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KAC-MOODY LIE ALGEBRAS AND THE CLASSIFICATION OF NILPOTENT LIE ALGEBRAS OF MAXIMAL RANK

L. J. SANTHAROUBANE

Introduction. The natural problem of determining all the Lie algebras of finite dimension was broken in two parts by Levi's theorem:

1) the classification of semi-simple Lie algebras (achieved by Killing and Cartan around 1890)

2) the classification of solvable Lie algebras (reduced to the classification of nilpotent Lie algebras by Malcev in 1945 (see [**10**])).

The Killing form is identically equal to zero for a nilpotent Lie algebra but it is non-degenerate for a semi-simple Lie algebra. Therefore there was a huge gap between those two extreme cases. But this gap is only illusory because, as we will prove in this work, a large class of nilpotent Lie algebras is closely related to the Kac-Moody Lie algebras. These last algebras could be viewed as infinite dimensional version of the semi-simple Lie algebras.

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All the structures are on an algebraically closed field K of characteristic 0.

1. Kac-Moody Lie algebras.

1.1. *Definition.* One calls *Generalized Cartan Matrix* (denoted G.C.M.) a matrix $A = (A_{ij})_{1 \leqq i,j \leqq l}$ with entries in $\mathbf{Z}$ satisfying:

(i) $A_{ii} = 2 \ \forall i = 1 \dots l$

(ii) $A_{ij} \leqq 0 \ \forall i, j = 1 \dots l, i \neq j$

(iii) $A_{ij} = 0 \Leftrightarrow A_{ji} = 0 \ \forall i, j = 1 \dots l.$

All through this paper the G.C.M. will be $l \times l$.

1.2. *Definition.* We will say that two G.C.M.s A and B are *equivalent*

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if there exists $\sigma \in \mathfrak{S}_l$ (permutation group of $\{1 \dots l\}$) such that $B_{ij} = A_{\sigma i \sigma j} \forall i, j = 1 \dots l$.

1.3. *Definition.* We will call *Kac-Moody Lie algebra associated to the* G.C.M. A, the Lie algebra $L(A)$ generated by a set $\{f_1 \dots f_l, h_1 \dots h_l, e_1 \dots e_l\}$ satisfying relations:

$$\forall i, j = 1 \dots l \, [h_i, h_j] = 0 \, [e_i, f_j] = \delta_{ij} h_i \, (\delta_{ij}\text{: Kronecker's symbol})$$

$$[h_i, e_j] = A_{ij} e_j, \, [h_i, f_j] = -A_{ij} f_j;$$

$$\forall i, j = 1 \dots l, i \neq j (\operatorname{ad} e_i)^{-A_{ij}+1} e_j = 0 \quad (\operatorname{ad} f_i)^{-A_{ij}+1} f_j = 0.$$

1.4. Let $\{\alpha_1 \dots \alpha_l\}$ be the canonical basis of $\mathbf{Z}^l$. For $\alpha \in \mathbf{N}^l \backslash \{0\}$, $\alpha = \sum d_i \alpha_i$ denote by L_α (resp. $L_{-\alpha}$) the subvector space of $L(A)$ generated by the elements $[e_{i_1} \dots e_{i_r}]$ (resp. $[f_{i_1} \dots f_{i_r}]$) where e_i (resp. f_i) appears d_i times ($[x_1 \dots x_n] = [x_1[x_2 \dots x_n] \dots]$). If $\alpha = \sum d_i \alpha_i \in \mathbf{Z}^l$ are such that all the d_i's are not of the same sign, let $L_\alpha = (0)$. Denote

$$L_0 = H = Kh_1 \oplus \dots \oplus Kh_l.$$

One calls root system of $L(A)$ the set

$$\Delta = \{\alpha \in \mathbf{Z}^l; \alpha \neq 0 \text{ and } L_\alpha \neq (0)\}.$$

The Lie algebra $L(A)$ is graded by

$$\Delta \cup \{0\} : L(A) = \bigoplus_{\alpha \in \Delta \cup \{0\}} L_\alpha \, [L_\alpha, L_\beta] \subset L_{\alpha+\beta} \forall \alpha, \beta \in \Delta \cup \{0\}.$$

One calls *positive root system* the set

$$\Delta_+ = \{\alpha \in \mathbf{N}^l; \alpha \neq 0 \text{ and } L_\alpha \neq (0)\}$$

and we let $\Delta^- = -\Delta^+$ (negative roots). We have then

$$\Delta = \Delta_- \cup \{0\} \cup \Delta_+.$$

Furthermore $L(A) = L_-(A) \oplus H \oplus L_+(A)$ where $L_+(A) = \oplus_{\alpha \in \Delta_+} L_\alpha$ is called the *positive part* and $L_-(A) = \bigoplus_{\alpha \in \Delta_-} L_\alpha$ the *negative part*. (For the proofs see [**8**] and [**11**].)

1.5. If $\alpha = \sum d_i \alpha_i$ let $|\alpha| = \sum d_i$ and call $|\alpha|$ the height of α. Denote

$$\Delta_+^n = \{\alpha \in \Delta_+; |\alpha| = n\} \text{ for all } n \in \mathbf{N}^*.$$

Remark that $\Delta_+^1 = \{\alpha_1 \dots \alpha_l\}$.

2. Root system for a nilpotent Lie algebra of maximal rank. All through Section 2, $\mathfrak{g}$ is a Lie algebra of finite dimension, Der$\mathfrak{g}$ and Aut$\mathfrak{g}$ denote its derivation algebra and automorphism group.

2.1. *Definition.* One calls a *torus* on $\mathfrak{g}$ a commutative subalgebra of Der$\mathfrak{g}$ which consists of semi-simple endomorphisms. A torus is said to be *maximal* if it is not contained strictly in any other torus.

2.2. A torus defines a representation in $\mathfrak{g}: T \times \mathfrak{g} \to \mathfrak{g}\ (t, x) \mapsto tx$. Since T is a commutative family of semi-simple endomorphisms and since the ground field is algebraically closed, the elements of T can be diagonalized simultaneously. In other words, $\mathfrak{g}$ is decomposed into a direct sum of root spaces for

$$T: \mathfrak{g} = \bigoplus_{\beta \in T^*} \mathfrak{g}^\beta$$

where T^* is the dual of the vector space T and

$$\mathfrak{g}^\beta = \{x \in \mathfrak{g}; tx = \beta(t)x \ \forall t \in T\}.$$

2.3. *Definition.* Let T be a maximal torus on $\mathfrak{g}$. One calls *root system of* $\mathfrak{g}$ *associated to* T, the set:

$$R(T) = \{\beta \in T^*; \mathfrak{g}^\beta \neq (0)\}.$$

2.4. LEMMA. *If* $\mathfrak{g}$ *is a nilpotent Lie algebra, the two following assertions are equivalent:*

(i) $(x_1 \dots x_l)$ *is a minimal system of generators;*

(ii) $(x_1 + C^2\mathfrak{g}, \dots, x_l + C^2\mathfrak{g})$ *is a basis for the vector space* $\mathfrak{g}/C^2\mathfrak{g}$ (*where* $C^2\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$).

Define the *type* of $\mathfrak{g}$ to be the dimension of $\mathfrak{g}/C^2\mathfrak{g}$.

Proof. See example 4 on page 119 of [**2**].

2.5. *Definition.* ($\mathfrak{g}$ nilpotent). Let T be a torus on $\mathfrak{g}$. One calls T-msg a minimal system of generators which consists of root vectors for T.

2.6. LEMMA. ($\mathfrak{g}$ *nilpotent*). *For any torus* T *on* $\mathfrak{g}$ *there exists a* T-msg.

Proof. Just take root vectors for T which form a basis for a T-stable supplement of $C^2\mathfrak{g}$.

2.7. LEMMA. ($\mathfrak{g}$ *nilpotent of type* l). *Let* T *be a maximal torus on* $\mathfrak{g}$, $(x_1 \dots x_l)$ *a* T-msg, β_i *the root of* x_i. *The dimension of* T *is equal to the rank of* $\{\beta_1 \dots \beta_l\}$.

Proof. Let $(t_1 \dots t_k)$ be a basis of T. The rank of $(\beta_1 \dots \beta_l)$ is equal to the rank of the matrix

$$(\beta_i(t_j))_{\substack{1 \leqq i \leqq l \\ 1 \leqq j \leqq k}}$$

whose value is k as one can see easily.

2.8. LEMMA. ($\mathfrak{g}$ *nilpotent of type* l). *The dimension of a maximal torus is an invariant of* $\mathfrak{g}$ *called the rank of* $\mathfrak{g}$. *If* k *is the rank, one has* $k \leqq l$.

Proof. By Mostow's theorem (4.1 of [**12**]), if T and T' are two maximal tori, there exists $\theta \in \operatorname{Aut} \mathfrak{g}$ such that $\theta T \theta^{-1} = T'$, therefore $\dim T = \dim T'$; by 2.7, $k \leqq l$.

2.9. *Definition.* ($\mathfrak{g}$ nilpotent of type l). One says that $\mathfrak{g}$ is of *maximal rank* if its rank is l.

2.10. THEOREM. *($\mathfrak{g}$ nilpotent of maximal rank and of type l). Let T be a maximal torus on $\mathfrak{g}$, $R(T)$ the associated root system, $(x_1 \dots x_l)$ a T-msg and $(\beta_1 \dots \beta_l)$ the corresponding roots.*

(i) *The set $\{\beta_1 \dots \beta_l\}$ is a basis for the vector space T^*.*

(ii) *For any $\beta \in R(T)$ there exists $(d_1 \dots d_l) \in \mathbf{N}^l$ unique such that $\beta = \sum d_i \beta_i$.*

(iii) *Furthermore if we let $|\beta| = \sum d_i$ then $1 \leqq |\beta| \leqq p$ where p is the nilpotency of $\mathfrak{g}$.*

Proof. See p. 82 of [**5**].

3. Cartan matrix associated to a nilpotent Lie algebra of maximal rank.

3.1. LEMMA. *($\mathfrak{g}$ nilpotent of maximal rank and of type l). If T is a maximal torus on $\mathfrak{g}$ and if $(x_1 \dots x_l)$ and $(y_1 \dots y_l)$ are two T-msgs then there exist a unique $\sigma \in \mathfrak{S}_l$ and $(\lambda_1 \dots \lambda_l) \in (K\backslash(0))^l$ such that $y_i = \lambda_i x_{\sigma i}$, $1 \leqq i \leqq l$.*

Proof. Let

$$\{x_1 \dots x_l\} \cup \{[x_{i_1} \dots x_{i_r}]; r \geqq 2, (i_1 \dots i_r) \in I_r\}$$

a basis of the vector space $\mathfrak{g}$ generated by $\{x_1 \dots x_l\}$; there exist $y_{ij}, y_{ii_1 \dots i_r} \in K$ such that

$$y_i = \sum_{j=1}^{l} y_{ij} x_j + \sum_{r \geqq 2, (i_1 \dots i_r) \in I_r} y_{ii_1 \dots i_r} [x_{i_1} \dots x_{i_r}];$$

let β_i be the root of x_i and γ_i the root of y_i $(1 \leqq i \leqq l)$. For $t \in T$ one has:

$$ty_i = \gamma_i(t) y_i = \sum y_{ij} \gamma_i(t) x_j + \sum y_{ii_1 \dots i_r} \gamma_i(t) [x_{i_1} \dots x_{i_r}];$$

on the other hand:

$$ty_i = \sum y_{ij} \beta_j(t) x_j + \sum y_{ii_1 \dots i_r} (\beta_{i_1} + \dots + \beta_{i_r}](t) [x_{i_1} \dots x_{i_r}],$$

therefore:

$$y_{ij}(\beta_j - \gamma_i) = 0 \, \forall i, j = 1 \dots l$$

and

$$y_{ii_1 \dots i_r} (\beta_{i_1} + \dots + \beta_{i_r} - \gamma_i) = 0 \, \forall i = 1 \dots l \, \forall r \geqq 2 \quad \forall (i_1 \dots i_r) \in I_r.$$

By 2.4, for any $i = 1 \dots l$, there exists $j = 1 \dots l$ such that $y_{ij} \neq 0$ thus $\beta_j = \gamma_i$; the integer j is unique since the β_j's are all distinct, there-

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fore one defines a map $\sigma: \{1 \ldots l\} \to \{1 \ldots l\}$ by setting $\sigma i = j$. Now, assume there exist $i, r, (i_1 \ldots i_r)$ such that $y_{i i_1 \ldots i_r} \neq 0$ then

$$\gamma_i = \beta_{i_1} + \ldots + \beta_{i_r}$$

thus

$$\beta_{\sigma i} = \beta_{i_1} + \ldots + \beta_{i_r}$$

which is impossible by 2.10 (since $r \geqq 2$) thus $y_{i i_1 \ldots i_r} = 0$ for all $i = 1 \ldots l$ all $r \geqq 2$ and all $(i_1 \ldots i_r) \in I_r$ therefore $y_i = y_{i\sigma i} x_{\sigma i}$ with $y_{i\sigma i} \neq 0$; this implies that $\sigma \in \mathfrak{S}_l$ and that σ is unique; one then lets $\lambda_i = y_{i\sigma i}$.

3.2. THEOREM. *To any nilpotent Lie algebra of maximal rank and of type l, one can associate an $l \times l$ Cartan matrix A whose equivalence class is an invariant of $\mathfrak{g}$ and which is characterized by the following property: to any maximal torus T and any T-msg $(x_1 \ldots x_l)$, there exists $\sigma \in \mathfrak{S}_l$ such that for all $i, j = 1 \ldots l, i \neq j$:*

$$(\operatorname{ad} x_{\sigma i})^{-A_{ij}} x_{\sigma j} \neq 0 \text{ and } (\operatorname{ad} x_{\sigma i})^{-A_{ij}+1} x_{\sigma j} = 0.$$

3.3. *Definition.* With the preceding notations one says that $(x_1 \ldots x_l)$ is *ordered relatively to A* if $\sigma = \mathrm{Id}$.

3.4. *Proof of* 3.2. We will proceed in four steps:

(i) Let T be a maximal torus and $(y_1 \ldots y_l)$ a T-msg. Since $\operatorname{ad} y_i$ is nilpotent, for $j \neq i$ there exists $A_{ij} \in \mathbf{Z}_{\leqq 0}$ unique such that

$$(\operatorname{ad} y_i)^{-A_{ij}} y_j \neq 0 \text{ and } (\operatorname{ad} y_i)^{-A_{ij}+1} y_j = 0;$$

let $A_{ii} = 2$; obviously $A = (A_{ij})$ is a G.C.M.

(ii) Let $(x_1 \ldots x_l)$ be another T-msg. By 3.1 there exist $\sigma \in \mathfrak{S}_l$ and $(\lambda_1 \ldots \lambda_l) \in (K \backslash (0))^l$ such that $y_i = \lambda_i x_{\sigma i}$ therefore

$$(\operatorname{ad} x_{\sigma i})^{-A_{ij}} x_{\sigma j} \neq 0 \text{ and } (\operatorname{ad} x_{\sigma i})^{-A_{ij}+1} x_{\sigma j} = 0.$$

(iii) Let T' be another maximal torus. By Mostow's theorem (4.1 of [**12**]) there exists $\theta \in \operatorname{Aut} \mathfrak{g}$ such that $\theta T \theta^{-1} = T'$; obviously $(\theta y_1 \ldots \theta y_l)$ is a T'-msg; by (i) there exists a G.C.M. A' such that

$$(\operatorname{ad} \theta y_i)^{-A'_{ij}} \theta y_j \neq 0 \text{ and}$$
$$(\operatorname{ad} \theta y_i)^{-A'_{ij}+1} \theta y_j = 0 \text{ for all } i \neq j;$$

this is equivalent to

$$(\operatorname{ad} y_i)^{-A'_{ij}} y_j \neq 0 \text{ and}$$
$$(\operatorname{ad} y_i)^{-A'_{ij}+1} y_j = 0 \text{ for all } i \neq j;$$

thus $A = A'$ by unicity.

(iv) In (i) we associated to $(T, (y_1 \dots y_l))$ a G.C.M. A. In (ii) we modified only $(y_1 \dots y_l)$ and we obtained an equivalent G.C.M.; therefore the equivalence class of A depends only on T. In (iii) we modified T into T' and obtained for a suitable T'-msg the same G.C.M., thus the equivalence class of A does not depend on T either.

4. Universal property.

4.1. If $X = \{\epsilon_1 \dots \epsilon_l\}$ is a set, the free Lie algebra $F(X)$ generated by X is graded by $\mathbf{N}^l \backslash (0)$. If $(\alpha_1 \dots \alpha_l)$ is the canonical basis of $\mathbf{Z}^l$ and $\alpha = \sum d_i \alpha_i \in \mathbf{N}^l \backslash (0)$, denote by F^α the subvector space of $F(X)$ spanned by the $[\epsilon_{i_1} \dots \epsilon_{i_r}]$'s where ϵ_i appears d_i times for all $i = 1 \dots l$. One has then

$$F(X) = \bigoplus_{\alpha \in \mathbf{N}^l \backslash (0)} F^\alpha \text{ and } [F^\alpha, F^\beta] \subset F^{\alpha+\beta} \text{ for all}$$

$\alpha, \beta \in \mathbf{N}^l \backslash (0)$ (see [**2**], p. 22).

4.2. Let $\rho : X \to F(X)$ be the canonical imbedding ([**2**], p. 19). The pair $(\rho, F(X))$ satisfies the following universal property: for any Lie algebra $\mathfrak{g}$ and any map $f : X \to \mathfrak{g}$ there exists a unique homomorphism $\varphi : F(X) \to \mathfrak{g}$ such that $f = \varphi \circ \rho$ ([**2**], p. 18).

4.3. LEMMA. *With the notation of* 1.4 *we have:*

(i) *$L_+(A)$ is a Lie algebra generated by $\{e_1 \dots e_l\}$ satisfying only the relations*

$$(\text{ad } e_i)^{-A_{ij}+1} e_j = 0 \ \forall i \neq j.$$

(ii) *$L_+(A)$ is graded by*

$$\Delta_+ : L_+(A) = \bigoplus_{\alpha \in \Delta_+} L_\alpha, \ [L_\alpha, L_\beta] \subset L_{\alpha+\beta} \text{ for all } \alpha, \beta \in \Delta_+.$$

(iii) *There exists a unique homomorphism λ from $F(X)$ onto $L_+(A)$ such that $\lambda \epsilon_i = e_i$ and satisfying the following properties:* Ker λ *is generated by*

$$(\text{ad } \epsilon_i)^{-A_{ij}+1} \epsilon_j, \ i, j = 1 \dots l, \ i \neq j \text{ and}$$
$$\lambda F_\alpha(X) = L_\alpha \text{ for all } \alpha \in \mathbf{N}^l \backslash (0).$$

Proof. The proof is straightforward.

4.4. LEMMA. *With the above notation one has:*

$$C^n L_+(A) = \bigoplus_{|\alpha| \geqq n} L_\alpha$$

where $C^n L_+(A)$ is the nth *term of the central descending series.*

Proof. This, again, is straightforward.

4.5. LEMMA. *For all $\alpha \in \Delta_+ \backslash \{\alpha_1 \dots \alpha_l\}$ there exists $i \in \{1 \dots l\}$ such that $\alpha - \alpha_i \in \Delta_+$.*

Proof. This follows as in the semi-simple case.

4.6. LEMMA. *Let* $\Delta_+^k = \{\alpha \in \Delta_+; |\alpha| = k\}$. *If* $\Delta_+^p = \emptyset$ *for some* $p \in \mathbf{N}^*$ *then* $\Delta_+^{p+n} = \emptyset$ *for all* $n \in \mathbf{N}$.

Proof. This follows from 4.5.

4.7. LEMMA. *For all* $k \in \mathbf{N}$ *and all* $i, j \in \{1 \dots l\}$ *we have*

$$L_{\alpha_j + k\alpha_i} = K(\mathrm{ad}\, e_i)^k e_j.$$

Proof. This is clear from above.

4.8. Let $p \in \mathbf{N}^*$ and A a G.C.M. We will need in the sequel two conditions on p and A. By commodity we gather them here. As shown in 4.9 (ii) and (vi), without these two hypotheses, the Lie algebra $\mathfrak{m}_p(A)$ won't have the invariants p and A.

(H_1) either dim $L(A) = +\infty$ or $\dim L(A) < \infty$

and in this case $p \leqq p_A$ where p_A is the height of the highest root of $L_+(A)$.

(H_2) $p \geqq \mathrm{Sup}\,\{-A_{ij} + 1; i, j \in \{1 \dots l\}\}$.

4.9. PROPOSITION. *Let*

$$\mathfrak{m} = \mathfrak{m}_p(A) = L_+(A)/C^{p+1}L_+(A) \ (p \geqq 1) \textit{ and}$$

$$\mu : L_+(A) \rightarrow \mathfrak{m}_p(A) \ x \mapsto \bar{x}$$

the canonical map.

(i) *The restriction of* μ *to the vector spaces* L_α *such that* $|\alpha| \leqq p$ *is an isomorphism from* L_α *onto* $\bar{L}_\alpha$ *and* $\mathfrak{m}_p(A)$ *is graded by*

$$\{\alpha \in \Delta_+; |\alpha| \leqq p\} : \mathfrak{m}_p(A) = \bigoplus_{|\alpha| \leqq p} \bar{L}_\alpha [\bar{L}_\alpha, \bar{L}_\beta] \subset \bar{L}_{\alpha+\beta}.$$

(ii) *The Lie algebra* $\mathfrak{m}_p(A)$ *is nilpotent and under the hypothesis* H_1 *of* 4.8. *its nilpotency is* p.

(iii) *The set* $\{\bar{e}_1 \dots \bar{e}_l\}$ *is a minimal system of generators of* $\mathfrak{m}_p(A)$.

(iv) *Let* $t_i \in \mathrm{Der}\,\mathfrak{m}_p(A)$ $(1 \leqq i \leqq l)$ *defined by* $t_i \bar{e}_j = \delta_{ij} \bar{e}_j$; *then* $T = \bigoplus_{i=1}^l Kt_i$ *is a maximal torus on* $\mathfrak{m}_p(A)$ *and the nilpotent Lie algebra* $\mathfrak{m}_p(A)$ *is of maximal rank; furthermore* $(\bar{e}_1 \dots \bar{e}_l)$ *is a* T-msg.

(v) *Let* $(t^{*1} \dots t^{*l})$ *be the dual basis of* $(t_1 \dots t_l)$; *if we identify* t^{*i} *and* α_i *then the root space decomposition relative to* T *is identical to the decomposition*

$$\mathfrak{m}_p(A) = \bigoplus_{\alpha \in \Delta_+ |\alpha| \leqq p} \bar{L}_\alpha.$$

(vi) *Under the hypothesis* H_2 *of* 4.8 A *is a* G.C.M. *associated to* $\mathfrak{m}_p(A)$ *and* $(\bar{e}_1 \dots \bar{e}_l)$ *is ordered relative to* A.

Proof. (i) is obvious from 4.4.

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(ii) The lie algebra $\mathfrak{m}$ is obviously nilpotent of nilpotency $\leqq p$. By 4.4,

$$C^p\mathfrak{m} = \overline{\bigoplus_{|\alpha|=p} L_\alpha};$$

by (i) one has $C^p\mathfrak{m} = (0)$ if and only if

$$\bigoplus_{|\alpha|=p} L_\alpha = (0);$$

by the definition of L_α one has $\bigoplus_{|\alpha|=p} L_\alpha = (0)$ if and only if $\Delta_+^p = \emptyset$; by 4.6 we have $\Delta_+^p = \emptyset$ if and only if $\Delta_+^{p+n} = \emptyset\ \forall n \geqq 0$; since

$$C^pL_+(A) = \bigoplus_{n\geqq 0} \bigoplus_{|\alpha|=p+n} L_\alpha$$

we have $\Delta_+^{p+n} = \emptyset\ \forall n \geqq 0$ if and only if $C^pL_+(A) = (0)$.

If $\dim L(A) = +\infty$ then $C^pL_+(A) \neq (0)\ \forall p \geqq 1$; if $\dim L(A) < \infty$ then $L(A)$ is a semi-simple Lie algebra and $L_+(A)$ is the nilpotent part ([**11**], p. 230) of nilpotency p_A thus $C^pL_+(A) \neq (0)$ (since $p \leqq p_A$). In both cases $C^pL_+(A) \neq (0)$ therefore $C^p\mathfrak{m} \neq (0)$.

(iii) We have

$$\mathfrak{m}/C^2\mathfrak{m} \cong \bigoplus_{|\alpha|=1} \bar{L}_\alpha = \bigoplus_{i=1}^{l} K\bar{e}_i$$

thus $(\bar{e}_1 \ldots \bar{e}_l)$ is a minimal system of generators for $\mathfrak{m}$(2.4).

(iv) Obviously T is a torus on $\mathfrak{m}$. Since the dimension of T is equal to the type of $\mathfrak{m}$ (by (iii)), T is a maximal torus and $\mathfrak{m}$ is of maximal rank.

(v) Let

$$\mathfrak{m}_\alpha = \{\bar{x} \in \bar{L};\, t\bar{x} = \alpha(t)\bar{x}\ \forall t \in T\};$$

it is easy to prove both inclusions: $\mathfrak{m}_\alpha \subset \bar{L}_\alpha$ and $\bar{L}_\alpha \subset \mathfrak{m}_\alpha$.

(vi) By (iii) and (iv) $(\bar{e}_1 \ldots \bar{e}_l)$ is a T-msg of $\mathfrak{m}$. We have

$$(\operatorname{ad} \bar{e}_i)^{-A_{ij}+1}\, \bar{e}_j = 0.$$

Assume that $(\operatorname{ad} \bar{e}_i)^{-A_{ij}}\, \bar{e}_j = \bar{0}$ then

$$(\operatorname{ad} e_i)^{-A_{ij}}\, e_j \in \operatorname{Ker} \mu$$

thus

$$L_{\alpha_j - A_{ij}\alpha_i} \subset \bigoplus_{|\alpha|\geqq p+1} L_\alpha$$

(by 4.4 and 4.7) therefore $1 - A_{ij} \geqq p + 1$ which contradicts H_2.

4.10. With the notation of 4.1, 4.2, 4.3 and 4.9 denote

$$u = \mu \circ \lambda \circ \rho : X \to \mathfrak{m}_p(A)$$

i.e., $u(\epsilon_i) = \bar{e}_i$. We assume in the sequel that H_1 and H_2 of 4.8 are satisfied which implies that $\mathfrak{m}_p(A)$ is a nilpotent Lie algebra of nilpotency p and that A is a G.C.M. associated to $\mathfrak{m}_p(A)$.

4.11. PROPOSITION. (i) *The pair* $(u, \mathfrak{m}_p(A))$ *satisfies the following universal property: for any nilpotent Lie algebra of type* l, *of maximal rank, of nilpotency* q *such that* $q \leqq p$, *whose associated* G.C.M. B *is such that* $|B_{ij}| \leqq |A_{ij}| \forall i, j$; *and for any map* $f: X \rightarrow \mathfrak{g}$ *such that* $(f\epsilon_1 \dots f\epsilon_l)$ *is a* $T\mathfrak{g}$-msg *ordered relative to* B *(for a maximal torus* $T\mathfrak{g}$ *on* $\mathfrak{g}$), *there exists a unique homomorphism* φ *from* $\mathfrak{m}_p(A)$ *onto* $\mathfrak{g}$ *such that* $\varphi \circ u = f$.

(ii) *Let* $\mathfrak{m}_p'(A)$ *be a nilpotent Lie of nilpotency* p, *of type* l, *of maximal rank, whose associated* G.C.M. *is* A; *let* $u': X \rightarrow \mathfrak{m}_p'(A)$ *be a map such that the pair* $(u', \mathfrak{m}_p'(A))$ *satisfies the universal property of* (i); *then there exists an isomorphism* $\Psi: \mathfrak{m}_p(A) \rightarrow \mathfrak{m}_p'(A)$ *such that* $\Psi \circ u = u'$.

Proof. (i) By 4.2. there exists a unique homomorphism $f_1: F(X) \rightarrow \mathfrak{g}$ such that $f = f_1 \circ \rho$. Since $(f\epsilon_1 \dots f\epsilon_l)$ is ordered relative to B one has

$$(\text{ad}\, f\epsilon_i)^{-A_{ij}+1} f\epsilon_j = 0 \ \forall i \neq j,$$

therefore

$$f_1((\text{ad}\ \epsilon_i)^{-A_{ij}+1}\ \epsilon_j) = 0 \ \forall i \neq j$$

thus $\text{Ker} f_1 \subset \text{Ker}\ \lambda$, and this implies the existence of a unique homomorphism $f_2: L \rightarrow \mathfrak{g}$ such that $f_2 \circ \lambda = f_1$. Since $q \leqq p$ and $C^{p+1}\mathfrak{g} = (0)$ we have $f_2(\text{Ker}\ \mu) = 0$ therefore there exists a unique homomorphism $\varphi: \mathfrak{m} \rightarrow \mathfrak{g}$ such that $\varphi \circ \mu = f_2$. This yields $\varphi \circ u = f$.

(ii) Apply (i).

5. Classification theorem.

5.1. Recall that H_1 and H_2 of 4.8 are assumed. If $\mathfrak{a}$ is an ideal of $\mathfrak{m}_p(A)$ denote $\mathfrak{g} = \mathfrak{m}/\mathfrak{a}$ and $\pi: \mathfrak{m} \rightarrow \mathfrak{g}$ the canonical map; $\mathfrak{g}$ is a nilpotent Lie algebra of nilpotency less than p.

5.2. LEMMA. *The two following assertions are equivalent:*

(i) $\mathfrak{a} \subset C^2\mathfrak{m}$;

(ii) $(\pi\bar{e}_1 \dots \pi\bar{e}_l)$ *is a minimal system of generators of* $\mathfrak{g}$.

Proof. If $\mathfrak{a} \subset C^2\mathfrak{m}$ then

$$\bar{e}_j + C^2\mathfrak{m} \mapsto \pi\bar{e}_j + C^2\mathfrak{g},\ \mathfrak{m}/C^2\mathfrak{m} \rightarrow \mathfrak{g}/C^2\mathfrak{g}$$

is an isomorphism; one then applies 2.4. Conversely if $\mathfrak{a} \not\subset C^2\mathfrak{m}$ there exist $(\lambda_1 \dots \lambda_l) \in K^l \backslash (0)$ such that $\sum \lambda_i \bar{e}_i \in \mathfrak{a}$ and thus

$$\sum \lambda_i \pi \bar{e}_i = 0.$$

5.3. LEMMA. *If* $\mathfrak{a}$ *is homogenous and contained in* $C^2\mathfrak{m}$ *and if* T *is the maximal torus defined in* 4.9 *then:*

(i) *For any* $y \in T$ *there exists* $\tilde{\pi}(t) \in \text{Der}\ \mathfrak{g}$ *unique such that* $\pi \circ t = \tilde{\pi}(t) \circ \pi$.

(ii) *The nilpotent Lie algebra* $\mathfrak{g}$ *is of maximal rank with* $\tilde{\pi}(T)$ *as a maximal torus and* $(\pi\bar{e}_1 \ldots \pi\bar{e}_l)$ *as a* $\tilde{\pi}(T)$-msg.

(iii) *For any* $\tilde{\pi}(T)$-msg $(y_1 \ldots y_l)$ *there exists a unique* T-msg $(x_1 \ldots x_l)$ *of* $\mathfrak{m}$ *such that* $\pi x_i = y_i \, \forall i = 1 \ldots l$.

Proof. (i) By 4.9 (v) $\mathfrak{a}$ is homogenous if and only if $\mathfrak{a}$ is T-invariant; this allows us to define $\tilde{\pi}(t)$ by

$$\tilde{\pi}(t)(\pi x) = \pi t x \; \forall x \in \mathfrak{m}.$$

(ii) Since $\mathfrak{a} \subset C^2\mathfrak{m}$, $(\pi\bar{e}_1 \ldots \pi\bar{e}_l)$ is a minimal system of generators of $\mathfrak{g}$ (by 5.2.) thus $\mathfrak{g}$ is of type l. Obviously $\tilde{\pi}(T)$ is a torus on $\mathfrak{g}$ with root vectors $(\pi\bar{e}_1 \ldots \pi\bar{e}_l)$; let $\lambda_1 \ldots \lambda_l \in K$ such that

$$\sum \lambda_i \tilde{\pi}(t_i) = 0$$

then $\lambda_j \pi e_j = 0$ thus $\lambda_j = 0$ therefore $\dim \tilde{\pi}(T) = l$ and by 2.8 $\mathfrak{g}$ is of maximal rank and $\tilde{\pi}(T)$ is a maximal torus.

(iii) Let

$$W = \bigoplus_{i=1}^{l} K\bar{e}_i;$$

it is easy to see that

$$\mathfrak{g} = \pi W \oplus C^2\mathfrak{g} \text{ and } W \cong \pi W;$$

let $(y_1 \ldots y_l)$ a $\tilde{\pi}(T)$ − msg of $\mathfrak{g}$; there exist $x_i \in W$ unique and $z_i \in C^2\mathfrak{g}$ unique such that $y_i = \pi x_i + z_i$. If β_i is the root of y_i it is easy to see (by using the preceeding decomposition of $\mathfrak{g}$) that

$$t x_i = \beta_i(\tilde{\pi} t) x_i \text{ and } z_i \in \mathfrak{g}^{\beta_i} \cap C^2\mathfrak{g} = (0).$$

5.4. LEMMA. *If* $\mathfrak{a}$ *is homogenous and if*

$$(\text{ad } \bar{e}_i)^{-A_{ij}} \bar{e}_j \notin \mathfrak{a} \; \forall i, j = 1 \ldots l, i \neq j$$

then $\mathfrak{g}$ *is of maximal rank and* A *is a* G.C.M. *associated to* $\mathfrak{g}$.

Proof. By simple arguments one can prove that $\mathfrak{a} \subset C^2\mathfrak{m}$; by applying 5.3 (ii) it suffices to prove that

$$(\text{ad } \pi\bar{e}_i)^{-A_{ij}} \pi\bar{e}_j \neq 0 \text{ and } (\text{ad } \pi\bar{e}_i)^{-A_{ij}+1} \pi\bar{e}_j = 0 \; \forall i \neq j$$

which is obvious.

5.5. LEMMA. *The two following assertions are equivalent:*

(i) $\mathfrak{g}$ *is of nilpotency* p.

(ii) $C^p\mathfrak{m} \not\subset \mathfrak{a}$.

Proof. This is straightforward.

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5.6. *Definition.* We call the automorphism group of the G.C.M. A the group

$$\mathfrak{S}_l(A) = \{\sigma \in \mathfrak{S}_l;\ A_{\sigma i \sigma j} = A_{ij}\ \forall i, j = 1 \dots l\}.$$

5.7. LEMMA. *Let* $\sigma \in \mathfrak{S}_l$. *There exists* $\tilde{\sigma} \in \operatorname{Aut} \mathfrak{m}$ *such that* $\tilde{\sigma}\bar{e}_i = \bar{e}_{\sigma i}$ $\forall i = 1 \dots l$ *if and only if* $\sigma \in \mathfrak{S}_l(A)$. *Write*

$$\tilde{\mathfrak{S}}_l(A) = \{\tilde{\sigma} \in \operatorname{Aut} \mathfrak{m};\ \sigma \in \mathfrak{S}_l(A)\}.$$

Proof. We can define a bijective linear map $\tilde{\sigma}: \mathfrak{m} \to \mathfrak{m}$ by setting

$$\sigma\bar{e}_i = \bar{e}_{\sigma i}\ \forall i = 1 \dots l.$$

We have $\tilde{\sigma} \in \operatorname{Aut} \mathfrak{m}$ if and only if

$$(\operatorname{ad} \bar{e}_{\sigma i})^{-A_{ij}+1}\, \bar{e}_{\sigma j} = 0\ \forall i \neq j$$

$$\text{i.e., } (\operatorname{ad} e_{\sigma i})^{-A_{ij}+1}\, e_{\sigma j} \in C^{p+1}L_+(A)\ \forall i \neq j.$$

Assume that $\tilde{\sigma} \in \operatorname{Aut} \mathfrak{m}$ and let (i, j) be such that

$$(\operatorname{ad} e_{\sigma i})^{-A_{ij}+1}\, e_{\sigma j} \neq 0$$

then

$$\alpha_{\sigma j} + (-A_{ij} + 1)\alpha_{\sigma i} \in \Delta_+$$

and we have

$$|\alpha_{\sigma j} + (-A_{ij} + 1)\alpha_{\sigma i}| \geqq p + 1;$$

since $p \geqq -A_{\sigma i \sigma j} + 1$ (by H_1 of 4.8) it follows that $A_{\sigma i \sigma j} \geqq A_{ij}$; now let (i, j) such that

$$(\operatorname{ad} e_{\sigma i})^{-A_{ij}+1}\, e_{\sigma j} = 0;$$

then $-A_{\sigma i \sigma j} + 1 \leqq -A_{ij} + 1$ and thus $A_{\sigma i \sigma j} \geqq A_{ij}$; in both cases we have $A_{\sigma i \sigma j} \geqq A_{ij}$ and therefore

$$A_{ij} \leqq A_{\sigma i \sigma j} \leqq A_{\sigma^2 i \sigma^2 j} \leqq \dots;$$

there exists $n \in \mathbf{N}^*$ such that $\sigma^n = 1$ therefore

$$A_{ij} \leqq A_{\sigma i \sigma j} \leqq \dots \leqq A_{ij}$$

which implies that $A_{ij} = A_{\sigma i \sigma j}$ $\forall i \neq j$ thus $\sigma \in \mathfrak{S}_l(A)$. The converse is obvious.

5.8. LEMMA. *The set*

$$\mathfrak{J} = \mathfrak{J}_p(A) = \{\mathfrak{a} \text{ homogenous ideal of } \mathfrak{m};\ C^p\mathfrak{m} \not\subset \mathfrak{a} \text{ and } (\operatorname{ad} \bar{e}_i)^{-A_{ij}}\, \bar{e}_j \notin \mathfrak{a}\ \forall i \neq j\}$$

is stable under $\tilde{\mathfrak{S}}_l(A)$.

Proof. This is clear.

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5.9. PROPOSITION. *Let $\mathfrak{g}$ be a nilpotent Lie algebra of maximal rank, of nilpotency p and such that A is an associated* G.C.M.

(i) *There exists $\mathfrak{a} \in \mathfrak{J}$ such that $\mathfrak{g} \cong \mathfrak{m}/\mathfrak{a}$.*

(ii) *If $\mathfrak{a}' \in \mathfrak{J}$ is such that $\mathfrak{g} \cong \mathfrak{m}/\mathfrak{a}'$ then there exists $\tilde{\sigma} \in \tilde{\mathfrak{S}}_l(A)$ such that $\tilde{\sigma}\mathfrak{a} = \mathfrak{a}'$.*

Proof. (i) Let $(x_1 \dots x_l)$ be a $T\mathfrak{g}$ — msg ordered relative to A (where $T\mathfrak{g}$ is a maximal torus on $\mathfrak{g}$). Let $f:X \to \{x_1 \dots x_l\}$ be a map defined by $f\epsilon_i = x_i$; by 4.11 there exists a homomorphism π from $\mathfrak{m}$ onto $\mathfrak{g}$ such that $\pi\bar{e}_i = x_i$. Let $\mathfrak{a} = \operatorname{Ker} \pi$ then $\mathfrak{g} \cong \mathfrak{m}/\mathfrak{a}$. Let us prove that $\mathfrak{a} \in \mathfrak{J}$. By 5.5 we have $C^p\mathfrak{m} \not\subset \mathfrak{a}$. Secondly, we have

$$(\operatorname{ad} \bar{e}_i)^{-A_{ij}} \bar{e}_j \notin \mathfrak{a}$$

since

$$(\operatorname{ad} x_i)^{-A_{ij}} x_j \neq 0.$$

Finally to prove that $\mathfrak{a}$ is homogenous one uses 2.10: let

$$\sum_{(i_1 \dots i_r) \in I} \lambda_{i_1 \dots i_r} [\bar{e}_{i_1} \dots \bar{e}_{i_r}] \in \mathfrak{a} \backslash (0)$$

with $\lambda_{i_1 \dots i_r} \neq 0$ and $[\bar{e}_{i_1} \dots \bar{e}_{i_r}] \notin \mathfrak{a} \ \forall (i_1 \dots i_r) \in I$, we have then that

$$\sum \lambda_{i_1 \dots i_r} [x_{i_1} \dots x_{i_r}] = 0$$

with $[x_{i_1} \dots x_{i_r}] \neq 0 \quad \forall (i_1 \dots i_r) \in I$ therefore there exists $\beta = \sum d_i\beta_i \in R(T)$ such that

$$\beta = \beta_{i_1} + \dots + \beta_{i_r} \ \forall (i_1 \dots i_r) \in I.$$

(β_i is the root of x_i which implies that $\beta_{i_1} + \dots + \beta_{i_r}$ is the root of $[x_{i_1} \dots x_{i_r}]$.) Let $d_{ii_1 \dots i_r}$ be the number of times that i appears in $(i_1 \dots i_r)$. We have

$$\sum_i d_i\beta_i = \sum_i d_{ii_1 \dots i_r}\beta_i;$$

therefore, by 2.10,

$$d_i = d_{ii_1 \dots i_r} \ \forall i = 1 \dots l \ \forall (i_1 \dots i_r) \in I;$$

thus $\bar{e}_i$ appears d_i times in $[\bar{e}_{i_1} \dots \bar{e}_{i_r}] \ \forall (i_1 \dots i_r) \in I$ which means that

$$[\bar{e}_{i_1} \dots \bar{e}_{i_r}] \in \bar{L}_\alpha \quad \forall (i_1 \dots i_r) \in I$$

where $\alpha = \sum d_i\alpha_i$; therefore $\mathfrak{a}$ is homogenous.

(ii) Let $\mathfrak{a}' \in \mathfrak{J}$ be such that $\mathfrak{g} \cong \mathfrak{m}/\mathfrak{a}'$. Let us make the following identification:

$$\mathfrak{g} = \mathfrak{m}/\mathfrak{a} = \mathfrak{m}/\mathfrak{a}'.$$

Let $\pi' : \mathfrak{m} \to \mathfrak{g}$ associated to $\mathfrak{a}'$ (see (i)). By 5.2 we have $\mathfrak{a} \subset C^2\mathfrak{m}$ and $\mathfrak{a}' \subset C^2\mathfrak{m}$. By 5.3 (ii), $\tilde{\pi}(T)$ and $\tilde{\pi}'(T)$ are maximal tori on $\mathfrak{g}$ and by 4.1 of [**12**] there exists $\varphi \in \text{Aut } \mathfrak{g}$ such that

$$\varphi\tilde{\pi}(T)\varphi^{-1} = \tilde{\pi}'(T).$$

By 5.2 (ii) $(\pi\bar{e}_1 \ldots \pi\bar{e}_l)$ is a $\tilde{\pi}(T)$-msg and therefore $(\varphi\pi\bar{e}_1 \ldots \varphi\pi\bar{e}_l)$ is a $\tilde{\pi}'(T)$-msg. By 5.3 (iii) there exists a T-msg $(\bar{e}_1' \ldots \bar{e}_l')$ of $\mathfrak{m}$ such that

$$\pi'\bar{e}_i' = \varphi\pi\bar{e}_i \,\forall i = 1 \ldots l.$$

By 3.1 there exist $\sigma \in \mathfrak{S}_l$ and $(\lambda_1 \ldots \lambda_l) \in (K\backslash(0))^l$ such that

$$\bar{e}_i' = \lambda_i \bar{e}_{\sigma i} \,\forall i = 1 \ldots l.$$

Since $(\bar{e}_1' \ldots \bar{e}_l')$ and $(\bar{e}_1 \ldots \bar{e}_l)$ are ordered relative to A we have $\sigma \in \mathfrak{S}_l(A)$ and therefore we define $\theta \in \text{Aut } \mathfrak{m}$ by setting $\theta\bar{e}_i = \bar{e}_i'$; we then have $\varphi\pi' = \pi'\theta$ and thus $\theta\mathfrak{a} \subset \mathfrak{a}'$; since θ is one to one and $\dim \mathfrak{a} = \dim \mathfrak{a}'$ this implies $\theta\mathfrak{a} = \mathfrak{a}'$; on the other hand $\theta\mathfrak{a} = \tilde{\sigma}\mathfrak{a}$ thus $\tilde{\sigma}\mathfrak{a} = \mathfrak{a}'$ with $\tilde{\sigma} \in \tilde{\mathfrak{S}}_l(A)$.

5.10. THEOREM. *The isomorphism classes of nilpotent Lie algebras of maximal rank, of nilpotency p and such that A is an associated* G.C.M. *are in bijection with the orbits of $\mathfrak{J}_p(A)$ under the action of $\tilde{\mathfrak{S}}_l(A)$.*

Proof. To each $\mathfrak{a} \in \mathfrak{J}$ associate the isomorphism class of $\mathfrak{m}/\mathfrak{a}$; by the preceeding results this gives the bijection.

6. Model of nilpotent Lie algebra.

6.1. LEMMA. *Let $\mathfrak{m}(l, p) = F(X)/C^{p+1}F(X)$ ($p \geqq 1$) and $\pi : F(X) \to \mathfrak{m}$ $x \mapsto \tilde{x}$ be the canonical map.*

(i) *The restriction of π to the subspaces F^α such that $|\alpha| \leqq p$ is an isomorphism from F^α onto $\tilde{F}^\alpha$ and $\mathfrak{m}(l, p)$ is graded by $\{\alpha \in \mathbf{N}^l\backslash(0) ; |\alpha| \leqq p\}$:*

$$\mathfrak{m}(l, p) = \bigoplus_{|\alpha| \leqq p} \tilde{F}^\alpha \text{ and } [\tilde{F}^\alpha, \tilde{F}^\beta] \subset \tilde{F}^{\alpha+\beta}.$$

(ii) *$\mathfrak{m}(l, p)$ is a nilpotent Lie algebra of nilpotency p.*

(iii) *$(\tilde{\epsilon}_1 \ldots \tilde{\epsilon}_l)$ is a minimal system of generators of $\mathfrak{m}(l, p)$.*

(iv) *Let $D_i \in \text{Der } \mathfrak{m}$ $(1 \leqq i \leqq l)$ be defined by*

$$D_i\tilde{\epsilon}_j = \delta_{ij}\tilde{\epsilon}_j,$$

*then $D = \bigoplus_{i=1}^l KD_i$ is a maximal torus on $\mathfrak{m}(l, p)$ and $\mathfrak{m}(l, p)$ is of maximal rank; furthermore $(\tilde{\epsilon}_1 \ldots \tilde{\epsilon}_l)$ is a D-*msg.

(v) *Let $(D^{*1} \ldots D^{*l})$ be the dual basis of D^*. If we identify D_i and α_i then the root space decomposition relative to D is identical to the decomposition of* (i).

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(vi) *Define the* G.C.M. A *by*

$$A_{ij} = -p + 1 \; \forall i \neq j.$$

Then A is associated to $\mathfrak{m}(l, p)$ and $(\tilde{\epsilon}_1 \dots \tilde{\epsilon}_l)$ is ordered relative to A.

Proof. (i), ..., (v) as for $\mathfrak{m}(l, p)$. Since $F(X)$ is free one has

$$(\text{ad } \tilde{\epsilon}_i)^{p-1} \tilde{\epsilon}_j \neq 0;$$

on the other hand

$$(\text{ad } \tilde{\epsilon}_i)^{p} \tilde{\epsilon}_j = 0.$$

This proves (vi).

6.2. PROPOSITION. *Let A be a* G.C.M. *such that*

$$A_{ij} = -p + 1 \; \forall i \neq j.$$

One has the following graded isomorphism $\mathfrak{m}_p(A) \cong \mathfrak{m}(l, p)$ i.e., $\bar{L}_\alpha \cong \tilde{F}^\alpha$ for all $\alpha \in \mathbf{N}^l \backslash (0)$ such that $|\alpha| \leqq p$.

Proof. It is easy to check that H_1 and H_2 of 4.8 are satisfied for p and A. One uses now the universal property of $\mathfrak{m}_p(A)$ (4.11) and of $\mathfrak{m}(l, p)$: any nilpotent Lie algebra of nilpotency p and of type l is a quotient of $\mathfrak{m}(l, p)$ (which comes from 4.2).

6.3. PROPOSITION. ([**1**], [**5**], [**6**]). *The isomorphism classes of nilpotent Lie algebras of nilpotency p and of type l are in bijection with the orbits of $\mathfrak{J}(l, p)$ under the action of $\tilde{\mathfrak{S}}_l$ (We denote by $\mathfrak{J}(l, p)$ the set of homogenous ideals contained in $C^2\mathfrak{m}$ and not contained in $C^p\mathfrak{m}$; the action of $\mathfrak{S}_l$ is $\sigma\tilde{\epsilon}_i = \tilde{\epsilon}_{\sigma i}$.)*

Proof. This follows from 5.10 and 6.2.

7. Examples.

7.1. We shall refer to the tables given in [**4**]. We drop the obvious study of algebras with direct factor.

7.2. *Dimension* 3.

Definition.

$$\mathfrak{g}_3 = Kx_1 \oplus \dots \oplus Kx_3 : [x_1x_2] = x_3.$$

Maximal torus:

$$T = Kt_1 \oplus Kt_2, \; t_ix_j = \delta_{ij}x_j, \; i, j = 1, 2.$$

T-msg: (x_1x_2). Roots:

$$(\beta_1\beta_2), \; \beta_i(t_j) = \delta_{ij}, \; i, j = 1, 2.$$

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Root space decomposition:

$$\mathfrak{g} = \mathfrak{g}^{\beta_1} \oplus \mathfrak{g}^{\beta_2} \oplus \mathfrak{g}^{\beta_1+\beta_2}$$

with

$$\mathfrak{g}^{\beta_1} = Kx_1,\ \mathfrak{g}^{\beta_2} = Kx_2,\ \mathfrak{g}^{\beta_1+\beta_2} = Kx_3.$$

Type: $l = 2$. Nilpotency: $p = 2$. G.C.M. $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = A_2$. Dynkin diagram: (see [7]) $\underset{1}{0} — \underset{2}{0}$. Conclusion:

$$\mathfrak{g}_3 = \mathfrak{m}_2(A_2)/(0) = L_+(A_2).$$

7.3. *Dimension* 4.

Definition:

$$\mathfrak{g}_4 = Kx_1 \oplus \ldots \oplus Kx_4 : [x_1x_2] = x_3,\ [x_1x_3] = x_4.$$

Maximal torus:

$$T = Kt_1 \oplus Kt_2;\ t_ix_j = \delta_{ij}x_j,\ i, j = 1, 2.$$

T-msg: (x_1x_2). Roots: $(\beta_1\beta_2)$:

$$\beta_i(t_j) = \delta_{ij},\ i, j = 1, 2.$$

Root space decomposition:

$$\mathfrak{g} = \mathfrak{g}^{\beta_1} \oplus \mathfrak{g}^{\beta_2} \oplus \mathfrak{g}^{\beta_1+\beta_2} \oplus \mathfrak{g}^{2\beta_1+\beta_2},$$

with

$$\mathfrak{g}^{\beta_1} = Kx_1,\ \mathfrak{g}^{\beta_2} = Kx_2,\ \mathfrak{g}^{\beta_1+\beta_2} = Kx_3,\ \mathfrak{g}^{2\beta_1+\beta_2} = Kx_4.$$

Type: $l = 2$. Nilpotency: $p = 3$. G.C.M.: $A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} = B_2$ Dynkin diagram: $\underset{1}{0} \Rightarrow \underset{2}{0}$. Conclusion:

$$\mathfrak{g}_4 = \mathfrak{m}_3(B_2)/(0) = L_+(B_2).$$

7.4. *Dimension* 5.

7.4.1. *Definition.*

$$\mathfrak{g}_{5,2} = Kx_1 \oplus \ldots \oplus Kx_5 : [x_1x_2] = x_4,\ [x_2x_3] = x_5$$

(we made the following change of notation: $x_1 \leftrightarrow x_2\, x_4 \to -\, x_4$). Maximal torus:

$$T = Kt_1 \oplus \ldots \oplus Kt_3 : t_ix_j = \delta_{ij}x_j,\ i, j = 1, 2, 3.$$

T-msg: $(x_1x_2x_3)$. Roots: $(\beta_1\beta_2\beta_3)$:

$$\beta_i(t_j) = \delta_{ij},\ i, j = 1, 2, 3.$$

Root space decomposition:

$$\mathfrak{g} = \mathfrak{g}^{\beta_1} \oplus \mathfrak{g}^{\beta_2} \oplus \mathfrak{g}^{\beta_3} \oplus \mathfrak{g}^{\beta_1+\beta_2} \oplus \mathfrak{g}^{\beta_2+\beta_3}$$

with

$$\mathfrak{g}^{\beta_1} = Kx_1,\ \mathfrak{g}^{\beta_2} = Kx_2,\ \mathfrak{g}^{\beta_3} = Kx_3,\ \mathfrak{g}^{\beta_1+\beta_2} = Kx_4,\ \mathfrak{g}^{\beta_2+\beta_3} = Kx_5.$$

Type: $l = 2$. Nilpotency: $p = 2$. G.C.M.:

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = A_3.$$

Dynkin diagram:

$$\begin{matrix} 0 & — & 0 & — & 0 \\ 1 & & 2 & & 3. \end{matrix}$$

Conclusion: $\mathfrak{g}_{5,2} = \mathfrak{m}_2(A_3)/(0)$ with

$$\mathfrak{m}_2(A_3) = L_+(A_3)/L_{\alpha_1+\alpha_2+\alpha_3}.$$

7.4.2. *Definition.*

$$\mathfrak{g}_{5,4} = Kx_1 \oplus \ldots \oplus Kx_5 : [x_1x_2] = x_3,\ [x_1x_3] = x_4,\ [x_2x_3] = x_5.$$

Maximal torus:

$$T = Kt_1 \oplus Kt_2 : t_ix_j = \delta_{ij}x_j,\ i, j = 1, 2.$$

T-msg: (x_1x_2). Roots:

$$(\beta_1\beta_2) : \beta_i(t_j) = \delta_{ij},\ i, j = 1, 2.$$

Root space decomposition:

$$\mathfrak{g} = \mathfrak{g}^{\beta_1} \oplus \mathfrak{g}^{\beta_2} \oplus \mathfrak{g}^{\beta_1+\beta_2} \oplus \mathfrak{g}^{2\beta_1+\beta_2} \oplus \mathfrak{g}^{\beta_1+2\beta_2}$$

with

$$\mathfrak{g}^{\beta_1} = Kx_1,\ \mathfrak{g}^{\beta_2} = Kx_2,\ \mathfrak{g}^{\beta_1+\beta_2} = Kx_3,$$
$$\mathfrak{g}^{2\beta_1+\beta_2} = Kx_4,\ \mathfrak{g}^{\beta_1+2\beta_2} = Kx_5.$$

Type: $l = 2$. Nilpotency: $p = 3$. G.C.M.: $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = A_1^{(1)}$. Dynkin diagram: $\begin{matrix} 0 \equiv 0 \\ 1 \quad 2 \end{matrix}$. Conclusion:

$$\mathfrak{g}_{5,4} = \mathfrak{m}_3(A_1^{(1)})/(0)$$

with

$$\mathfrak{m}_3(A_1^{(1)}) = L_+(A_1^{(1)})/\bigoplus_{|\alpha| \geqq 4} L_\alpha.$$

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7.4.3. *Definition.*

$$\mathfrak{g}_{5,5} = Kx_1 \oplus \ldots \oplus Kx_5 : [x_1x_2] = x_3[x_1x_3] = x_4,\ [x_1x_4] = x_5.$$

Maximal torus:

$$T = Kt_1 \oplus Kt_2 : t_ix_j = \delta_{ij}x_j,\ i, j = 1, 2.$$

T-msg (x_1x_2). Roots: $(\beta_1\beta_2)$:

$$\beta_i(t_j) = \delta_{ij},\ i, j = 1, 2.$$

Root space decomposition:

$$\mathfrak{g} = \mathfrak{g}^{\beta_1} \oplus \mathfrak{g}^{\beta_2} \oplus \mathfrak{g}^{\beta_1+\beta_2} \oplus \mathfrak{g}^{2\beta_1+\beta_2} \oplus \mathfrak{g}^{3\beta_1+\beta_2}$$

with

$$\mathfrak{g}^{\beta_1} = Kx_1,\ \mathfrak{g}^{\beta_2} = Kx_2,\ \mathfrak{g}^{\beta_1+\beta_2} = Kx_3,\ \mathfrak{g}^{2\beta_1+\beta_2} = Kx_4,\ \mathfrak{g}^{3\beta_1+\beta_2} = Kx_5.$$

Type: $l = 2$. Nilpotency: $p = 4$. G.C.M.: $A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = G_2$.
Dynkin diagram: $\underset{1}{0} \Rrightarrow \underset{2}{0}$. Conclusion:

$$\mathfrak{g}_{5,5} = \mathfrak{m}_3(G_2)$$

with

$$\mathfrak{m}_3(G_2) = L_+(G_2)/L_{3\alpha_1+2\alpha_2}.$$

8. The semi-simple and the Euclidian (of rank 2) case.

8.1. All through Section 8 we assume that A is of semi-simple type i.e.,

$$A \in \{A_lB_lC_lD_lE_6E_7E_8F_4G_2\}$$

(see [**7**]) or Euclidian (of rank 2) type i.e., $A \in \{A_1^{(1)}, A_2^{(2)}\}$ with

$$A_1^{(1)} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix},\ A_2^{(2)} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$$

(see [**9**]). Those types have in common the fact that $\dim L_\alpha = 1\ \forall \alpha \in \Delta$ (the converse is true). We assume also that H_1 and H_2 of 4.8. hold.

8.2. Denote by $\mathcal{N}_p(A)$ the set of isomorphism classes of nilpotent Lie algebras of maximal rank, of nilpotency p such that A is an associated G.C.M.

8.3. Let $\mathfrak{a}$ be an homogenous ideal of $\mathfrak{m}_p(A)$; then

$$\mathfrak{a} = \bigoplus_{\alpha \in \Delta_p} \mathfrak{a} \cap \bar{L}_\alpha$$

where

$$\Delta_p = \{\alpha \in \Delta_+ ; |\alpha| \leqq p\};$$

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since $\mathfrak{a} \cap \bar{L}_\alpha = (0)$ or $\bar{L}_\alpha$ we have

$$\mathfrak{a} = \bigoplus_{\alpha \in \Delta_p(\mathfrak{a})} \bar{L}_\alpha$$

with

$$\Delta_p(\mathfrak{a}) = \{\alpha \in \Delta_p : \mathfrak{a} \cap \bar{L}_\alpha \neq (0)\}.$$

By 1.5 and 4.4, $C^p\mathfrak{m} \not\subset \mathfrak{a}$ is equivalent to $\Delta_+{}^p \not\subset \Delta_p(\mathfrak{a})$. By 4.7

$$(\mathrm{ad}\, \bar{e}_i)^{-A_{ij}} \bar{e}_j \notin \mathfrak{a}$$

is equivalent to

$$\alpha_j - A_{ij}\alpha_i \notin \Delta_p(\mathfrak{a}).$$

Let E be a subset of Δ_p, and call E ideal of Δ_p if for all $\alpha \in E$ and all $i = 1 \dots l$ such that $\alpha + \alpha_i \in \Delta_p$ one has $\alpha + \alpha_i \in E$. Obviously $\mathfrak{a}$ is an ideal of $\mathfrak{m}$ if and only if $\Delta_p(\mathfrak{a})$ is an ideal of Δ_p. Define

$$\mathfrak{j}_p(A) = \{E \text{ ideal of } \Delta_p; \quad (\mathrm{a})\, \Delta_+{}^p \not\subset E \ (b)\, \alpha_j - A_{ij}\alpha_i \notin E\ \forall i \neq j\}.$$

By the above remarks the map

$$\mathscr{F}_p(A) \to \mathfrak{j}_p(A) \quad \mathfrak{a} \mapsto \Delta_p(\mathfrak{a})$$

is a bijection with inverse

$$E \mapsto \mathfrak{a}_E = \bigoplus_{\alpha \in E} \bar{L}_\alpha.$$

The group $\mathfrak{S}_l(A)$ operates on Δ_p by

$$\sigma\left(\sum d_i\alpha_i\right) = \sum d_i\alpha_{\sigma i}.$$

Denote by $\bar{\mathfrak{j}}_p(A)$ the set of orbits. With the notation of 5.9 (i) and by 5.9 (ii), $\mathfrak{S}_l(A) \cdot \Delta_p(\mathfrak{a})$ does not depend on $\mathfrak{a}$. By 5.10 one gets

8.4. THEOREM. *If A is of semi-simple type or Euclidian type of rank* 2 *and if p satisfies* H_1 *and* H_2 *of* 4.8 *then the $\mathfrak{S}_l(A)$-orbits of $\mathfrak{j}_p(A)$ classify canonically the elements of $\bar{\mathscr{N}}_p(A)$. More precisely, the map*

$$\bar{\mathscr{N}}_p(A) \to \bar{\mathfrak{j}}_p(A) \quad \bar{\mathfrak{g}} \to \mathfrak{S}_l(A) \cdot \Delta_p(\mathfrak{a})$$

($\mathfrak{a}$ *defined in* 5.9 (i)) *is a bijection and* $\mathfrak{S}_l(A) \cdot E \to \overline{(\mathfrak{m}/\mathfrak{a}_E)}$ *is the inverse* ($\mathfrak{a}_E$ *defined in* 8.3).

8.5. *Semi-simple case of rank* 2.

8.5.1. Start with the G.C.M. $A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. The root system is

$$\Delta_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}.$$

The hypotheses H_1 and H_2 give $2 \leqq p \leqq 2$ thus $p = 2$, $\Delta_2 = \Delta_+$, $\Delta_+{}^2 = \{\alpha_1 + \alpha_2\}$. The conditions $\Delta_+{}^2 \not\subset E$ and E ideal of Δ_2 imply $E = \emptyset$. Thus

$\mathfrak{j}_2(A_2) = \{\emptyset\}$ and therefore $\mathfrak{J}_2(A_2) = \{(0)\}$ which gives $\mathcal{N}_2(A_2) = \{\mathfrak{m}_2(A_2)\}$. Since $\mathfrak{m}_2(A_2) = L_+(A_2) = \mathfrak{g}_3$ (7.2) we have the following

THEOREM. *Up to isomorphism* $\mathfrak{g}_3$ *defined in* 7.2 *is the only nilpotent Lie algebra of maximal rank with* A_2 *as an associated* G.C.M.

8.5.2. Case

$$B_2 = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}.$$

Root system:

$$\Delta_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}.$$

Hypothesis: $3 \leqq p \leqq 3$. Consequences:

$$p = 3,\ \Delta_3 = \Delta_+,\ \Delta_+^3 = \{2\alpha_1 + \alpha_2\},\ \mathfrak{j}_3(B_2) = \{\emptyset\},$$
$$\mathcal{F}_3(B_2) = \{(0)\},\ \mathcal{N}_3(B_2) = \{\mathfrak{g}_4\}\ (7.3).$$

THEOREM. *Up to isomorphism* $\mathfrak{g}_4$ *defined in* 7.4 *is the only nilpotent Lie algebra of maximal rank with* B_2 *as an associated* G.C.M.

8.5.3. Case

$$G_2 = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

Root system:

$$\Delta_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}.$$

Hypothesis: $4 \leqq p \leqq 5$. Consequences: $p = 4$ or 5,

$$\Delta_4 = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\},\ \Delta_5 = \Delta_+,$$
$$\Delta_+^4 = \{3\alpha_1 + \alpha_2\},\ \Delta_+^5 = \{3\alpha_1 + 2\alpha_2\},$$
$$\mathfrak{j}_4(G_2) = \mathfrak{j}_5(G_2) = \{\emptyset\},\ \mathfrak{J}_4(G_2) = \mathfrak{J}_5(G_2) = \{(0)\},$$
$$\mathcal{N}_4(G_2) = \{\mathfrak{g}_{5,5}\}\ (7.4.3),\ \mathcal{N}_5(G_2) = \{L_+(G_2)\}.$$

THEOREM. *Up to isomorphism* $\mathfrak{g}_{5,5}$ *defined in* 7.4.3 *and*

$$L_+(G_2) = Kx_1 \oplus \ldots \oplus Kx_6 : [x_1x_2] = x_3,\ [x_1x_3] = x_4,$$
$$[x_1x_4] = x_5,\ [x_2x_5] = [x_3x_4] = x_6$$

are the only nilpotent Lie algebras of maximal rank with G_2 *as an associated* G.C.M.

8.6. *The case of* $A_1^{(1)}$.

8.6.1. We use the presentation of [**9**], Section 3. Let $K[t]$ be the vector space of polynomials with one indeterminate, $K_m[t]$ the vector space of polynomials of degree $<m$ and $sl(2, K) = Kf + Kh + Ke$ with brackets

$[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. If

$$A = A_1^{(1)} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

then

$$L_+(A) = Ke \otimes 1 + sl(2, K) \otimes tK[t];$$

the brackets in $L_+(A)$ are defined by:

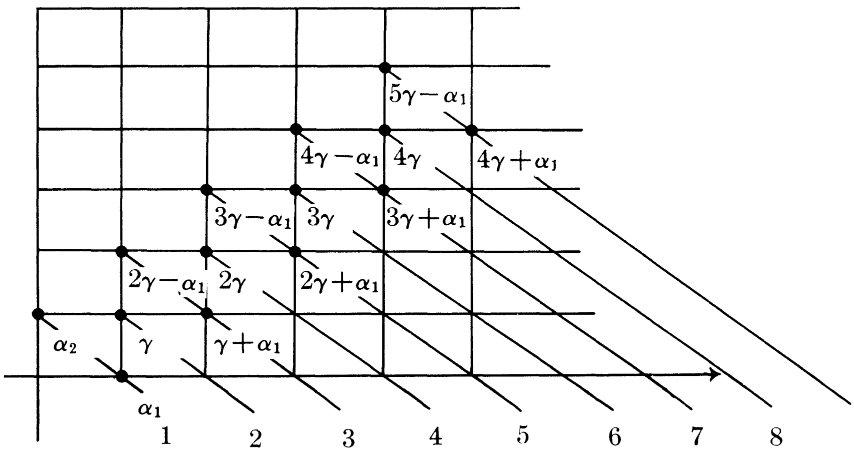
$$[x \otimes t^i, y \otimes t^j] = [x, y] \otimes t^{i+j}.$$

The root spaces are:

$$L_{\alpha_1} = Ke \otimes 1,$$
$$L_{i\gamma-\alpha_1} = Kf \otimes t^i, \ (\gamma = \alpha_1 + \alpha_2)$$
$$L_{i\gamma} = Kh \otimes t^i,$$
$$L_{i\gamma+\alpha_1} = Ke \otimes t^i, \ i \geqq 1.$$

The set of positive roots is

$$\Delta_+ = \{\alpha_1\} \cup \{i\gamma - \alpha_1, i\gamma, i\gamma + \alpha_1; i \geqq 1\}.$$



Picture of Δ_9 for $A_1^{(1)} (\gamma = \alpha_1 + \alpha_2)$

8.6.2. LEMMA. *We have*

$$\mathfrak{j}_{2q}(A_1^{(1)}) = \{\emptyset\} \text{ and}$$
$$\mathfrak{j}_{2q+1}(A_1^{(1)}) = \{\emptyset, \{q\gamma + \alpha_1\}, \{(q + 1)\gamma - \alpha_1\}\}$$

with $q \geqq 2$.

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Proof. The condition $\Delta_+^p \not\subset E$ implies $q\gamma \notin E$ if $p = 2q$ and $\{q\gamma + \alpha_1, (q+1)\gamma - \alpha_1\} \not\subset E$ if $p = 2q + 1$. By the picture, the fact that an element E of $\mathfrak{j}_{2q}$ is an ideal gives $E = \emptyset$; if $E \in \mathfrak{j}_{2q+1}$ then, obviously,

$$E \subsetneq \{q\gamma + \alpha_1, (q+1)\gamma - \alpha_1\},$$

thus

$$E = \emptyset, \{q\gamma + \alpha_1\}, \{(q+1)\gamma - \alpha_1\}.$$

Since $\operatorname{Sup}\{-A_{ij} + 1; i \neq j\} = 2$ we have $q \geqq 2$.

8.6.3. THEOREM. *Up to isomorphism, there exist exactly* 3 *infinite series of nilpotent Lie algebras of maximal rank such that* $A_1^{(1)}$ *is an associated* G.C.M.: (*we write down respectively the algebra* $\mathfrak{g}$, *the nilpotency* p, *the dimension* n, *the element* $\Delta_p(\mathfrak{a})$ *in* $\mathfrak{j}_p(A)$ (8.3), *and the root system* R):

(1) $A_{1,q,1}^{(1)} = Ke \otimes 1 + \bigoplus_{i=1}^{q} sl(2,K) \otimes t^i + Kf \otimes t^{q+1}, q \geqq 1,$

$p = 2q + 1, n = 3q + 2, \Delta_p(\mathfrak{a}) = \emptyset,$

$R_1 = \{\alpha_1\} \cup \{i\gamma - \alpha_1, i\gamma, i\gamma + \alpha_1; 1 \leqq i \leqq q\} \cup \{(q+1)\gamma - \alpha_1\}.$

(2) $A_{1,q,2}^{(1)} = A_{1,q,1}^{(1)} + Kh \otimes t^{q+1}, q \geqq 1,$

$p = 2q + 2, n = 3q + 3, \Delta_p(\mathfrak{a}) = \emptyset,$

$R_2 = R_1 \cup \{(q+1)\gamma\}.$

(3) $A_{1,q,3}^{(1)} = A_{1,q,2}^{(1)} + Ke \otimes t^{q+1}, q \geqq 1,$

$p = 2q + 3, n = 3q + 4, \Delta_p(\mathfrak{a}) = \{(q+2)\gamma - \alpha_1\},$

$R_3 = R_2 \cup \{(q+1)\gamma + \alpha_1\}.$

(*Notations are such that* $\dim A_{1,q,r}^{(1)} = 3q + r + 1$.)

Proof. The ideals $\{q\gamma + \alpha_1\}$ and $\{(q+1)\gamma - \alpha_1\}$ of 8.6.2 are interchanged by non-trivial element of $\mathfrak{S}_l(A)$. We then apply 8.4.

8.6.4. *Remark*. The algebra $\mathfrak{g}_{5,4}$ (7.4.2) given by [**4**] is the first term of the series $(A_{1,q,1})_{q \geqq 1}$.

8.7. *The case* $A_2^{(2)}$.

8.7.1. Let $\mathscr{S} = sl(3, K) = Kf_2 + Kf_1 + Kh_1 + Kh_2 + Ke_1 + Ke_2$ be the Kac-Moody Lie algebra associated to $A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ (1.3). The group $\mathfrak{S}_2(A_2)$ $(= \mathfrak{S}_2 = \{1, \sigma\}\ \sigma: 1 \leftrightarrow 2)$ operates on $\mathscr{S}$ by $\sigma e_i = e_{\sigma i}$, $\sigma f = f_{\sigma i}$, $\sigma h_i = h_{\sigma i}$; the eigenvalues of σ are ± 1 and the eigenspaces are

$$\mathscr{S}_{\pm 1} = \{a \in \mathscr{S}; \sigma a = \pm a\}.$$

We have:

$$\mathcal{S}_{+1} = K(f_1 + f_2) + K(h_1 + h_2) + K(e_1 + e_2)$$

and

$$\mathcal{S}_{-1} = K[f_1 f_2] + K(f_1 - f_2) + K(h_1 - h_2) + K(e_1 - e_2) + K[e_1 e_2].$$

We have

$$L_+(A_2^{(2)}) = K(e_1 + e_2) \otimes 1 + \bigoplus_{i \geqq 1} \mathcal{S}_{+1} \otimes t^{2i} + \bigoplus_{i \geqq 0} \mathcal{S}_{-1} \otimes t^{2i+1}$$

where the brackets in $L_+(A_2^{(2)})$ are defined as in $L_+(A_1^{(1)})$ (8.6.1). The root spaces are

$$L_{\alpha_1} = K(e_1 + e_2) \otimes 1,$$
$$L_{(2i+1)\gamma - 2\alpha_1} = K[f_1 f_2] \otimes t^{2i+1},\ \gamma = 2\alpha_1 + \alpha_2,$$
$$L_{(2i+1)\gamma - \alpha_1} = K(f_1 - f_2) \otimes t^{2i+1},$$
$$L_{(2i+1)\gamma} = K(h_1 - h_2) \otimes t^{2i+1},$$
$$L_{(2i+1)\gamma + \alpha_1} = K(e_1 - e_2) \otimes t^{2i+1},$$
$$L_{(2i+1)\gamma + 2\alpha_1} = K[e_1 e_2] \otimes t^{2i+1},\ (\text{with } i \geqq 0),$$
$$L_{2j\gamma - \alpha_1} = K(f_1 + f_2) \otimes t^{2j},$$
$$L_{2j\gamma} = K(h_1 + h_2) \otimes t^{2j},$$
$$L_{2j\gamma + \alpha_1} = K(e_1 + e_2) \otimes t^{2j},\ (\text{with } j \geqq 1).$$

The set of positive roots is

$$\Delta_+ = \{\alpha\} \cup \{2i\gamma + k\alpha_1;\ i \geqq 1,\ k = 0, 1\} \cup \{(2i + 1)\gamma + k\alpha_1;\ i \geqq 0,\ k = 0,\ \pm 1,\ \pm 2\}.$$

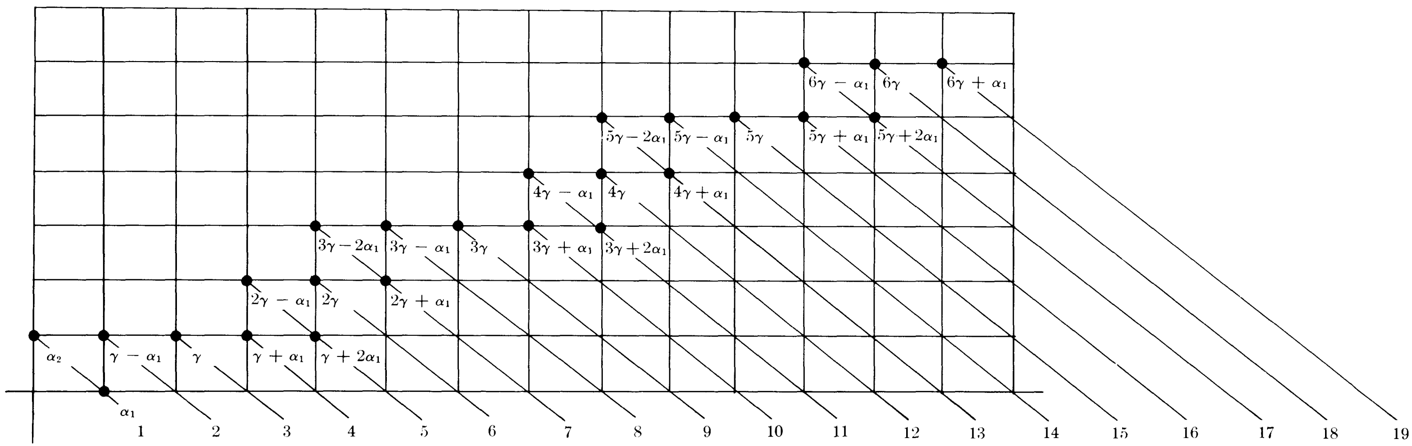
(See [**9**] for details.)

8.7.2. LEMMA. *We have*

$$\mathfrak{j}_{6q+r} = \{\emptyset\} \text{ for } q \geqq 1,\ r = 0, 2, 3, 4,$$
$$\mathfrak{j}_{6q+1} = \{\emptyset, \{2q\gamma + \alpha_1\}, \{(2q + 1)\gamma - 2\alpha_1\}\} \text{ for } q \geqq 1 \text{ and}$$
$$\mathfrak{j}_{6q+5} = \{\emptyset, \{(2q + 1)\gamma + 2\alpha_1\}, \{(2q + 2)\gamma - \alpha_1\}\} \text{ for } q \geqq 1.$$

Proof. This follows as for $A_1^{(1)}$ (8.6.2) with the help of the picture.

8.7.3. THEOREM. *Up to isomorphism there exist exactly* 10 *infinite series of nilpotent Lie algebras of maximal rank such that* $A_2^{(2)}$ *is an associated*



Picture of $\Delta_{19}(\gamma = 2\alpha_1 + \alpha_2)$.

G.C.M. (*we use same notations as in* 8.6.3):

(1) $A^{(2)}_{2,q,1} = K(e_1 + e_2) \otimes 1 + \bigoplus_{i=1}^{q} \mathscr{S}_{+1} \otimes t^{2i} + \bigoplus_{i=0}^{q} \mathscr{S}_{-1} \otimes t^{2i+1}, q \geqq 0,$
$p = 6q + 5, n = 8q + 6, \Delta_p(\mathfrak{a}) = \{(2q + 2)\gamma - \alpha_1\},$
$R_1 = \{\alpha_1\} \cup \{2i\gamma + k\alpha_1; 1 \leqq i \leqq q, k = 0, \pm 1\}$
$\cup \{(2i + 1)\gamma + k\alpha_1; 0 \leqq i \leqq q, k = 0, \pm 1, \pm 2\}.$

(2) $A^{(2)}_{2,q,2} = A^{(2)}_{2,q,1} + K(f_1 + f_2) \otimes t^{2q+2}, q \geqq 0,$
$p = 6q + 5, n = 8q + 7, \Delta_p(\mathfrak{a}) = \emptyset,$
$R_2 = R_1 \cup \{(2q + 2)\gamma - \alpha_1\}.$

(3) $A^{(2)}_{2,q,3} = A^{(2)}_{2,q,2} + K(h_1 + h_2) \otimes t^{2q+2},$
$p = 6q + 6, n = 8q + 8, \Delta_p(\mathfrak{a}) = \emptyset,$
$R_3 = R_2 \cup \{(2q + 2)\gamma\}.$

(4) $A^{(2)}_{2,q,4} = A^{(2)}_{2,q,3} + K(e_1 + e_2) \otimes t^{2q+2},$
$p = 6q + 7, n = 8q + 9, \Delta_p(\mathfrak{a}) = \{(2q + 3)\gamma - 2\alpha_1\},$
$R_4 = R_3 \cup \{(2q + 2)\gamma + \alpha_1\}.$

(5) $A^{(2)}_{2,q,5} = A^{(2)}_{2,q,3} + K[f_1, f_2] \otimes t^{2q+3}, q \geqq 0,$
$p = 6q + 7, n = 8q + 9, \Delta_p(\mathfrak{a}) = \{(2q + 2)\gamma + \alpha_1\},$
$R_5 = R_3 \cup \{(2q + 3)\gamma - 2\alpha_1\}.$

(6) $A^{(2)}_{2,q,6} = A^{(2)}_{2,q,3} + K(e_1 + e_2) \otimes t^{2q+2} + K[f_1, f_2] \otimes t^{2q+3}, q \geqq 0,$
$p = 6q + 7, n = 8q + 10, \Delta_p(\mathfrak{a}) = \emptyset,$
$R_6 = R_3 \cup \{(2q + 2)\gamma + \alpha_1, (2q + 3)\gamma - 2\alpha_1\}.$

(7) $A^{(2)}_{2,q,7} = A^{(2)}_{2,q,6} + K(f_1 - f_2) \otimes t^{2q+3}, q \geqq 0$
$p = 6q + 8, n = 8q + 11, \Delta_p(\mathfrak{a}) = \emptyset,$
$R_7 = R_6 \cup \{(2q + 3)\gamma - \alpha_1\}.$

(8) $A^{(2)}_{2,q,8} = A^{(2)}_{2,q,7} + K(h_1 - h_2) \otimes t^{2q+3}, q \geqq 0,$
$p = 6q + 9, n = 8q + 12, \Delta_p(\mathfrak{a}) = \emptyset,$
$R_8 = R_7 \cup \{(2q + 3)\gamma\}.$

(9) $A^{(2)}_{2,q,9} = A^{(2)}_{2,q,8} + K(e_1 - e_2) \otimes t^{2q+3}, q \geqq 0$
$p = 6q + 10, n = 8q + 13, \Delta_p(\mathfrak{a}) = \emptyset,$
$R_9 = R_8 \cup \{(2q + 3)\gamma + \alpha_1\}.$

(10) $A^{(2)}_{2,q,10} = A^{(2)}_{2,q,9} + K(f_1 + f_2) \otimes t^{2q+4}, q \geqq 0,$
$p = 6q + 11, n = 8q + 14, \Delta_p(\mathfrak{a}) = \{(2q + 3)\gamma + 2\alpha_1\},$
$R_{10} = R_9 \cup \{(2q + 4)\gamma - \alpha_1\}.$

Proof. This follows as for $A_1^{(1)}$.

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University of Poitiers,
Poitiers, Cedex, France

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