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IMPULSIVE PERIODIC SOLUTIONS FOR SINGULAR PROBLEMS VIA VARIATIONAL METHODS

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Abstract

In this paper we study impulsive periodic solutions for second-order nonautonomous singular differential equations. Our proof is based on the mountain pass theorem. Some recent results in the literature are extended.

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1. Introduction

In this paper we discuss periodic solutions for second-order nonautonomous singular problems

$$\begin{cases} u'' - \dfrac{b(t)}{u^{\alpha}} = e(t), \quad \text{a.e. } t \in (0, T), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \tag{1.1}$$

under the impulse conditions

$$\Delta u'(t_j) = I_j(u(t_j)), \quad j = 1, 2, \ldots, p-1, \tag{1.2}$$

where $\alpha > 1$, $b \in \mathbb{C}^1([0, T], (0, \infty))$ and $e \in L^2([0, T], \mathbb{R})$ are T-periodic, t_j, $j = 1, 2, \ldots, p-1$, are the instants when the impulses occur and $0 = t_0 < t_1 < t_2 < \cdots < t_{p-1} < t_p = T$, $I_j : \mathbb{R} \to \mathbb{R}$ $(j = 1, 2, \ldots, p-1)$ are continuous.

Impulse effects occur widely in many evolution processes in which their states are changed abruptly at certain moments in time. For the general aspects of impulsive differential equations, we refer the reader to the classical monograph [18]. Applications of impulsive differential equations with or without delays occur in medicine, population dynamics, chaos theory, and so on; see [8, 16, 17, 21].

Impulsive differential equations have been studied by many authors [4, 11, 19, 20]. Some classical tools have been used to study such problems. These classical techniques include the coincidence degree theory of Mawhin [24], the method of upper and lower solutions [6], some fixed point theorems [7] and variational methods [22, 26, 27]. In 2009, Nieto and O'Regan [22] developed the variational framework for impulsive problems and established existence results for a class of impulsive differential equations with Dirichlet boundary conditions. Sun *et al.* [26] obtained multiple periodic solutions for second-order perturbed Hamiltonian systems with impulse effects via variational methods.

Singular problems without impulse effects have also been investigated extensively in the literature [2, 3, 5, 9, 10, 12–14] by using topological methods and variational methods. For example, Boucherif and Daoudi-Merzagui [5] considered a class of singular differential equations and obtained the existence of periodic solutions when the nonlinearity is bounded from above on u by using the mountain pass theorem.

Inspired by [5, 22], in this paper we shall study the existence of periodic solutions for impulsive singular problems. The study of impulsive singular problems is more recent and the number of references is small; see [1, 11, 25]. The tools used in all these references are topological methods. In this paper we prove that problem (1.1)–(1.2) has at least one periodic solution by applying variational methods.

Our result is presented as follows.

Theorem 1.1. *Assume that:*

(S_1) $b \in \mathbb{C}^1([0, T], (0, \infty))$ *is* T*-periodic and* $b'(t) \geq 0$ *for all* $t \in [0, T]$*;*

(S_2) $e \in L^2([0, T], \mathbb{R})$ *is* T*-periodic and* $\int_0^T e(t)\, dt < 0$*;*

(S_3) *there exist two constants* m*,* M *such that, for any* $t \in \mathbb{R}$*,*

$$m \leq I_j(t) \leq M, \quad j = 1, 2, \ldots, p-1,$$

where $m < 0$ *and* $0 \leq M < -1/(p-1) \int_0^T e(t)\, dt$*;*

(S_4) *for any* $t \in \mathbb{R}$*,*

$$\int_0^t I_j(s)\, ds \geq 0, \quad j = 1, 2, \ldots, p-1.$$

Then problem (1.1)–(1.2) has at least one solution.

Remark 1.2. In fact, it is not difficult to find some functions I_j satisfying (S_3) and (S_4). For example,

$$I_j(t) = \sin t, \quad t \in \mathbb{R}.$$

Remark 1.3. Obviously Theorem 1.1 also holds if there is no impulse. In [15], Daoudi-Merzagui studied the existence of periodic solutions for singular differential equations without impulsive effects. In order to apply the method of upper and lower functions, he assumed that the singular nonlinearity $f(t, \cdot)$ is unbounded from above and from below. However, in our paper we consider the case where $f(t, \cdot)$ is only bounded from above (here $f(t, u) = -b(t)/u^\alpha$). So we extend the result in [15]. Moreover, we also extend the result in [5] to the impulsive case.

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2. Preliminaries

Let

$H_T^1 = \{u : [0, T] \to \mathbb{R} \mid u$ is absolutely continuous, $u(0) = u(T)$ and $u' \in L^2([0, T], \mathbb{R})\}$

with the inner product

$$(u, v) = \int_0^T u(t)v(t)\,dt + \int_0^T u'(t)v'(t)\,dt, \quad \forall u, v \in H_T^1.$$

The corresponding norm is defined by

$$\|u\|_{H_T^1} = \left(\int_0^T |u(t)|^2\,dt + \int_0^T |u'(t)|^2\,dt\right)^{1/2}, \quad \forall u \in H_T^1.$$

Then H_T^1 is a Banach space. (In fact, it is a Hilbert space.)

If $u \in H_T^1$, then u is absolutely continuous and $u' \in L^2([0, T], \mathbb{R})$. In this case, $\Delta u'(t) = u'(t^+) - u'(t^-) = 0$ is not necessarily valid for every $t \in (0, T)$ and the derivative u' may have some discontinuities. This may lead to impulse effects.

Following the ideas of [22], take $v \in H_T^1$ and multiply the two sides of the equality

$$-u'' + \frac{b(t)}{u^\alpha} + e(t) = 0$$

by v and integrate from 0 to T:

$$\int_0^T \left(-u'' + \frac{b(t)}{u^\alpha} + e(t)\right) v\,dt = 0. \tag{2.1}$$

Note that, since $u'(0) - u'(T) = 0$,

$$\begin{aligned}\int_0^T u''(t)v(t)\,dt &= \sum_{j=0}^{p-1} \int_{t_j}^{t_{j+1}} u''(t)v(t)\,dt = \sum_{j=0}^{p-1} (u'(t_{j+1}^-)v(t_{j+1}^-) - u'(t_j^+)v(t_j^+)) \\ &\quad - \sum_{j=0}^{p-1} \int_{t_j}^{t_{j+1}} u'(t)v'(t)\,dt \\ &= u'(T)v(T) - u'(0)v(0) - \sum_{j=1}^{p-1} \Delta u'(t_j)v(t_j) - \int_0^T u'(t)v'(t)\,dt \\ &= -\sum_{j=1}^{p-1} I_j(u(t_j))v(t_j) - \int_0^T u'(t)v'(t)\,dt.\end{aligned}$$

Combining with (2.1),

$$\int_0^T u'(t)v'(t)\,dt + \sum_{j=1}^{p-1} I_j(u(t_j))v(t_j) + \int_0^T \frac{b(t)}{u^\alpha} v(t)\,dt + \int_0^T e(t)v(t)\,dt = 0.$$

As a result, we introduce the following concept of a weak solution for problem (1.1)–(1.2).

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Definition 2.1. We say that a function $u \in H_T^1$ is a weak solution of problem (1.1)–(1.2) if

$$\int_0^T u'(t)v'(t)\,dt + \sum_{j=1}^{p-1} I_j(u(t_j))v(t_j) + \int_0^T \frac{b(t)}{u^\alpha} v(t)\,dt + \int_0^T e(t)v(t)\,dt = 0$$

holds for any $v \in H_T^1$.

Define the functional $\Phi : H_T^1 \to \mathbb{R}$ by

$$\Phi(u) := \frac{1}{2}\int_0^T |u'(t)|^2\,dt + \sum_{j=1}^{p-1}\int_0^{u(t_j)} I_j(s)\,ds + \int_0^T b(t)\Big(\int_1^{u(t)} \frac{1}{s^\alpha}\,ds\Big)\,dt + \int_0^T e(t)u(t)\,dt \tag{2.2}$$

for every $u \in H_T^1$. It is easy to verify that Φ is well defined on H_T^1, continuously differentiable and weakly lower semicontinuous. Moreover, the critical points of Φ are the weak solutions of problem (1.1)–(1.2).

In the next section, the following version of the mountain pass theorem will be used in our argument.

Theorem 2.2 [23, Theorem 4.10]. *Let X be a Banach space and let $\varphi \in C^1(X, \mathbb{R})$. Assume that there exist $x_0, x_1 \in X$ and a bounded open neighbourhood Ω of x_0 such that $x_1 \in X \setminus \overline{\Omega}$ and*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf_{x\in\partial\Omega} \varphi(x).$$

Let

$$\Gamma = \{h \in C([0, 1], X) : h(0) = x_0,\ h(1) = x_1\}$$

and

$$c = \inf_{h\in\Gamma} \max_{s\in[0,1]} \varphi(h(s)).$$

If φ satisfies the (PS)-condition (that is, a sequence $\{u_n\}$ in X satisfying $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$ has a convergent subsequence), then c is a critical value of φ and $c > \max\{\varphi(x_0), \varphi(x_1)\}$.

3. Proof of Theorem 1.1

In order to study problem (1.1)–(1.2), for any $\lambda \in (0, 1)$ we consider the modified problem

$$\begin{cases} u'' + b(t)f_\lambda(u(t)) = e(t), & \text{a.e. } t \in (0, T),\\ \Delta u'(t_j) = I_j(u(t_j)), & j = 1, 2, \ldots, p-1,\\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \tag{3.1}$$

where $f_\lambda : [0, T] \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f_\lambda(u) = \begin{cases} -\dfrac{1}{u^\alpha}, & u \geq \lambda, \\ -\dfrac{1}{\lambda^\alpha}, & u < \lambda. \end{cases}$$

Let $F_\lambda(u) = \int_1^u f_\lambda(s)\, ds$ and consider the functional

$$\Phi_\lambda : H_T^1 \to \mathbb{R}$$

defined by

$$\begin{aligned} \Phi_\lambda(u) := \frac{1}{2} \int_0^T |u'(t)|^2\, dt + \sum_{j=1}^{p-1} \int_0^{u(t_j)} I_j(s)\, ds - \int_0^T b(t) F_\lambda(u(t))\, dt \\ + \int_0^T e(t) u(t)\, dt. \end{aligned} \tag{3.2}$$

Clearly, Φ_λ is well defined on H_T^1, continuously differentiable and weakly lower semicontinuous. Moreover, the critical points of Φ_λ are the weak solutions of problem (3.1).

Proof. The proof is divided into four steps.

Step 1. We verify that the functional Φ_λ satisfies the Palais–Smale condition.

Let a sequence $\{u_n\}$ in H_T^1 be such that $\Phi_\lambda(u_n)$ is bounded and $\Phi'_\lambda(u_n) \to 0$. That is, there exist a constant $c_1 > 0$ and a sequence $\{\epsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $\epsilon_n \to 0$ as $n \to +\infty$ such that, for all n,

$$\left| \int_0^T \left(\frac{1}{2} |u_n'(t)|^2 - b(t) F_\lambda(u_n(t)) + e(t) u_n(t) \right) dt + \sum_{j=1}^{p-1} \int_0^{u_n(t_j)} I_j(s)\, ds \right| \leq c_1, \tag{3.3}$$

and, for every $v \in H_T^1$,

$$\left| \int_0^T [u_n'(t) v'(t) - b(t) f_\lambda(u_n(t)) v(t) + e(t) v(t)]\, dt + \sum_{j=1}^{p-1} I_j(u_n(t_j)) v(t_j) \right| \leq \epsilon_n \|v\|_{H_T^1}. \tag{3.4}$$

By a standard argument, it suffices to show that $\{u_n\}$ is bounded when verifying the (PS)-condition.

Taking $v(t) \equiv -1$ in (3.4),

$$\left| \int_0^T (b(t) f_\lambda(u_n(t)) - e(t))\, dt - \sum_{j=1}^{p-1} I_j(u_n(t_j)) \right| \leq \epsilon_n \sqrt{T} \quad \text{for all } n.$$

By (S_3),

$$\left|\int_0^T b(t)f_\lambda(u_n(t))\,dt\right| \le \epsilon_n\sqrt{T} + \left|\int_0^T e(t)\,dt\right| + \sum_{j=1}^{p-1}|I_j(u_n(t_j))|$$
$$\le \epsilon_n\sqrt{T} + \left|\int_0^T e(t)\,dt\right| + (p-1)M := c_2.$$

Note that, for any $t \in [0, T]$, $b(t)f_\lambda(u_n(t)) < 0$. Thus

$$\int_0^T |b(t)f_\lambda(u_n(t))|\,dt = \left|\int_0^T b(t)f_\lambda(u_n(t))\,dt\right| \le c_2.$$

On the other hand, take, in (3.4),

$$v(t) \equiv w_n(t) := u_n(t) - \bar{u}_n, \quad \text{where } \bar{u}_n = \frac{1}{T}\int_0^T u_n(t)\,dt.$$

By [23, Proposition 1.1],

$$\begin{aligned} c_3\|w_n\|_{H_T^1} &\ge \int_0^T (w_n'(t)^2 - b(t)f_\lambda(u_n(t))w_n(t) + e(t)w_n(t))\,dt + \sum_{j=1}^{p-1} I_j(u_n(t_j))w_n(t_j) \\ &\ge \|w_n'\|_{L^2}^2 - (c_2 + \|e\|_{L^1})\|w_n\|_{L^\infty} + (p-1)m\|w_n\|_{L^\infty} \\ &= \|w_n'\|_{L^2}^2 - (c_2 + \|e\|_{L^1} - (p-1)m)\|w_n\|_{L^\infty} \\ &\ge \|w_n'\|_{L^2}^2 - c_4\|w_n\|_{H_T^1}, \end{aligned}$$

where c_3 and c_4 are two positive constants. Thus,

$$\|w_n'\|_{L^2}^2 \le (c_3 + c_4)\|w_n\|_{H_T^1}.$$

Consequently, using the Wirtinger inequality, we see that there exists $c_5 > 0$ such that

$$\|u_n'\|_{L^2}^2 \le c_5. \tag{3.5}$$

Now suppose that

$$\|u_n\|_{H_T^1} \to +\infty \quad \text{as } n \to +\infty.$$

Since (3.5) holds, we have, passing to a subsequence if necessary, that either

$$\begin{aligned} M_n &:= \max u_n \to +\infty \quad \text{as } n \to +\infty \quad \text{or} \\ m_n &:= \min u_n \to -\infty \quad \text{as } n \to +\infty. \end{aligned}$$

(i) Assume that the first possibility occurs. By (S_3) and the fact that $f_\lambda < 0$,

$$\int_0^T (b(t)F_\lambda(u_n(t)) - e(t)u_n(t))\,dt - \sum_{j=1}^{p-1}\int_0^{u_n(t_j)} I_j(s)\,ds$$
$$\ge \int_0^T \left(\left(\int_1^{u_n(t)} b(t)f_\lambda(s)\,ds\right) - e(t)u_n(t)\right)dt - (p-1)MM_n$$

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$$
\begin{aligned}
&= \int_0^T \Big(\Big(\int_1^{M_n} b(t) f_\lambda(s)\, ds - \int_{u_n(t)}^{M_n} b(t) f_\lambda(s)\, ds \Big) - e(t)u_n(t) \Big)\, dt - (p-1)MM_n \\
&= \int_0^T b(t) F_\lambda(M_n)\, dt - \int_0^T M_n e(t)\, dt - \int_0^T \Big(\int_{u_n(t)}^{M_n} (b(t) f_\lambda(s) - e(t))\, ds \Big)\, dt \\
&\quad - (p-1)MM_n \\
&\geq F_\lambda(M_n) \int_0^T b(t)\, dt - M_n \int_0^T e(t)\, dt + \int_0^T (M_n - u_n(t)) e(t)\, dt \\
&\quad - (p-1)MM_n \\
&\geq F_\lambda(M_n) \int_0^T b(t)\, dt - M_n \int_0^T e(t)\, dt - \|e\|_{L^1} \|M_n - u_n\|_C - (p-1)MM_n.
\end{aligned}
$$

Thus, using Sobolev and Poincaré's inequalities,

$$
\begin{aligned}
&-\Big((p-1)M + \int_0^T e(t)\, dt \Big) M_n \\
&\qquad \leq \int_0^T \Big(b(t) F_\lambda(u_n(t)) - e(t)u_n(t) \Big)\, dt - \sum_{j=1}^{p-1} \int_0^{u_n(t_j)} I_j(s)\, ds + \sqrt{T} \|e\|_{L^1} \|u_n'\|_{L^2} \\
&\qquad\quad - F_\lambda(M_n) \int_0^T b(t)\, dt \\
&\qquad = \int_0^T \Big(b(t) F_\lambda(u_n(t)) - e(t)u_n(t) \Big)\, dt - \sum_{j=1}^{p-1} \int_0^{u_n(t_j)} I_j(s)\, ds + \sqrt{T} \|e\|_{L^1} \|u_n'\|_{L^2} \\
&\qquad\quad - \frac{\int_0^T b(t)\, dt}{\alpha - 1} \Big(\frac{1}{M_n^{\alpha-1}} - 1 \Big).
\end{aligned}
$$

From (3.3), (3.5) and the fact that $1/M_n^{\alpha-1} \to 0$ as $n \to +\infty$, we see that the right-hand side of the above inequality is bounded, which is a contradiction.

(ii) Assume that the second possibility occurs, that is, $m_n \to -\infty$ as $n \to +\infty$. We replace M_n by m_n in the preceding arguments, and we also get a contradiction.

Therefore, Φ_λ satisfies the Palais–Smale condition.

Step 2. Let

$$
\Omega = \Big\{ u \in H_T^1 \;\Big|\; \min_{t \in [0,T]} u(t) > 1 \Big\},
$$

and

$$
\partial\Omega = \{ u \in H_T^1 | u(t) \geq 1 \text{ for all } t \in (0,T),\ \exists t_u \in (0,T) : u(t_u) = 1 \}.
$$

We show that there exists $d > 0$ such that $\inf_{u \in \partial\Omega} \Phi_\lambda(u) \geq -d$ whenever $\lambda \in (0,1)$.

For any $u \in \partial\Omega$, there exists some $t_u \in (0,T)$ such that $\min_{t \in [0,T]} u(t) = u(t_u) = 1$.

By (3.2), (S_4) and extending the functions by T-periodicity,

$$\begin{aligned}\Phi_\lambda(u) &= \int_{t_u}^{t_u+T}\Big(\frac{1}{2}|u'(t)|^2 - b(t)F_\lambda(u(t)) + e(t)u(t)\Big)\,dt + \sum_{j=1}^{p-1}\int_0^{u(t_j)} I_j(s)\,ds \\ &\geq \frac{1}{2}\int_{t_u}^{t_u+T}|u'(t)|^2\,dt + \frac{1}{\alpha-1}\int_{t_u}^{t_u+T} b(t)\Big(1-\frac{1}{u(t)^{\alpha-1}}\Big)\,dt \\ &\quad + \int_{t_u}^{t_u+T} e(t)(u(t)-1)\,dt + \int_{t_u}^{t_u+T} e(t)\,dt \\ &\geq \frac{1}{2}\int_{t_u}^{t_u+T}|u'(t)|^2\,dt + \int_{t_u}^{t_u+T} e(t)(u(t)-1)\,dt + \int_{t_u}^{t_u+T} e(t)\,dt.\end{aligned}$$

By the Schwarz inequality,

$$\Phi_\lambda(u) \geq \tfrac{1}{2}\|(u(\cdot)-1)'\|_{L^2}^2 - \|e\|_{L^2}\cdot\|(u(\cdot)-1)\|_{L^2} - \|e\|_{L^1}.$$

Applying Poincaré's inequality to $u(\cdot)-1$,

$$\Phi_\lambda(u) \geq \tfrac{1}{2}\|u'\|_{L^2}^2 - \gamma\|e\|_{L^2}\cdot\|u'\|_{L^2} - \|e\|_{L^1},$$

where $\gamma = \gamma(t_u)$. The above inequality shows that

$$\Phi_\lambda(u) \to +\infty \quad as\ \|u'\|_{L^2} \to +\infty.$$

Since $\min_{t\in[0,T]} u(t) = 1$, we have that $\|u(\cdot)-1\|_{H_T^1} \to +\infty$ is equivalent to $\|u'\|_{L^2} \to +\infty$. Hence

$$\Phi_\lambda(u) \to +\infty \quad \text{as } \|u\|_{H_T^1} \to +\infty,\ \forall u \in \partial\Omega,$$

which shows that Φ_λ is coercive. Thus it has a minimising sequence. The weak lower semicontinuity of Φ_λ yields

$$\inf_{u\in\partial\Omega}\Phi_\lambda(u) > -\infty.$$

It follows that there exists $d > 0$ such that $\inf_{u\in\partial\Omega}\Phi_\lambda(u) > -d$ for all $\lambda \in (0,1)$.

Step 3. We prove that there exists $\lambda_0 \in (0,1)$ with the property that, for every $\lambda \in (0,\lambda_0)$, any solution u of problem (3.1) satisfying $\Phi_\lambda(u) > -d$ is such that $\min_{u\in[0,T]} u(t) \geq \lambda_0$, and hence u is a solution of problem (1.1)–(1.2).

Assume, to the contrary, that there are sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ such that:

(i) $\lambda_n \leq 1/n$;
(ii) u_n is a solution of (3.1) with $\lambda = \lambda_n$;
(iii) $\Phi_{\lambda_n}(u_n) \geq -d$;
(iv) $\min_{t\in[0,T]} u_n(t) < 1/n$.

Since $f_{\lambda_n} < 0$ and $\int_0^T (b(t)f_{\lambda_n}(u_n(t)) - e(t))\,dt = 0$,

$$\|b(\cdot)f_{\lambda_n}(u_n(\cdot))\|_{L^1} \leq c_6, \qquad \text{for some constant } c_6 > 0.$$

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Hence

$$\|u_n'\|_{L^\infty} \le c_7, \quad \text{for some constant } c_7 > 0. \tag{3.6}$$

From $\Phi_{\lambda_n}(u_n) \ge -d$, it follows that there must exist two constants l_1 and l_2, with $0 < l_1 < l_2$ such that

$$\max\{u_n(t); t \in [0, T]\} \subset [l_1, l_2].$$

If not, u_n would tend uniformly to 0 or $+\infty$. In both cases, by (S_2)–(S_3) and (3.6),

$$\Phi_{\lambda_n}(u_n) \to -\infty \quad \text{as } n \to +\infty,$$

which contradicts $\Phi_{\lambda_n}(u_n) \ge -d$.

Let τ_n^1, τ_n^2 be such that, for n large enough,

$$u_n(\tau_n^1) = \frac{1}{n} < l_1 = u_n(\tau_n^2).$$

Multiplying the differential equation in (3.1) by u_n' and integrating the equation on $[\tau_n^1, \tau_n^2]$, or on $[\tau_n^2, \tau_n^1]$,

$$\begin{aligned}\Psi &:= \int_{\tau_n^1}^{\tau_n^2} u_n''(t)u_n'(t)\,dt + \int_{\tau_n^1}^{\tau_n^2} b(t)f_{\lambda_n}(u_n(t))u_n'(t)\,dt \\ &= \int_{\tau_n^1}^{\tau_n^2} e(t)u_n'(t)\,dt.\end{aligned} \tag{3.7}$$

It is easy to verify that

$$\Psi = \Psi_1 + \tfrac{1}{2}(u_n'^2(\tau_n^2) - u_n'^2(\tau_n^1)),$$

where

$$\Psi_1 = \int_{\tau_n^1}^{\tau_n^2} b(t)f_{\lambda_n}(u_n(t))u_n'(t)\,dt.$$

From (S_2), (3.6) and (3.7) it follows that Ψ is bounded, and consequently Ψ_1 is bounded.

On the other hand, it is easy to see that

$$b(t)f_{\lambda_n}(u_n(t))u_n'(t) = \frac{d}{dt}(b(t)F_{\lambda_n}(u_n(t))) - b'(t)F_{\lambda_n}(u_n(t)).$$

Thus, by (S_1),

$$\begin{aligned}\Psi_1 &= b(\tau_n^2)F_{\lambda_n}(l_1) - b(\tau_n^1)F_{\lambda_n}\Big(\frac{1}{n}\Big) - \int_{\tau_n^1}^{\tau_n^2} b'(t)F_{\lambda_n}(u_n(t))\,dt \\ &\le b(\tau_n^2)F_{\lambda_n}(l_1) - b(\tau_n^1)F_{\lambda_n}\Big(\frac{1}{n}\Big) - \frac{1}{\alpha-1}\int_{\tau_n^1}^{\tau_n^2} b'(t)\Big(\frac{1}{l_2{}^{\alpha-1}} - 1\Big)\,dt.\end{aligned}$$

From the fact that $F_{\lambda_n}(1/n) \to +\infty$ as $n \to +\infty$, we obtain $\Psi_1 \to -\infty$, that is, Ψ_1 is unbounded. This is a contradiction.

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Step 4. We show that Φ has a mountain pass geometry for $\lambda \le \lambda_0$.

Fix $\lambda \in (0, \lambda_0]$. Then

$$\begin{aligned} F_\lambda(0) &= \int_1^0 f_\lambda(s)\,ds = -\int_0^1 f_\lambda(s)\,ds \\ &= -\int_0^\lambda f_\lambda(s)\,ds - \int_\lambda^1 f_\lambda(s)\,ds \\ &= \frac{1}{\lambda^{\alpha-1}} - \int_\lambda^1 f_\lambda(s)\,ds, \end{aligned}$$

which implies that

$$F_\lambda(0) > -\int_\lambda^1 f_\lambda(s)\,ds = \int_1^\lambda f_\lambda(s)\,ds = F_\lambda(\lambda).$$

Hence

$$\begin{aligned} \Phi_\lambda(0) &= -F_\lambda(0)\int_0^T b(t)\,dt < -F_\lambda(\lambda)\int_0^T b(t)\,dt \\ &= -\frac{\int_0^T b(t)\,dt}{\alpha-1}\Big(\frac{1}{\lambda^{\alpha-1}} - 1\Big). \end{aligned} \tag{3.8}$$

Consider $\lambda \in (0, \lambda_0]$ such that

$$\frac{1}{\lambda^{\alpha-1}} > 1 + \frac{d(\alpha-1)}{\int_0^T b(t)\,dt}.$$

Thus it follows from (3.8) that $\Phi_\lambda(0) < -d$.

Also, using (S_3), we can choose $R > 1$ large enough that

$$-\Big(M(p-1) + \int_0^T e(t)\,dt\Big)R - \frac{\int_0^T b(t)\,dt}{\alpha-1}\Big(1 - \frac{1}{R^{\alpha-1}}\Big) > d.$$

Then,

$$\begin{aligned} \Phi_\lambda(R) &= \sum_{j=1}^{p-1}\int_0^R I_j(s)\,ds - F_\lambda(R)\int_0^T b(t)\,dt + R\int_0^T e(t)\,dt \\ &\le M(p-1)R + \frac{1}{\alpha-1}\Big(1 - \frac{1}{R^{\alpha-1}}\Big)\int_0^T b(t)\,dt + R\int_0^T e(t)\,dt \\ &= \Big(M(p-1) + \int_0^T e(t)\,dt\Big)R + \frac{\int_0^T b(t)\,dt}{\alpha-1}\Big(1 - \frac{1}{R^{\alpha-1}}\Big) < -d. \end{aligned}$$

Since Ω is a neighbourhood of R, $0 \notin \Omega$ and

$$\max\{\Phi_\lambda(0), \Phi_\lambda(R)\} < \inf_{x\in\partial\Omega} \Phi_\lambda(u).$$

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Steps 1 and 2 imply that Φ_λ has a critical point u_λ such that

$$\Phi_\lambda(u_\lambda) = \inf_{h\in\Gamma} \max_{s\in[0,1]} \Phi_\lambda(h(s)) \geq \inf_{x\in\partial\Omega} \Phi_\lambda(u),$$

where

$$\Gamma = \{h \in C([0, 1], H_T^1) : h(0) = 0, h(1) = R\}.$$

Since $\inf_{u\in\partial\Omega} \Phi_\lambda(u_\lambda) \geq -d$, it follows from Step 3 that u_λ is a solution of problem (1.1)–(1.2). The proof of the main result is complete. □

4. An example

Consider the impulsive singular problem

$$\begin{cases} u'' - \dfrac{b(t)}{u^\alpha} = e(t), & \text{a.e. } t \in (0, T), \\ \Delta u'(t_j) = I_j(u(t_j)), & j = 1, 2, \ldots, p-1, \\ u(0) - u(T) = u'(0) - u'(T) = 0, & \end{cases} \tag{4.1}$$

where $\alpha > 1$, $p \geq 2$ and $T > 0$. Take $b \in \mathbb{C}^1([0, T], (0, \infty))$ such that $b'(t) \geq 0$ for all $t \in [0, T]$, $I_j(t) = \sin t$, $t \in \mathbb{R}$ and $e \in L^2([0, T], \mathbb{R})$ such that $\int_0^T e(t)\, dt < -(p-1)$. Choose $m = -1$ and $M = 1$. Then (S_1)–(S_4) in Theorem 1.1 hold. Therefore, problem (4.1) has at least one periodic solution.

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