EQUATIONS FOR PERIODIC SOLUTIONS OF
A LOGISTIC DIFFERENCE EQUATION

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Abstract

The paper is concerned with periodic solutions of the difference equation $u_{n+1} = 2au_n - bu_n^2$, where $a$ and $b$ are constants, with $a > \frac{1}{2}$ and $b > 0$. A new method is developed for dealing with this problem and, for period lengths up to 6, polynomial equations are given which allow the periodic solutions to be determined in a precise and practical manner. These equations apply whether the periodic solutions are stable or unstable and the elements of the cycle can be determined with an accuracy which is not affected by instability of the cycle.

A simple transformation puts the equation into the form $w_{n+1} = w_n^2 - A$, where $A = a^2 - a$, and the detailed discussion is based on this simpler form. The discussion includes details such as the number of cyclic solutions for a given value of $A$, the pattern of the cycles and their stability. For practical purposes, it is enough to consider a restricted range of values of $A$, namely $-\frac{1}{4} < A < 2$, although the equations obtained are valid for $A > 2$.

1. Introduction

Difference equations of the form

$$u_{n+1} = (2a - bu_n)u_n$$

are well known in population dynamics. They can be regarded as a finite analogue of the logistic differential equation

$$N'(t) = N(t)(C - BN(t)),$$

where $C$ and $B$ are positive constants and $N(t)$ is the population size at time $t$. If we replace $N'(t)$ by $(1/h)(N(t + h) - N(t))$, with $h > 0$, and write $u_n$ for

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Then equation (1.1) is obtained with

\[ 2a = 1 + hC > 1, \quad b = hB > 0. \tag{1.3} \]

In this case \( u_{n+1} - u_n \) has the same sign as \( N' \) for a given population size and, in particular, the equilibrium values are the same. Indeed, equation (1.1) is the Cauchy–Euler approximation to (1.2) and the solutions behave in the same way for \( h \) small (Hurewicz [4]). However, the solutions of the difference equation show a much richer variety of possible behaviour for larger values of \( h \). In a review article [8], May has drawn attention to these possibilities and suggested that the difference equation provides a more appropriate model than the differential equation in a number of practical applications. In particular, the difference equation can have solutions which oscillate above and below the equilibrium level \( N^* = C/B \), whereas solutions of equation (1.2) approach \( N^* \) from one side only.

A change of scale, with \( v_n = bu_n \), is enough to eliminate \( b \) from the difference equation. The solutions then depend on the value of \( a \) and the variety of solutions available becomes greater as \( a \) increases. Periodic solutions can arise and a good deal of information is available about the existence and stability of these periodic solutions for different values of \( a \) (see May [8], Hoppensteadt and Hyman [3]). For longer periods, much of this information comes from computational studies. In the present paper, algebraic equations are given which allow the periodic solutions to be determined for periods up to 6. These equations are valid whether the solutions are stable or unstable, although information about the stability comes out as a by-product. For periods 5 and 6, the solutions are only stable within narrow intervals of the parameter \( a \) and outside these intervals iteration of equation (1.1) can give an apparent periodicity when the initial value is not an element of a periodic solution or an apparent drift away from periodicity when the initial value is a good approximation to an element of a periodic solution.

Section 2 introduces a preliminary transformation which simplifies the discussion. Some known results are cited and the general problem is outlined. Section 3 considers solutions of period 5 in detail, to illustrate the approach that is used. Each periodic solution is identified by the sum of its elements, \( \alpha \), and, for a given value of \( a \), \( \alpha \) is determined from a polynomial equation of degree 6. The elements of the periodic solution are then found as the zeros of a polynomial of degree 5.

Corresponding equations for solutions with periods 2, 3, 4 and 6 are summarized in Section 4. In the most complicated case, for solutions with period 6, appropriate values of \( \alpha \) come from a polynomial equation of degree 9 and the elements of the periodic solution are the roots of an equation of degree 6.
The results for solutions with periods 3 and 4 are discussed in Section 5 and Section 6 gives a similar discussion for solutions of periods 5 and 6. The results are summarized in Table 1.

2. Preliminary transformation and notation

Starting from equation (1.1), with $2a > 1$ and $b > 0$, we can obtain a simpler form by using $w_n = a - bu_n$, which gives

$$w_{n+1} = w_n^2 - A = F(w_n),$$

where $A = a^2 - a$. This is essentially the form used by Chaundy and Phillips [1], who considered the convergence of the sequence $\{w_n\}$. They showed that, as $n \to \infty$,

(i) for $|w_0| > a$, $w_n \to \infty$ monotonically,
(ii) for $|w_0| < a$ and $\frac{1}{2} < a < \frac{3}{2}$, $w_n \to 1 - a$,
(iii) for $|w_0| < a$ and $\frac{3}{2} < a < 2$, $w_n$ oscillates finitely, with $|w_n| < a$, and
(iv) for $|w_0| < a$ and $a > 2$, $w_n \to \infty$ in general.

The exceptional cases in (iv) are the values of $w_0$ which lead to the equilibrium values ($w_n = a$ and $w_n = 1 - a$) or to a periodic solution. Note also that $w_0 = \pm a$ gives $w_n = a$ for $n > 1$. Thus, in looking for periodic solutions of practical interest, we can impose the restriction that $|w_0| < a$ and consider only sequences with $\frac{3}{2} < a < 2$, that is $0.75 < A < 2$. (Periodic solutions certainly exist for $a > 2$ but result (iv) indicates that they are unstable.)

For $A = 2$, Lorenz [7] gives an exact solution and this is of considerable help in checking results. If we write $w_0 = 2 \cos \phi$, then $w_n = 2 \cos(2^n \phi)$ and the condition for $w_n$ to equal $w_0$ becomes

$$2^n \pm 1)\phi = 2N\pi,$$  

for any integer $N$. It should be noted that this condition gives not only the solutions with basic period $n$ but also those with basic period $n/p$, where $p$ is a factor of $n$. (When $p = n$ we get the equilibrium values.) These exact solutions will be referred to later as the trigonometric solutions.

For $a > \frac{1}{2}$, the relationship $A = a^2 - a$ can be inverted to give

$$a = \left\{ \frac{1}{2} \right\} \{1 + \sqrt{(1 + 4A)}\},$$

which means that we can treat $A$ as the relevant parameter in discussing equation (2.1) and occasionally refer back to $a$ when comparing results with those found by other authors. Also we can write, for $m = 2, 3, \ldots$,

$$w_{n+m} = F_m(w_n) = F \{ F_{m-1}(w_n) \},$$  

(2.3)
where $F_1 = F$. Thus $F_m$ is the $m$th iterate of equation (2.1) and $F_m(w_n)$ will be a polynomial of degree $2^m$ in $w_n$. Setting $w_{n+m} = w_n$ gives a polynomial equation of degree $2^m$, namely

$$G_m(w_n) = F_m(w_n) - w_n = 0.$$  

(2.4)

Since $F'(w_n) = 2w_n$, we can write

$$F'_m = 2w_{m+n-1}F'_{m-1} = 2^m w_{m+n-1} w_{m+n-2} \cdots \cdot w_n.$$  

(2.5)

The equation $G_m(w_n) = 0$ must hold for a solution with period $m$ although for $m > 1$ the degree of the equation can be reduced slightly. When $m$ is a prime number, the degree of the equation can be reduced by 2, since $G_m(w_n)$ must include factors $w_n - a$ and $w_n - (1 - a)$, which correspond to the equilibrium solutions. The product of these two factors, $w_n^2 - w_n - A$, can be removed from $G_m$, leaving an equation of degree $2^m - 2$. When $m$ is a composite number, with factor $p$, then $G_p(w_n)$ must be a factor of $G_m(w_n)$ and the degree of equation (2.4) can be reduced by $2^p$, with the possibility of a further reduction from the cofactor of $p$. Despite these reductions, the dominant effect is that the degree of the equation increases exponentially with the length of the period and a direct solution soon becomes impracticable.

3. Equations for solutions of period 5

For convenience, we shall denote a solution with basic period 5 as a $C5$ solution, with a similar notation for solutions with other basic periods. Also we write $b_1, b_2, b_3, b_4$ and $b_5$ for the elements of the solution, with

$$h_5(x) = \prod_{i=1}^{5} (x - b_i) = x^5 - ax^4 + bx^3 - \gamma x^2 + \delta x - e.$$  

(3.1)

This equation defines $\alpha, \beta, \gamma, \delta$ and $e$ as symmetric functions of the $b$'s, with full permutational symmetry. It turns out that $\beta, \gamma, \delta$ and $e$ can be expressed in terms of $\alpha$, and so a knowledge of $\alpha$ defines $h_5(x)$. (Since it does not matter which element of the $C5$ solution we take as $b_1$, it is appropriate to identify $h_5(x)$ by a symmetric function of the elements, such as $\alpha$. The values of $b_i$ can be determined by solving $h_5(x) = 0$ once $\alpha$ is known.)

If we write $G_5(x) = (x^2 - x - A)H_5(x)$, then $H_5$ is a polynomial in $x$ of degree 30 and $h_5$ must be a factor of $H_5$. At most we can expect six $C5$ solutions for a given value of $A$, corresponding to a factorization of $H_5$ into fifth-degree polynomials of the form $h_5(x)$. To avoid carrying out this factorization explicitly, we derive a sixth-degree equation for $\alpha$ and look for real roots of this equation. A fair amount of algebra is required to derive the equation for $\alpha$ but solving it
for a given value of $A$ is straightforward. For each real $\alpha$ we can then calculate $\beta, \gamma, \delta$ and $\epsilon$ and solve $h_s(x) = 0$.

The condition for local stability in the general case is that $|F''_m(w_n)| < 1$ and for our $C5$ solution this becomes

$$-\frac{1}{32} < \epsilon < \frac{1}{32}, \quad (3.2)$$

using equation (2.5) and noting that $\epsilon = b_1b_2b_3b_4b_5$. Thus the stability of the solution is determined by the value of $\epsilon$.

The basic equations for the elements $b_i$ are

$$b_{i+1} = b_i^2 - A, \quad i = 1, 2, 3, 4, 5, \quad (3.3)$$

where $b_6 = b_1$, and we want to use these equations to express $\beta, \gamma, \delta$ and $\epsilon$ in terms of $\alpha$ and then to form an equation for $\alpha$. From the theory of equations (Uspensky [10] or other books on the topic) any symmetrical function of the $b_i$ can be expressed in terms of $\alpha, \beta, \gamma, \delta$ and $\epsilon$ and if we can relate two symmetrical functions by means of equations (3.3) we get an equation relating $\alpha, \beta, \gamma, \delta$ and $\epsilon$. For example, taking summations over $i = 1$ to 5 and using equation (3.3)

$$\sum b_i^2 = \sum (A + b_{i+1}) = 5A + \alpha, \quad (3.4)$$

$$\sum b_i^4 = \sum (A + b_{i+1})^2 = 5A^2 + 2A\alpha + (5A + \alpha). \quad (3.5)$$

From the theory of equations

$$\sum b_i^2 = \alpha^2 - 2\beta \quad \text{and} \quad \sum b_i^4 = \alpha^4 - 4\alpha^2\beta + 2\beta^2 + 4\alpha\gamma - 4\delta, \quad (3.6)$$

and we get

$$\alpha^2 - 2\beta = \alpha + 5A$$

and

$$\alpha^4 - 4\alpha^2\beta + 2\beta^2 + 4\alpha\gamma - 4\delta = \alpha + 5A + 2\alpha\alpha + 5A^2. \quad (3.7)$$

These examples typify the procedure, that is $b_i^2$ is replaced by $b_{i+1} + A$ wherever it occurs and this links a symmetric function of higher degree with a function of lower degree.

One snag which arises is that the function of lower degree may not have full permutational symmetry, since equations (3.3) have cyclic symmetry rather than permutational symmetry. To deal with this, it is convenient to introduce functions with cyclic symmetry in the $b$'s. If we use $\sum_0$ to denote cyclic summation (over 1 to 5 in this case), we can write

$$\beta_1 = \sum_0 b_1b_2 = b_1b_2 + b_2b_3 + b_3b_4 + b_4b_5 + b_5b_1, \quad (3.8)$$

$$\beta_2 = \sum_0 b_1b_3 = \sum_0 b_1b_4 = \beta - \beta_1. \quad (3.9)$$
These functions arise naturally in equations such as
\[
\sum b_i^3 = \Sigma_0 b_i(A + b_2) = A\alpha + \beta_1, \tag{3.10}
\]
\[
\sum b_i^5 = \Sigma_0 b_i(A + b_2)^2 = \Sigma_0 b_i(A^2 + 2Ab_2 + A + b_3)
= A^2\alpha + 2A\beta_1 + A\alpha + \beta_2. \tag{3.11}
\]
Combining these with standard relationships from the theory of equations gives
\[
\alpha^3 - 3\alpha\beta + 3\gamma = \Sigma_0 b_i^3 = \beta_1 + \alpha A, \tag{3.12}
\]
\[
\alpha^5 - 5\alpha^3\beta + 5\alpha^2\gamma - 5\beta\gamma + 5\alpha\beta^2 - 5\alpha\delta + 5\epsilon
= \Sigma_0 b_i^5 = \beta_2 + A\alpha + 2A\beta_1 + A^2\alpha. \tag{3.13}
\]
Equations (3.7), (3.9), (3.12) and (3.13) suffice to express \(\beta, \gamma, \delta\) and \(\epsilon\) in terms of \(\alpha\) and \(\beta_1\). The detailed expressions are
\[
2\beta = \alpha^2 - \alpha - 5A, \quad 6\gamma = \alpha^3 - 3\alpha^2 - 13\alpha A + 2\beta_1, \tag{3.14}
\]
\[
24\delta = \alpha^4 - 6\alpha^3 + \alpha^2(3 - 22A) + \alpha(18A - 6) + 45A^2 - 30A + 8\alpha\beta_1, \tag{3.15}
\]
\[
120\epsilon = \alpha^5 - 10\alpha^4 + \alpha^3(15 - 30A) + \alpha^2(70A - 18) + \alpha(149A^2 - 126A - 12)
- 60A + \beta_1(20\alpha^2 - 20\alpha - 24 - 52A). \tag{3.16}
\]
At this stage we need two more equations if we are to solve for \(\beta_1\) and \(\alpha\). From equation (3.3),
\[
\Pi_0(b_3 - b_2) = \Pi_0(b_2^2 - b_2),
\]
where \(\Pi_0\) is used as a cyclic product sign, and if we take the \(b\)’s as distinct this gives
\[
1 = \Pi_0(b_1 + b_2). \tag{3.17}
\]
Similarly,
\[
1 = \Pi_0(b_1 + b_3). \tag{3.18}
\]
This looks promising but in fact both these equations lead to the same relationship
\[
0 = (5A^2 + 5A - 1) + \alpha(1 + 4A + 2A^2) + \beta(1 + 3A)
+ \gamma(1 + 2A) + \delta + 2\epsilon. \tag{3.19}
\]
An equivalent form is
\[
48\epsilon = -\alpha^4 + (2 - 8A)\alpha^3 + (10A - 3)\alpha^2 + (56A^2 - 26A - 6)\alpha
+ (24 - 30A + 15A^2) - 8\beta_1(\alpha + 1 + 2A). \tag{3.20}
\]
An additional equation can be obtained in different ways, one being to use the symmetrical function
\[
S = \alpha\delta - 5\epsilon + (\alpha^2 - 2\beta)\gamma = \sum (b_i^3b_jb_k + 2b_i^3b_jb_kb_m). \tag{3.21}
\]
where the summation is over 1 to 5 for \( i, j, k \) and \( m \), with \( i \neq j \neq k \neq m \). Using equations (3.3),
\[
S = \gamma(2 + 7A) + 5\delta + \beta(1 + 4A) + 2\alpha + 5\alpha A + 10A + \beta_1(1 + A),
\]
and equating the two expressions for \( S \) leads to
\[
120\epsilon = \alpha^5 - 7\alpha^4 + (13 - 30A)\alpha^3 + (52A - 9)\alpha^2
\]
\[
+ (149A^2 - 88A - 6)\alpha + 15A^2 - 30A
\]
\[
+ \beta_1(8\alpha^2 - 32\alpha - 40 - 40A). \tag{3.23}
\]
Equations (3.16), (3.20) and (3.23) can be used to eliminate \( \epsilon \) and \( \beta_1 \). This gives
\[
0 = (2\alpha - 5)K(\alpha, A), \tag{3.24}
\]
where
\[
K(\alpha, A) \equiv \alpha^6 + \alpha^5 + (3 - 11A)\alpha^4 + (11 - 18A)\alpha^3 + (44 - 19A + 19A^2)\alpha^2
\]
\[
+ (36 + 24A + 17A^2)\alpha + (32 - 28A + 40A^2 - 9A^3). \tag{3.25}
\]
The solution \( \alpha = \frac{5}{2} \) in equation (3.24) arises because equations (3.17) and (3.18) are satisfied when \( b_1 = b_2 = b_3 = b_4 = b_5 = \frac{1}{2} \) and these elements also provide an equilibrium solution of equations (3.3) in the limiting case as \( a \to \frac{1}{2} \). Thus the effective equation for \( \alpha \) is \( K(\alpha, A) = 0 \). For a given value of \( A \) we can look for real roots of this equation and for each real root we evaluate \( \beta, \beta_1, \gamma, \delta \) and \( \epsilon \) and obtain the elements of the \( C5 \) solution, if there is one, from \( h_5(x) = 0 \). In solving for \( \beta_1 \), the equation used in practice was
\[
\beta_1(40\alpha^2 - 8 - 24A) = -2\alpha^5 + 15\alpha^4 + (20A - 20)\alpha^3 + (21 - 90A)\alpha^2
\]
\[
- (18A^2 - 122A + 6)\alpha + (120 - 30A + 75A^2), \tag{3.26}
\]
obtained by eliminating \( \epsilon \) between equations (3.16) and (3.20).

4. Equations for periods 2, 3, 4 and 6

For \( C2 \) solutions, the results are well known (May [8], Levin and May [7]) and require little comment. In our notation, the basic equations are, for elements \( b_1 \) and \( b_2 \),
\[
b_2 = b_1^2 - A \quad \text{and} \quad b_1 = b_2^2 - A. \tag{4.1}
\]
Subtracting one from the other gives \( \alpha = b_1 + b_2 = -1 \), assuming \( b_1 \neq b_2 \), while adding them gives
\[
\beta = b_1b_2 = \left(\frac{1}{2}\right)(\alpha^2 - \alpha - 2A) = 1 - A. \tag{4.2}
\]
Thus \( b_1 \) and \( b_2 \) are solutions of the equation
\[
0 = x^2 - \alpha x + \beta = x^2 + x + 1 - A
\] (4.3)
and the condition for real, distinct roots is \( A > \frac{3}{4} \). From equation (4.2), the solutions are locally stable if \(-1 < 4(1 - A) < 1\), that is, for \( \frac{3}{4} < A < \frac{5}{4} \). Levin and May [7] note that in this case local stability implies global stability for \( |w_0| < a \).

For solutions with period 3, we can write \( G_3(x) = (x^2 - x - A)H_3(x) \), where \( H_3 \) is a polynomial of degree 6. A typical \( C_3 \) solution will give a factor
\[
h_3(x) = (x - b_1)(x - b_2)(x - b_3) = x^3 - \alpha x^2 + \beta x - \gamma
\] (4.4)
and at most we can have two factors of this type in \( H_3 \). Detailed analysis along the same lines as in Section 3 gives
\[
2\beta = \alpha^2 - \alpha - 3A \quad \text{and} \quad 6\gamma = \alpha^3 - 2\alpha^2 - \alpha(1 + 7A) - 3A,
\] (4.5)
with \( \alpha \) satisfying the quadratic equation
\[
\alpha^2 + \alpha + 2 - A = 0.
\] (4.6)
For \( A < \frac{7}{4} \) this equation does not have real roots but for \( A > \frac{7}{4} \) there are real roots and corresponding \( C_3 \) solutions, obtained by solving \( h_3(x) = 0 \). The \( b \)'s satisfy a relationship
\[
(b_1 + b_2)(b_2 + b_3)(b_3 + b_4) = 1,
\] (4.7)
which corresponds to equations (3.17) and (3.18) of the \( C_5 \) case. The condition for local stability for the \( C_3 \) solutions is \( |\gamma| < \frac{1}{5} \).

For solutions with period 4, we can write \( G_4(x) = G_2(x)H_4(x) \), where \( H_4 \) is a polynomial of degree 12. A typical \( C_4 \) solution gives a factor
\[
h_4(x) = (x - b_1)(x - b_2)(x - b_3)(x - b_4) = x^4 - \alpha x^3 + \beta x^2 - \gamma x + \delta
\] (4.8)
and \( H_4 \) can at most have three factors of this type. Thus we can expect \( \alpha \) to satisfy a cubic equation and the analysis confirms this. For our typical \( C_4 \) solution, there are two equations similar to (4.7). They are
\[
(b_1 + b_3)(b_2 + b_4) = -1
\] (4.9)
and
\[
(b_1 + b_2)(b_2 + b_3)(b_3 + b_4)(b_4 + b_1) = 1.
\] (4.10)
Equation (4.9) gives
\[
\beta_1 = \sum_0 b_1b_2 = b_1b_2 + b_2b_3 + b_3b_4 + b_4b_1 = -1,
\] (4.11)
so, although the distinction between cyclic symmetry and permutational symmetry begins to appear at this stage, it causes no difficulty. As in Section 3, we can

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use the basic equations for the $C_4$ solution to obtain

$$2\beta = \alpha^2 - \alpha - 4A, \quad 6\gamma = \alpha^3 - 3\alpha^2 - 10\alpha A - 2,$$

and

$$24\delta = \alpha^4 - 6\alpha^3 + \alpha^2(3 - 16A) + \alpha(12A - 14) + 24(A^2 - A),$$

giving $\beta$, $\gamma$ and $\delta$ in terms of $\alpha$. The equation for $\alpha$ is $J(\alpha, A) = 0$, with

$$J(\alpha, A) = \alpha^3 + (3 - 4A)\alpha + 4.$$  

This equation has at least one real solution for any value of $A$ but, as in the case of $C_2$ solutions, the corresponding equation $h_A(x) = 0$ does not always have real solutions. (A simple example of this is $A = 0$, $\alpha = -1$ and $h_A(x) = x^4 + x^3 + x^2 + x + 1$.) In fact, real $C_4$ solutions occur only for $A > \frac{5}{4}$. In the border-line case, $A = \frac{5}{4}$, equation (4.14) has a single real root, $\alpha = -2$, and the equation $h_A(x) = 0$ gives a $C_2$ solution which is at the limit for local stability, a result which agrees with the folk-lore for the subject. For the $C_4$ solutions, the condition for local stability is $|\delta| < \frac{1}{16}$.

For solutions with period 6 the algebra becomes heavier but the procedure follows the same lines. We can write

$$G_6(x) = (x^2 - x - A)(x^2 + x + 1 - A)H_3(x)H_6(x),$$

where $H_6$ is a polynomial of degree 54. The first three factors cover, respectively, the equilibrium solutions, the $C_2$ solutions and the $C_3$ solutions, which are included as degenerate cases in the equation $w_{n+6} = w_n$. The $C_6$ solutions come from the remaining factor, that is from $H_6(x) = 0$. As before, a $C_6$ solution with elements $b_1$ to $b_6$ will be represented in $H_6$ by a factor

$$h_6(x) = \prod_{i=1}^{6} (x - b_i) = x^6 - \alpha x^5 + \beta x^4 - \gamma x^3 + \delta x^2 - \varepsilon x + \theta,$$  

and at most there can be nine factors of this type. In agreement with this it turns out that $\alpha$ satisfies the equation $L(\alpha, A) = 0$, where

$$L(\alpha, A) \equiv \alpha^9 - \alpha^8 + (2 - 24A)\alpha^7 + (14 - 8A)\alpha^6$$

$$+ (49 - 16A + 144A^2)\alpha^5 + (175 - 16A + 112A^2)\alpha^4$$

$$+ (140 + 136A + 160A^2 - 256A^3)\alpha^3$$

$$+ (196 - 552A + 480A^2 - 256A^3)\alpha^2$$

$$+ (448 - 416A - 304A^2 + 256A^3)\alpha + (384A - 592A^2 + 256A^3).$$  

(4.17)

Also, $\beta$, $\gamma$, $\delta$, $\varepsilon$ and $\theta$ can be expressed in terms of $\alpha$ and $\beta_1$, where

$$\beta_1 = \sum_0 b_1 b_2 = b_1 b_2 + b_2 b_3 + b_3 b_4 + b_4 b_5 + b_5 b_6 + b_6 b_1.$$  

(4.18)
The relevant equations are, for \( \alpha + 1 \neq 0 \),

\[
2\beta = \alpha^2 - \alpha - 6A, \quad 6\gamma = \alpha^3 - 3\alpha^2 - 16A\alpha + 2\beta_1, \tag{4.19}
\]
\[
24\delta = \alpha^4 - 6\alpha^3 + (3 - 28A)\alpha^2 + (24A - 6)\alpha + 72A^2 - 36A + 8\alpha\beta_1, \tag{4.20}
\]
\[
120e = \alpha^5 - 10\alpha^4 + (15 - 40A)\alpha^3 + (100A - 22)\alpha^2
+ (264A^2 - 156A - 8)\alpha + (8 - 32A)
+ (20\alpha^2 - 20\alpha - 24 - 72A)\beta_1 + 16\{(\beta_1 + 2A - 2)/(1 + \alpha)\},
\]
\[
(4.21)
\]
\[
720\theta = \alpha^6 - 11\alpha^5 + (35 - 50A)\alpha^4 + (200A - 37)\alpha^3
+ (544A^2 - 326A + 64)\alpha^2 - (264A^2 - 424A + 4)\alpha
- (80 + 136A - 1080A^2 + 720A^3) - 112\{(\beta_1 + 2A - 2)/(1 + \alpha)\}
+ \beta_1\{20\alpha^3 - 100\alpha^2 - (104 + 272A)\alpha - 80 - 48A\}. \tag{4.22}
\]

The cyclic summation \( \beta_1 \) can be written as a rational function of \( \alpha \) in two ways and the equation for \( \alpha \) comes from equating these two expressions for \( \beta_1 \). More precisely, for \( \alpha \neq 3 \),

\[
0 = \beta_1M_1(\alpha) + M_2(\alpha) \quad \text{and} \quad 0 = \beta_1N_1(\alpha) + N_2(\alpha), \tag{4.23}
\]

where

\[
M_1 = 5\alpha^3 + 10\alpha^2 + (9 - 8A)\alpha + 8 - 8A, \tag{4.24}
\]
\[
M_2 = \left(\frac{1}{4}\right)(\alpha^6 - 4\alpha^5 - 2\alpha^3 + 15\alpha^2 + 150\alpha + 96)
- A(5\alpha^4 + 24\alpha^2 + 52\alpha + 15) + 16A^2(\alpha^2 + \alpha), \tag{4.25}
\]
\[
N_1 = 5\alpha^5 + (35 - 60A)\alpha^4 + (49 - 148A)\alpha^3 + (87 - 376A + 240A^2)\alpha^2
+ (208 - 548A + 320A^2)\alpha + (80 - 180A + 80A^2), \tag{4.26}
\]

and

\[
N_2 = \left(\frac{1}{4}\right)\alpha^8 - 2\alpha^7 - \alpha^6 + (30A - 3)\alpha^5
+ \{36A^2 + 36A - (127/4)\}\alpha^4 + (312A^2 - 126A - 50)\alpha^3
- \{160A^3 - 856A^2 + 482A + \left(\frac{87}{2}\right)\}\alpha^2
+ (99 - 1060A + 1560A^2 - 640A^3)\alpha
+ (240 - 990A + 1140A^2 - 480A^3). \tag{4.27}
\]

Eliminating \( \beta_1 \) from equations (4.23) gives

\[
0 = 4(M_1N_2 - M_2N_1) = 15(\alpha + 1)(4A - 3)L(\alpha, A). \tag{4.28}
\]

Thus \( A = \frac{3}{4}, \alpha = -1 \) and \( \alpha = 3 \) have to be examined separately, but it turns out that the equation \( L(\alpha, A) = 0 \) covers these special cases.
With the equations above, the C6 solutions can be explored systematically. For a given value of $A$, the real roots of $L(\alpha, A) = 0$ can be found to any desired accuracy and for each real root equations (4.19) to (4.27) can be used to evaluate $\beta_1, \beta, \gamma, \delta, \epsilon$ and $\theta$. The elements of the C6 solution then come from solving $h_6(x) = 0$. The criterion for stability is that $|\theta| < \frac{1}{4}$.

For the C6 solutions, the equations corresponding to (3.17) and (3.18) are

$$(b_1 + b_4)(b_2 + b_5)(b_3 + b_6) = -1,$$  \hspace{1cm} (4.29)

$$1 = \prod_0(b_1 + b_2) \quad \text{and} \quad 1 = \prod_0(b_1 + b_3),$$  \hspace{1cm} (4.30)

where the cyclic product $\prod_0$ is now taken over 1 to 6. Equations (4.23) for $\beta_1$ come essentially from equations (4.30).

5. Discussion of $C3$ and $C4$ solutions

We can take the results for the C2 solutions as sufficiently well known not to require further discussion and look briefly at the C3 and C4 solutions, which illustrate some of the ideas that occur in the C5 and C6 solutions. For the C3 solutions, the equation for $\alpha$, equation (4.6), has no real solutions for $A < \frac{7}{4}$ but, for $A > \frac{7}{4}$, the equation has real, distinct roots, say $\alpha_1$ and $\alpha_2$, with $\alpha_1 < -\frac{1}{2} < \alpha_2$. For the critical value $A = \frac{7}{4}$, the equation has a double root $\alpha = -\frac{1}{2}$. It can be shown that there is a C3 solution corresponding to each real value of $\alpha$. Thus we get C3 solutions for $A > A^* = 1.75$ and we can refer to $A^*$ as the critical value for $A$, with $\alpha^*$ as the corresponding double root value for $\alpha$. This notation will be used in other cases where a similar situation arises.

For $A = \frac{7}{4}$ and $\alpha = -\frac{1}{2}, \gamma = -\frac{1}{8}$ and the corresponding C3 solution represents a limiting case for local stability. For $A > \frac{7}{4}, \alpha_1$ decreases and $\alpha_2$ increases as $A$ increases. Now, from equations (4.5) and (4.6), we can write

$$\gamma = -(\alpha^3 + 2\alpha^2 + 3\alpha + 1),$$

which gives $d\gamma/d\alpha = -(3\alpha^2 + 4\alpha + 3) < 0$. For the solution corresponding to the $\alpha_1$ roots, $\gamma$ increases with $A$ and these solutions are unstable. On the other hand, $\gamma$ decreases with $A$ for the $\alpha_2$ family of solutions and there is a range of $A$ for which $-\frac{1}{3} < \gamma < \frac{1}{8}$ and the solutions are locally stable. It is easy to check that the equation $\gamma = -(\alpha^3 + 2\alpha^2 + 3\alpha + 1) = -\frac{1}{8}$ gives a unique real value of $\alpha$ and a corresponding value for $A$, say $A^{**}$, with local stability for $A^* < A < A^{**}$.

This type of behaviour occurs for C4, C5 and C6 solutions also, in the sense that real solutions for $\alpha$ mostly occur in pairs for $A > A^*$, where $A^*$ is a critical value of $A$. At $A = A^*$ there is a double root, $\alpha = \alpha^*$, and the corresponding
solution is on the limit of stability. As \( A \) increases from \( A^* \), one family of \( \alpha \) values gives unstable solutions while the other family gives local stability for \( A^* < A < A^{**} \), where \( A^{**} \) is an upper critical value.

For \( C4 \) solutions, \( \alpha \) is obtained from \( J(\alpha, A) = 0 \), with \( J(\alpha, A) \) as defined in equation (4.14). The equation has a negative root for all values of \( A \) but these negative roots do not lead to real solutions for \( b_1, b_2, b_3 \) and \( b_4 \) as long as \( A < \frac{3}{4} \). This can be proved algebraically. For \( A = \frac{3}{4}, \alpha = -2 \) and the equation \( h_4(x) = 0 \) gives \( b_1 = b_3 = \frac{1}{2}(-1 + \sqrt{2}) \) and \( b_2 = b_4 = \frac{1}{2}(-1 - \sqrt{2}) \), that is, a \( C2 \) solution with \( b_1b_2 = -\frac{1}{4} \). For \( A > \frac{3}{4}, \) the negative root for \( \alpha \) gives a real \( C4 \) solution which is locally stable for a small range of values of \( A \). In Table 1, where the results are summarized, the negative roots are denoted by \( \alpha_1 \) and the critical value \( A^* \) is given as 1.25, in the sense that the values of \( \alpha_1 \) lead to real \( C4 \) solutions for \( A > A^* \).

Positive roots for \( \alpha \) appear for \( A > A^* = \frac{3}{4}(1 + (2)^{2/3}) \), with a double root for \( A = A^* \) and distinct roots \( \alpha_2 \) and \( \alpha_3 \) for \( A > A^* \). It can be shown that each positive root leads to a real \( C4 \) solution. Taking \( \alpha_2 < \alpha_3 \), the table indicates that the \( \alpha_3 \) values lead to locally stable solutions for \( A^* < A < A^{**} \), while the \( \alpha_2 \) values lead to unstable solutions.

The last column of Table 1 shows values of the \( b_i \) for a typical solution and it will be observed that the solution corresponding to negative values of \( \alpha \) is similar to a \( C2 \) solution in its behaviour. It might be described as a split-level \( C2 \) solution. The solutions corresponding to positive values of \( \alpha \) show a decrease from \( b_1 \) to \( b_4 \), with a sharp increase from \( b_4 \) to \( b_1 \) at the beginning of the next cycle. In terms of the population problem, this represents a gradual increase in the population over three seasons (or generations) with a catastrophic drop at the end of that time, a pattern reminiscent of some of the "plague" outbreaks described by Elton [2].

The critical values for the \( C3 \) solutions agree with those given by May [8, Table 3]. For the \( C4 \) solutions, May's table could be misleading since he records only the critical values corresponding to the \( \alpha_3 \) family of solutions. (It should be noted that May's parameter \( a \) corresponds to twice the parameter \( a \) as defined in this paper.)

6. Discussion of \( C5 \) and \( C6 \) solutions

Table 1 shows similar results for \( C5 \) and \( C6 \) solutions. The roots are arranged in order of increasing magnitude, with \( \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha_5 < \alpha_6 \) when six real, distinct roots occur in the \( C5 \) case. In particular, all six values of \( \alpha \) were determined for \( A = 2 \) and found to agree with the trigonometric solutions.
**Table 1**

Critical values for $C_3$, $C_4$, $C_5$ and $C_6$ solutions

<table>
<thead>
<tr>
<th>Roots</th>
<th>Critical values</th>
<th>Stable limit</th>
<th>Typical periodic solution $b_1, b_2, \ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$A^*$</td>
<td>$a^*$</td>
<td>$A^{**}$</td>
</tr>
<tr>
<td>$C_3$</td>
<td>$\alpha_1, \alpha_2$</td>
<td>1.75</td>
<td>-0.50</td>
</tr>
<tr>
<td></td>
<td>$\alpha_1$</td>
<td>1.25</td>
<td>-2.00</td>
</tr>
<tr>
<td></td>
<td>$\alpha_2, \alpha_3$</td>
<td>1.940551</td>
<td>1.259921</td>
</tr>
<tr>
<td>$C_4$</td>
<td>$\alpha_1$</td>
<td>1.624395</td>
<td>-2.501045</td>
</tr>
<tr>
<td></td>
<td>$\alpha_2, \alpha_3$</td>
<td>1.860587</td>
<td>-0.914998</td>
</tr>
<tr>
<td></td>
<td>$\alpha_4, \alpha_5$</td>
<td>1.985410</td>
<td>+3.217952</td>
</tr>
<tr>
<td>$C_5$</td>
<td>$\alpha_1, \alpha_2$</td>
<td>1.474695</td>
<td>-3.462728</td>
</tr>
<tr>
<td></td>
<td>$\alpha_3, \alpha_4$</td>
<td>1.907251</td>
<td>-1.200622</td>
</tr>
<tr>
<td></td>
<td>$\alpha_5$</td>
<td>1.768529</td>
<td>-0.727756</td>
</tr>
<tr>
<td></td>
<td>$\alpha_6, \alpha_7$</td>
<td>1.966764</td>
<td>+0.932874</td>
</tr>
<tr>
<td></td>
<td>$\alpha_8, \alpha_9$</td>
<td>1.996375</td>
<td>+5.211297</td>
</tr>
</tbody>
</table>
mentioned in Section 2. The corresponding periodic solutions also agreed with
the trigonometric solutions. A similar check was made for the C6 solutions for
\( A = 2 \).

For the C5 solutions, a computer programme was used to evaluate \( K(\alpha, A) \)
for different values of \( A \) and it was evident that the real roots occurred in pairs
and led to real C5 solutions, except in one special case. The special case arose
for \( A = \frac{4}{3} \) and \( \alpha = -1 \), which gives a minimum value of \( K(\alpha, A) \), with \( K = 0 \) at
the minimum. For neighbouring values of \( A \) and \( \alpha \), \( K \) is positive and a slight
increase in \( A \) does not lead to two real, distinct solutions for \( \alpha \). Equation (3.26)
for \( \beta_1 \) gives an indeterminate form at \( A = \frac{4}{3} \) and \( \alpha = -1 \) but it can be shown
that

\[
\beta_1^2 = (2 + 2A)\beta + 4\alpha A + 5A^2 + 2\delta - \beta_1(1 + 2A),
\]

(6.1)

which leads to a quadratic equation for \( \beta_1 \). In the general case, this provides a
useful check on the solution for \( \beta, \beta_1 \) and \( \delta \). For \( A = \frac{4}{3} \) and \( \alpha = -1 \), equation
(6.1) gives \( \beta_1^2 + \frac{12}{5} \beta_1 + \frac{6\delta}{5} = 0 \) and we get complex solutions for \( \beta_1 \), which
shows that this special case does not lead to a real C5 solution.

In the critical cases listed in Table 1, \( \alpha = \alpha^* \) and \( A = A^* \) gives a point where
\( K = 0 \) and \( \partial K/\partial \alpha = 0 \) but \( \partial K/\partial A \neq 0 \). In practice, it was fairly easy to find
approximate solutions for \( \alpha^* \) and \( A^* \) and then to improve the approximation by
iteration, using the equations \( K = 0 \) and \( \partial K/\partial \alpha = 0 \). In each case, the \( (\alpha^*, A^*) \)
solution gave \( \varepsilon = 0.03125 \), thus corresponding to a limiting value of \( \varepsilon \) for local
stability.

The C5 solutions corresponding to \( \alpha_1, \alpha_2, \alpha_3 \) and \( \alpha_4 \) show similar characteris-
tics, in that \( b_1 > b_2 > b_3 \) but there is a jump upward from \( b_3 \) to \( b_4 \) and then a
drop to \( b_5 \). In a paper by May and Oster [9] the theorem by Li and Yorke [6]
about chaotic behaviour is mentioned and there is a statement that the proof of
the theorem applies if there exists any cycle with an odd period. This claim is
doubtful since the C5 pattern above is not covered by the statement of Li and
Yorke's theorem.

For solutions of period 6, the equation for \( \alpha \) is \( L(\alpha, A) = 0 \), with \( L(\alpha, A) \)
given by equation (4.17). Since \( L(\alpha, A) \) is a polynomial of odd degree in \( \alpha \), it
must have at least one real zero for all values of \( A \). This zero is indicated by \( \alpha_5 \)
in Table 1 and its value decreases from zero at \( A = 0 \) to \(-0.6 \) at \( A = 1 \) and to
\(-0.76 \) at \( A = 2 \). It corresponds to the \( \alpha_1 \) family of roots in the C4 case and has
similar characteristics. For \( A < A^* \), the \( \alpha_5 \) values do not lead to real C6
solutions. For \( A = A^* = 1.768529 \), the equation \( h_6(\alpha) = 0 \) gives a degenerate
C6 solution, with \( b_1 = b_4 = 1.3494, b_2 = b_5 = 0.0525, b_3 = b_6 = -1.7658 \) and
\( b_1b_2b_3 = -0.125 \). In fact, this is the C3 solution which is at the limit for local
stability, as indicated by the value for \( A^{**} \) in the C3 entry in Table 1. For
\( A > A^* \), the \( \alpha_5 \) roots lead to real C6 solutions whose form is similar to that for
C3 solutions and which are locally stable for $1.768529 < A < 1.777222$. As mentioned earlier, the stability criterion is that $|\theta| < \frac{1}{\alpha_4}$.

The other real roots occur in pairs and lead to real C6 solutions except in one special case similar to that which arose when considering the C5 solutions. The special case occurs when $\alpha = \alpha_0 = 0.597634$ and $A = A_0 = 1.4097134$. For these values of $\alpha$ and $A$, the polynomials $M_1$, $M_2$, $N_1$ and $N_2$, as defined in equations (4.24) to (4.27), are all zero. This means that both equations for $\beta_1$ are indeterminate (equation 4.23) and it is easy to verify from equation (4.28) that $L$, $\partial L/\partial \alpha$ and $\partial L/\partial A$ are all zero at $(\alpha_0, A_0)$. As in the corresponding C5 case, $L$ has a minimum at $(\alpha_0, A_0)$ and is positive for neighbouring points, so $(\alpha_0, A_0)$ is not a starting point for families of real roots of $L(\alpha, A) = 0$.

Although equations (4.23) give indeterminate forms for $\beta_1$, there are quadratic equations available which can be used to show that $\beta_1$, $\gamma$ and $\delta$ are complex when $\alpha = \alpha_0$ and $A = A_0$. Thus the corresponding values for $b_1, b_2, \ldots, b_6$ cannot all be real.

As for the C5 solutions, a computer programme was used to evaluate $L(\alpha, A)$ for different values of $A$ and to examine the solutions of $h_6(x) = 0$ corresponding to zeros of $L(\alpha, A)$. It will be seen from Table 1 that $L(\alpha, A) = 0$ has at most three real roots for $A < 1.9$ but additional pairs of roots appear as $A$ increases from 1.9 to 2.0. For $A = 2$,

$$L(\alpha, 2) = \alpha^9 - \alpha^8 - 46\alpha^7 - 2\alpha^6 + 593\alpha^5 + 591\alpha^4 - 996\alpha^3 - 1036\alpha^2 + 448\alpha + 448 = (\alpha^2 - 1)(\alpha^3 - 21\alpha - 28)(\alpha^2 + p\alpha - 4)(\alpha^2 - (p + 1)\alpha - 4),$$

where $p$ is the positive root of $p^2 + p = 16$. From this, $L(\alpha, 2) = 0$ gives nine real roots which are, in ascending order,

$$\alpha_1 = -4.43, \quad \alpha_2 = -3.65, \quad \alpha_3 = -1.49, \quad \alpha_4 = -1.00, \quad \alpha_5 = -0.76, \quad \alpha_6 = 0.90, \quad \alpha_7 = 1.00, \quad \alpha_8 = 5.14, \quad \alpha_9 = 5.29.$$  

In terms of the trigonometric solutions in Section 2, the corresponding values of $\phi$ are, respectively, $22\pi/65, 22\pi/63; 10\pi/63, 10\pi/65; 14\pi/65, 6\pi/65, 6\pi/63; 2\pi/63, 2\pi/65$. Thus the $\phi$ values pair off in the same way as the $\alpha$ values in Table 1, with the larger $\phi$ value in each pair corresponding to the unstable $\alpha$ sequence. For $\alpha_2, \alpha_3, \alpha_7$ and $\alpha_8$, $\theta = +1$ and this checks that these are the $\alpha$ sequences for which $\theta$ increases as $A$ increases. For $\alpha_1, \alpha_4, \alpha_5, \alpha_6$ and $\alpha_9, \theta = -1$, which also agrees with the conclusions about stability. A similar check can be used for the C3, C4 and C5 entries in Table 1.

Typical C6 solutions are shown in the last column of Table 1. For $\alpha_1$ and $\alpha_2$, the solution oscillates above and below the equilibrium value $w = 1 - \alpha$, in the
manner of a $C_2$ solution but with three distinct levels on either side of the equilibrium level. For $\alpha_3$, $\alpha_4$, $\alpha_6$ and $\alpha_7$, the solutions show the same type of behaviour, with a decrease from $b_1$ to $b_4$, a partial recovery at $b_5$ and a sharp drop to $b_6$. For $\alpha_8$ and $\alpha_9$, the typical solution shows a decrease from $b_1$ to $b_6$, followed by a sharp increase to start the next cycle. As mentioned before, the solutions corresponding to $\alpha_5$ are reminiscent of $C_3$ solutions.

For the $C_5$ and $C_6$ solutions, the critical values agree with those given by May [8, Table 3], although May’s table does not have an entry corresponding to the “solitary” root, $\alpha_5$, in his values for cycles with period 6.

7. Conclusion and acknowledgments

This work arose from looking at papers in which it was hard to tell what had been proved and what had been deduced from numerical simulation, so to some extent the aim was to set up equations which could be regarded as exact. The idea of “factorizing” the equation $H_n(x) = 0$ emerged from this and it will be seen that for $n = 4$, 5 and 6 it results in a considerable simplification of the original problem. To continue the method to $C_7$ solutions would involve finding an equation of degree 18 for $\alpha$ and an equation of degree 7 for the elements of the corresponding periodic solution. It seems likely that this could be done without any additional technical problems but it would require a fair amount of time and patience.

The original draft of the paper contained some additional algebraic details but at the suggestion of the referees the paper has been reduced in length and corroboration of some less important points has been omitted.

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References


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