in \mathrm{Irr}(I)$ and, by Lemma 2.2(b), $(\theta_0\tau)^{G'} \in \mathrm{Irr}(G')$, for all $\tau \in \mathrm{Irr}(I/M)$. For $\tau(1) = 300 \in \mathrm{cd}(U_5(2))$, we find that $5 \cdot 300 \cdot \theta_0(1) = 2^2 \cdot 3 \cdot 5^3$ divides some degrees of G, which is a contradiction.

Case 6. $H_0 = Fi_{22}$. By Lemma 3.1(b), one of the following holds.

(i) $U/M \cong 2 \cdot U_6(2)$ and $t\varphi_i(1)$ divides one of $3 \cdot 5 \cdot 11$, $2^2 \cdot 3 \cdot 5 \cdot 11$, $2^4 \cdot 5 \cdot 7$ or 3^5, for all i.

As U/M is perfect, the centre of U/M lies in every maximal subgroup of U/M and so the indices of maximal subgroups of U/M and those of $U_6(2)$ are the same. By inspecting the list of maximal subgroups of $U_6(2)$ in [5, pages 115–121], the index of a maximal subgroup of $U_6(2)$ does not divide $3 \cdot 5 \cdot 11$, $2^2 \cdot 3 \cdot 5 \cdot 11$, $2^4 \cdot 5 \cdot 7$ or 3^5. Thus $t = 1$ and hence $I = U$. Let $M \leqslant L \leqslant I$ such that L/M is isomorphic to the centre of I/M and let $\lambda \in \mathrm{Irr}(L \mid \theta)$. As $L \trianglelefteq I$, for any $\varphi \in \mathrm{Irr}(I \mid \lambda)$, $\varphi(1)$ divides $3 \cdot 5 \cdot 11$, $2^2 \cdot 3 \cdot 5 \cdot 11$, $2^4 \cdot 5 \cdot 7$ or 3^5. As above, we deduce that λ is I-invariant. Let $L \leqslant T \leqslant I$ such that $T/L \cong U_5(2)$. It follows that λ is T-invariant and, since the Schur multiplier of $T/L \cong U_5(2)$ is trivial, λ extends to $\lambda_0 \in \mathrm{Irr}(T)$. By Lemma 2.1(b), $\tau\lambda_0$ is an irreducible constituent of λ^T for every $\tau \in \mathrm{Irr}(T/L)$. Choose $\tau \in \mathrm{Irr}(T/L)$ with $\tau(1) = 2^{10}$ and let $\gamma = \tau\lambda_0 \in \mathrm{Irr}(T \mid \lambda)$. If $\chi \in \mathrm{Irr}(I)$ is any irreducible constituent of γ^I, then $\chi(1) \geq \gamma(1)$, by Frobenius reciprocity [10, Lemma 5.2], and $\chi(1)$ divides $3 \cdot 5 \cdot 11$, $2^2 \cdot 3 \cdot 5 \cdot 11$, $2^4 \cdot 5 \cdot 7$ or 3^5, which implies that $\gamma(1)2^{10}\lambda(1) \leqslant \chi(1) \leqslant 660$, which is impossible.

(ii) $U/M \cong O_8^+(2) : S_3$ and $t\varphi_i(1)$ divides six, for all i.

Let $M \trianglelefteq W \trianglelefteq U$ such that $W/M \cong O_8^+(2)$. Then $M \trianglelefteq IW \trianglelefteq I$ and $M \trianglelefteq IW \leqslant W$. Assume that $W \nleqslant I$. Then $I \lneq WI \leqslant U$ and $t = |U : I| = |U : WI| \cdot |WI : I|$. Now $|WI : I| = |W : WI| > 1$ and t is divisible by $|W : W \cap I|$. As $W/M \cong O_8^+(2)$, t is divisible by the index of some maximal subgroup of $O_8^+(2)$. Thus some index of a maximal subgroup of $O_8^+(2)$ divides six, which is impossible, by [5, pages 85–88]. Thus $W \leqslant I \leqslant U$. Write $\theta^W = \sum_{i=1}^{l} f_i\mu_i$, where $\mu_i \in \mathrm{Irr}(W|\theta)$ for $i = 1, 2, \ldots, l$. As $W \trianglelefteq I$, $\mu_i(1)$ divides six for every i. If $f_j = 1$ for some j, then θ extends to $\theta_0 \in \mathrm{Irr}(W)$. By Lemma 2.1(b), $\tau\theta_0$ is an irreducible constituent of θ^W for every $\tau \in \mathrm{Irr}(W/M)$, and so $\tau(1)\theta_0(1) = \tau(1)$ divides six. However we can choose $\tau \in \mathrm{Irr}(W/M)$ with $\tau(1) = 28$ and this degree does not divide six. Therefore $f_i > 1$, for all i. We deduce that, for each i, f_i is the degree of a nontrivial proper irreducible projective representation of $O_8^+(2)$. As $\mu_i(1) = f_i\theta(1) = f_i$,

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each f_i divides six. This is impossible as the smallest nontrivial proper projective degree of $O_8^+(2)$ is eight.

(iii) $U/M \cong 2^{10} : M_{22}$ and $t\varphi_i(1)$ divides six, for all i.

Let $M \trianglelefteq L \trianglelefteq U$ such that $L/M \cong 2^{10}$. We have that $L \trianglelefteq U$ and $U/L \cong M_{22}$. The same argument as in part (ii) shows that $U = IL$ since the minimal index of a maximal subgroup of M_{22} is 22, by [5, pages 39–41]. Hence $U/L \cong I/L_1 \cong M_{22}$, where $L_1 = L \cap I \trianglelefteq I$. Let $\lambda \in \mathsf{Irr}(L_1 \mid \theta)$. Then, for any $\varphi \in \mathsf{Irr}(I \mid \lambda)$, $\varphi(1)$ divides six. We conclude that λ is I-invariant as the index of a maximal subgroup of $I/L_1 \cong M_{22}$ is at least 22. Write $\lambda^I = \sum_{i=1}^{l} f_i\mu_i$, where $\mu_i \in \mathsf{Irr}(I \mid \lambda)$ for $i = 1, 2, \ldots, l$. Then $\mu_i(1)$ divides six, for each i. If $f_j = 1$ for some j, then λ extends to $\lambda_0 \in \mathsf{Irr}(I)$. By Lemma 2.1(b), $\tau\lambda_0$ is an irreducible constituent of λ^I for every $\tau \in \mathsf{Irr}(I/L_1)$, and so $\tau(1)\lambda_0(1) = \tau(1)$ divides six. However we can choose $\tau \in \mathsf{Irr}(I/L_1)$ with $\tau(1) = 21$ and this degree does not divide 6. Therefore $f_i > 1$, for all i. We deduce that, for each i, f_i is the degree of a nontrivial proper irreducible projective representation of M_{22}. As $\mu_i(1) = f_i\lambda(1) = f_i$, each f_i divides 6. However this is impossible as the smallest nontrivial proper projective degree of M_{22} is ten.

Case 7. $H_0 = Fi_{24}'$. By Lemma 3.1(b), $U/M \cong 2.Fi_{23}$ and, for each i, $t\varphi_i(1)$ divides one of the numbers in the set $\mathcal{A}$, defined as

$$\mathcal{A} := \{2^4 \cdot 5^2 \cdot 7 \cdot 17 \cdot 23, 2 \cdot 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 17, 2^2 \cdot 3 \cdot 11 \cdot 13 \cdot 17 \cdot 23,$$
$$2^3 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 23, 2^4 \cdot 3 \cdot 13 \cdot 17 \cdot 23, 2^2 \cdot 7 \cdot 11 \cdot 17 \cdot 23, 11 \cdot 13 \cdot 17 \cdot 23\}.$$

By inspecting the list of maximal subgroups of Fi_{23} in [5, pages 177–180], the index of a maximal subgroup of U/M divides no number in $\mathcal{A}$. Thus $t = 1$ and $I = U$. As the Schur multiplier of $I/M \cong Fi_{23}$ is trivial and θ is I-invariant, we deduce, from [10, Theorem 11.7], that θ extends to $\theta_0 \in \mathsf{Irr}(I)$. By Lemma 2.1(b), $\tau\theta_0$ is an irreducible constituent of θ^I for every $\tau \in \mathsf{Irr}(I/M)$, and so $\tau(1)\theta_0(1) = \tau(1)$ divides one of the numbers in $\mathcal{A}$. Choose $\tau \in \mathsf{Irr}(I/M) = \mathsf{Irr}(Fi_{23})$ with $\tau(1) = 559\,458\,900$. This degree divides none of the numbers in $\mathcal{A}$, which is a contradiction. □

Proposition 3.5. *Let G be a finite group with $\mathsf{cd}(G) = \mathsf{cd}(H)$, where H is an almost simple group whose socle is a sporadic simple group H_0. If G'/M is the chief factor of G, then $M = 1$, and hence $G' \cong H_0$.*

Proof. Here we deal with the groups mentioned in Remark 1.3, namely, $H = \mathsf{Aut}(H_0)$, where H_0 is one of the sporadic groups J_2, J_3, McL, HS, He, HN, Fi_{22}, Fi_{24}', $O'N$ and Suz. It follows, from Proposition 3.3, that G'/M is isomorphic to H_0 and, by Proposition 3.4, every linear character θ of M is G'-invariant. By applying Lemma 2.6, $|M/M'|$ divides the order of Schur Multiplier $\mathsf{M}(H_0)$ (see Table 1). Therefore, G'/M' is isomorphic to either H_0 or one of the groups in the third column of Table 5. In the latter case, we observe, by ATLAS [5], that G'/M' has a degree shown in the fifth column of Table 5 which must divide some degrees of H_0, which is a contradiction. Therefore, $|M/M'| = 1$ or, equivalently, M is perfect.

TABLE 5. Degrees of some groups related to sporadic simple groups H_0 in Proposition 3.5.

H_0	$\mathsf{Aut}(H_0)$	G'/M'	Degree of $\mathsf{Aut}(H_0)$	Degree of G'/M'	Largest degree of H_0
J_2	$J_2:2$	$2\cdot J_2$	28	64	336
HS	$HS:2$	$2\cdot HS$	308	616	3200
J_3	$J_3:2$	$3\cdot J_3$	170	1530	3078
McL	$McL:2$	$3\cdot McL$	1540	1980	10395
He	$He:2$	-	102	-	23324
Suz	$Suz:2$	$2\cdot Suz$, $3\cdot Suz$, $6\cdot Suz$	10010	60060	248832
$O'N$	$O'N:2$	$3\cdot O'N$	51832	63612	234080
Fi_{22}	$Fi_{22}:2$	$2\cdot Fi_{22}$, $3\cdot Fi_{22}$, $6\cdot Fi_{22}$	277200	235872	2729376
HN	$HN:2$	-	266	-	5878125
Fi'_{24}	$Fi'_{24}:2$	$3\cdot Fi'_{24}$	149674800	216154575	336033532800

The symbol '-' means that there is only one possibility for G'/M', which is H_0.

Suppose that M is nonabelian, and let $N \leqslant M$ be a normal subgroup of G' such that M/N is a chief factor of G'. Then $M/N \cong S^k$, for some nonabelian simple group S. It follows, from Lemma 2.4, that S possesses a nontrivial irreducible character φ such that $\varphi^k \in \mathsf{Irr}(M/N)$ extends to G'/N. By Lemma 2.1(b), we must have $\varphi(1)^k\tau(1) \in \mathsf{cd}(G'/N) \subseteq \mathsf{cd}(G')$, for all $\tau \in \mathsf{Irr}(G'/M)$. Now we can choose $\tau \in G'/M$ such that $\tau(1)$ is the largest degree of H_0, as in the last column of Table 5 and, since φ is nontrivial, $\varphi(1)^k\tau(1)$ divides no degrees of G, which is a contradiction. Therefore, M is abelian. Since $M = M'$, we conclude that $M = 1$ and G' is isomorphic to H_0. □

PROPOSITION 3.6. *Let G be a finite group with $\mathsf{cd}(G) = \mathsf{cd}(H)$, where H is an almost simple group whose socle is a sporadic simple group H_0. Then $G/Z(G) \cong H$.*

PROOF. By Remark 1.3, we only consider the case where $H = \mathsf{Aut}(H_0)$ with H_0 one of the sporadic groups J_2, J_3, McL, HS, He, HN, Fi_{22}, Fi'_{24}, $O'N$ and Suz. According to Proposition 3.5, G' is isomorphic to H_0. Let $A := C_G(G')$. Since $G' \cap A = 1$ and $G'A \cong G' \times A$, it follows that $G' \cong G'A/A \trianglelefteq G/A \leqslant \mathsf{Aut}(G')$. Thus G/A is isomorphic to H_0 or $\mathsf{Aut}(H_0) = H_0 : 2$. In the case where G/A is isomorphic to H_0, we must have $G \cong A \times H_0$. This is impossible as G possesses a character with the degree shown in the fourth column of Table 5, but H_0 has no such degree. Therefore, G/A is isomorphic to $\mathsf{Aut}(H_0)$. Note, also, that $G' \cap A = 1$. Thus $[G, A] = 1$ and $A = Z(G)$, as claimed. □

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SEYED HASSAN ALAVI, Department of Mathematics,
Faculty of Science, Bu-Ali Sina University, Hamedan, Iran
e-mail: alavi.s.hassan@gmail.com

ASHRAF DANESHKHAH, Department of Mathematics,
Faculty of Science, Bu-Ali Sina University, Hamedan, Iran
e-mail: daneshkhah.ashraf@gmail.com, adanesh@basu.ac.ir

ALI JAFARI, Department of Mathematics,
Faculty of Science, Bu-Ali Sina University, Hamedan, Iran
e-mail: a.jaefary@gmail.com