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A Locally Compact Non Divisible Abelian Group Whose Character Group Is Torsion Free and Divisible

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Abstract. It was claimed by Halmos in 1944 that if G is a Hausdorff locally compact topological abelian group and if the character group of G is torsion free, then G is divisible. We prove that such a claim is false by presenting a family of counterexamples. While other counterexamples are known, we also present a family of stronger counterexamples, showing that even if one assumes that the character group of G is both torsion free and divisible, it does not follow that G is divisible.

1 Introduction

Let G be an abelian group[1]. Given an integer n, we consider the subgroups of G defined by

$$nG = \{nx : x \in G\}, \quad G[n] = \{x \in G : nx = 0\}.$$

If G is an abelian topological group, then its *character group* $\widehat{G}$ is the abelian group of all continuous homomorphisms $\xi\colon G \to S^1$, where S^1 is the (multiplicative) circle group of unitary complex numbers; the group $\widehat{G}$ is endowed with the compact-open topology. The celebrated *Pontryagin duality theorem* (see, for instance, [7]) states that if G is a Hausdorff locally compact abelian topological group, then its character group $\widehat{G}$ is a Hausdorff locally compact abelian topological group as well and the character group of $\widehat{G}$ is G itself; more precisely, the map that associates each $x \in G$ with the evaluation map $\widehat{G} \ni \xi \mapsto \xi(x) \in S^1$ is a homeomorphic isomorphism between G and the character group of $\widehat{G}$.

If H is a subgroup of G, then the *annihilator* of H is the subgroup $\operatorname{ann}(H)$ of $\widehat{G}$ consisting of all characters $\xi\colon G \to S^1$ that are trivial over H. Clearly, given an integer n,

$$\operatorname{ann}(nG) = \widehat{G}[n].$$

In particular, if G is divisible, *i.e.,* if $nG = G$ for every nonzero integer n, then its character group $\widehat{G}$ is torsion free, *i.e.,* $\widehat{G}[n]$ is trivial for every nonzero integer n. It was claimed by Halmos [5] that the converse is true if G is Hausdorff locally compact. The argument presented in [5] has a gap: if $\widehat{G}$ is torsion free, then $\operatorname{ann}(nG)$ is trivial for every nonzero integer n, but that, in principle, implies only that nG is dense[2] in

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[1] Except for the circle group S^1, abelian groups will be written additively.

[2] If $\operatorname{ann}(nG)$ is trivial, then nG is indeed dense in G. Otherwise, Pontryagin duality would give us a nontrivial character on the (nontrivial) quotient of G by the closure of nG and such nontrivial character would correspond to a nontrivial element of $\operatorname{ann}(nG)$.

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G, not that $nG = G$. It should be observed, however, that the claim made by Halmos is true if G is either compact or discrete, and that the proof of his main result is not affected by the incorrect claim.

In Section 3, we will present a family of examples of Hausdorff locally compact abelian topological groups G such that nG is dense in G for every nonzero integer n, but such that $nG \neq G$ for some nonzero integer n. In particular, any such group G is an example of a Hausdorff locally compact abelian topological group that is not divisible, but whose character group is torsion free. While other examples of that phenomenon are known (see [1, 4.16]), in Section 4 we will also present a family of examples of Hausdorff locally compact abelian topological groups G that are *both* divisible and torsion free, but such that $\widehat{G}$ is (torsion free but) not divisible. In particular, by Pontryagin duality, it follows that $\widehat{G}$ is a Hausdorff locally compact abelian topological group whose character group (which is isomorphic to G) is *both* divisible and torsion free, but still $\widehat{G}$ is not divisible.

2 Extending the Topology of a Subgroup

Let us start by presenting a general construction of a topology on an abelian group from a topology on a given subgroup (the construction is well known; see, for instance, [2–4]). Let G be an abelian group and H be a subgroup of G. Assume that H is endowed with a topology that makes it into a topological group. We claim that there exists a unique topology on G such that

(i) G is a topological group;
(ii) the given topology of H is inherited from G;
(iii) H is open in G.

Such a topology is constructed as follows. Given $g \in G$, the coset $g + H$ of H can be endowed with a topology by requiring that the translation map

$$L_g \colon H \ni x \longmapsto g + x \in g + H$$

be a homeomorphism. The fact that the translation maps of H are homeomorphisms of H implies that the topology defined on the coset $g + H$ does not depend on the representative g of the coset. We topologize G by making it the topological sum of the cosets $g+H$, $g \in G$. That is, we say that U is open in G if $U \cap (g+H)$ is open in $g+H$ for every $g \in G$. One readily checks that such a topology is the only topology on G satisfying (i), (ii), and (iii). Notice that since the cosets of H are all homeomorphic to H and open in G, it follows that if H is Hausdorff, then so is G. Moreover, since every compact neighborhood of the neutral element in H is also a compact neighborhood of the neutral element in G, it follows that G is locally compact if H is locally compact.

3 The First Family of Counterexamples

Let A be a Hausdorff compact abelian topological group that is not divisible, and let B be a divisible abelian group such that A is a subgroup of B (for instance, let $B = S^1$ and A be a nontrivial finite subgroup of S^1 endowed with the discrete topology).

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Let B^ω denote the group of all sequences $(x_k)_{k\in\omega}$ of elements of B and let G denote the subgroup of B^ω consisting of those sequences $(x_k)_{k\in\omega}$ such that x_k is in A for k sufficiently large. Let $H = A^\omega$ denote the subgroup of G consisting of sequences in A. We endow H with the product topology and G with the unique topology satisfying (i), (ii), and (iii) of Section 2. Then H is a Hausdorff compact topological group and thus G is a Hausdorff locally compact topological group. If n is a nonzero integer, then the subgroup nG of G consists of those sequences $(x_k)_{k\in\omega}$ such that x_k is in nA for k sufficiently large. If n_0 is a nonzero integer such that $n_0A \neq A$, then $n_0G \neq G$ and therefore G is not divisible. We will show that if n is a nonzero integer, then nG is dense in G and from this it will follow from the discussion in the introduction that the character group $\widehat{G}$ is torsion free. Let J denote the subgroup of G consisting of sequences $(x_k)_{k\in\omega}$ in B that are trivial for k sufficiently large. Since J is obviously contained in nG for any nonzero integer n, it suffices to prove that J is dense in G in order to establish that nG is dense in G for every nonzero integer n. Clearly, $G = H + J$, so that J intersects every coset of H. Now let us prove that J is dense in G by proving that $J \cap (x + H)$ is dense in $x + H$ for every coset $x + H$ of H in G. Since the coset $x + H$ intersects J, we can assume that $x \in J$. Thus, the translation map $L_x\colon H \to x + H$ is a homeomorphism that carries $J \cap H$ to $J \cap (x + H)$. From the definition of the product topology, it is obvious that $J \cap H$ is dense in H and therefore $J \cap (x + H)$ is dense in $x + H$. This concludes the proof that the subgroup J is dense in G.

4 The Family of Stronger Counterexamples

We will now present an example of a Hausdorff locally compact abelian topological group G that is both divisible and torsion free, but such that its character group $\widehat{G}$ is not divisible. We need some preliminary lemmas.

Lemma 1 *Let G be an abelian divisible topological group. If there exists an open subgroup H of G, a nonzero integer n, and a* discontinuous *homomorphism $\phi\colon H \to S^1$ that is trivial over nH, then the character group $\widehat{G}$ is not divisible.*

Proof Since S^1 is divisible, ϕ extends to a (obviously discontinuous) homomorphism $\phi'\colon G \to S^1$. Consider the homomorphism $\xi\colon G \to S^1$ defined by $\xi(x) = \phi'(nx)$, for all $x \in G$. Then ξ is trivial over H and, since H is open, ξ is continuous. Assuming by contradiction that $\widehat{G}$ is divisible, we can find a continuous homomorphism $\alpha\colon G \to S^1$ such that $\alpha(x)^n = \alpha(nx) = \xi(x)$ for all $x \in G$. Then α and ϕ' are equal over nG and since G is divisible, we obtain that $\alpha = \phi'$, contradicting the continuity of α. ■

Lemma 2 *Let K be an abelian group endowed with a topology[3]. If K admits a proper dense subgroup D, then there exists a discontinuous homomorphism from K to S^1.*

Proof Since K/D is a nontrivial abelian group, there exists a nontrivial homomorphism $\phi\colon K/D \to S^1$ (start with a nontrivial S^1-valued homomorphism defined over

[3] It is not necessary that K be a topological group, *i.e.,* the continuity of the operations of K is not used in the proof.

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a nontrivial cyclic subgroup of K/D and then extend it to all of K/D using the fact that S^1 is divisible). The composition of ϕ with the quotient map $K \to K/D$ is a nontrivial homomorphism that is trivial over D, and therefore it must be discontinuous. ■

Corollary 3 *Let G be an abelian divisible topological group. If there exists an open subgroup H of G and a nonzero integer n such that H/nH (endowed with the quotient topology) has a proper dense subgroup, then the character group $\widehat{G}$ is not divisible.*

Proof By Lemma 2, there exists a discontinuous S^1-valued homomorphism over H/nH; its composition with the quotient map $H \to H/nH$ is a discontinuous S^1-valued homomorphism over H that is trivial over nH. The conclusion follows from Lemma 1. ■

The construction of our family of stronger counterexamples goes as follows. Let A be a Hausdorff compact abelian non divisible topological group and let B be a torsion free divisible abelian group such that A is a subgroup of B. A concrete example of groups A, B satisfying the required conditions will be supplied at the end of the section. Let $H = A^\omega$ denote the group of all sequences in A endowed with the product topology, and let $G = B^\omega$ be the group of all sequences in B, endowed with the unique topology satisfying (i), (ii), and (iii) of Section 2. The group H is Hausdorff compact and thus G is Hausdorff locally compact; moreover, like B, the group G is both divisible and torsion free. We use Corollary 3 to establish that the character group $\widehat{G}$ is not divisible. Let n be a nonzero integer such that $nA \neq A$. We claim that if H/nH is endowed with the quotient topology, then it has a proper dense subgroup. First, we check that the quotient topology of H/nH coincides with the product topology of $(A/nA)^\omega$, each factor A/nA being endowed with the quotient topology. Namely, if A/nA is endowed with the quotient topology, then the quotient map $A \to A/nA$ is continuous, open, and surjective; therefore, if $H/nH \cong (A/nA)^\omega$ is endowed with the product topology, then the quotient map $H \to H/nH$ is also continuous, open, and surjective and therefore it is a topological quotient map. This observation proves that the product topology of $(A/nA)^\omega$ coincides with the quotient topology of H/nH. Now it follows directly from the definition of the product topology that the subgroup of $H/nH \cong (A/nA)^\omega$ consisting of sequences $(x_k)_{k\in\omega}$ that are trivial for k sufficiently large is a (proper) dense subgroup. This concludes the proof that $\widehat{G}$ is not divisible.

Finally, let us present a concrete example of groups A, B satisfying the required conditions. Let A be the group of p-adic integers (where p is some fixed prime number) and B be the p-adic field. We have $pA \neq A$, so that A is not divisible; moreover, B is a field of characteristic zero, so that it is both torsion free and divisible as an abelian group. The fact that A can be made into a Hausdorff compact topological group follows from the observation that A is (isomorphic to) the character group of the discrete p-quasicyclic group $\mathbb{Z}(p^\infty)$ of elements of S^1 whose order is a power of p (see, for instance, [6, Proposition 3.1]) and that the character group of a discrete topological group is compact.

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