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AN ORDER-THEORETICAL CHARACTERIZATION OF VARIETIES OF BANDS

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ABSTRACT. We give a description of band varieties by properties of the compatibility of certain quasi-orders λ_n and ρ_n.

1. **Introduction.** Recently Gerhard and Petrich [1987a] have found a very fine way to characterize every member of the lattice of proper band varieties. This is based upon the inductive definition of three sequences of words G_n, H_n and I_n.

$$G_2 = x_2x_1, \quad H_2 = x_2, \quad I_2 = x_2x_1x_2$$
$$G_n = x_n\bar{G}_{n-1}, \quad T_n = G_nx_n\bar{T}_{n-1} \text{ for } T \in \{H, I\} \text{ and } n \geq 3$$

where $\bar{w}$, the dual word to w, is obtained from w by reversing the order of variables. That means if $w = x_1x_2\cdots x_n$ then $\bar{w} = x_nx_{n-1}\cdots x_1$.

Figure 1 shows the shape of the lattice of proper band varieties, determined independently by Birjukov [1970] and Gerhard [1970], and the characterization of the different varieties using the system of identities proved in Gerhard and Petrich [1987a].

Gerhard and Petrich [1987b] have given several charcterizations for each variety. The aim of this paper is to find a different description by (quasi-)ordertheoretical means (compare with Figure 2).

2. **A new characterization of band varieties.** We need the following relations introduced by Nambooripad [1975], which are closely connected with the well-known ordering of idempotents. Let S be a semigroup. Define relations ω^l, ω^r on $E(S)$ by

$$e\omega^l f \iff e = e.f \qquad e\omega^r f \iff e = f.e \qquad \omega := \omega^l \cap \omega^r$$

The proof of the following lemma is easy.

LEMMA 2.1. *ω^l and ω^r are quasi-orders (that is reflexive and transitive relations) on $E(S)$.* ■

We can now inductively define the following relations ($\mathcal{L}^0$ and $\mathcal{R}^0$ denote the greatest congruence contained in $\mathcal{L}$ resp. $\mathcal{R}$)

$$\lambda_1 := \omega^l,\ \rho_1 := \omega^r,\ \mathcal{L}_1 := \mathcal{L},\ \mathcal{R}_1 := \mathcal{R}$$

(2.2) for $n \geq 2$: $a\lambda_nb \iff \bar{a}\rho_{n-1}\bar{b}$ in $B/\mathcal{L}^0$; $\quad \mathcal{L}_n := \lambda_n \cap \lambda_n^{-1}$

$a\rho_nb \iff \bar{a}\lambda_{n-1}\bar{b}$ in $B/\mathcal{R}^O$; $\quad \mathcal{R}_n := \rho_n \cap \rho_n^{-1}$

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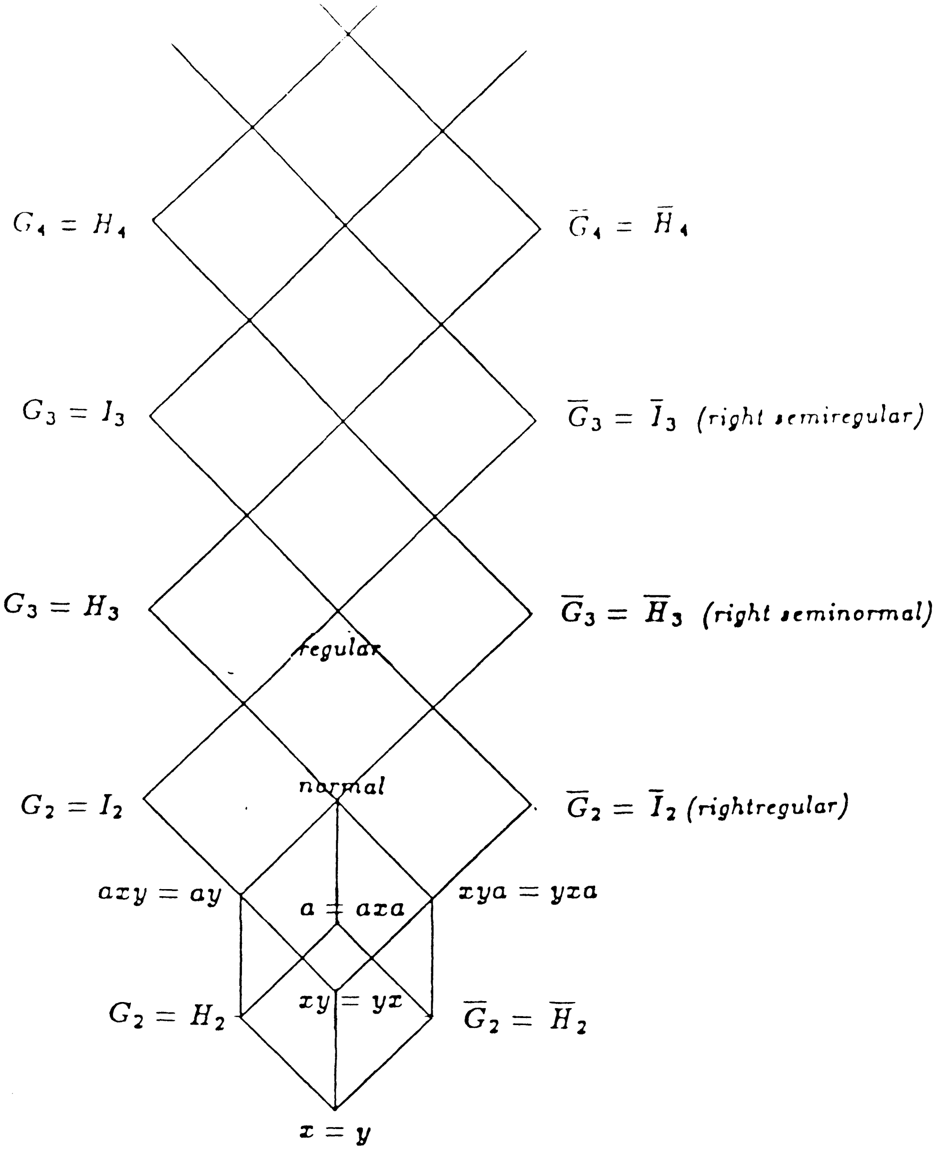

Figure 1

where $\bar{a}$ is the congruence class of a in $\mathcal{L}^0$ resp. in $\mathcal{R}^0$.

REMARK. $\mathcal{L}_1 = \lambda_1 \cap \lambda_1^{-1}$, $\mathcal{R}_1 = \rho_1 \cap \rho_1^{-1}$.

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LEMMA 2.3. *λ_n and ρ_n are quasi-orders for any regular semigroup S.*

PROOF. By induction: $n = 1$ is proved by Lemma 2.1.
$n \longrightarrow n+1$:

- •) reflexive: let $a \in B \Rightarrow \bar{a}\rho_n\bar{a}$ in $B/\mathcal{L}^0$ by inductive assumption, i.e., by definition $a\lambda_{n+1}a$ and λ_{n+1} are reflexive; it can be shown very similarly that ρ_{n+1} is reflexive.
- •) transitive: let $a\lambda_{n+1}b$ and $b\lambda_{n+1}c$, i.e., $\bar{a}\rho_n\bar{b}$ and $\bar{b}\rho_n\bar{c}$ in $B/\mathcal{L}^0$ and induction hypothesis gives $\bar{a}\rho_n\bar{c}$ in $B/\mathcal{L}^0$, i.e., by definition $a\lambda_{n+1}c$; similarly for ρ_{n+1}. ■

It follows from Lemma 2.3 that $\mathcal{L}_n$ and $\mathcal{R}_n$ are equivalence relations on S and we can define

$$\mathcal{D}_n := \mathcal{L}_n \vee \mathcal{R}_n \text{ for } n \geq 1 \text{ in the lattice of equivalence relations.}$$

PROPOSITION 2.4. *Let S be a regular semigroup. Then the following holds for any $n \in \mathbb{N}$:*

(i) λ_n is compatible on the right, if n is odd;
(ii) λ_n is compatible on the left, if n is even;
(iii) ρ_n is compatible on the left, if n is odd;
(iv) ρ_n is compatible on the right, if n is even.

PROOF. Let $a\lambda_1 b$, i.e., $a = ab$; then the hypothesis implies that for $c \in S : ac = abc = a(bc)(bc) = (ab)cbc = (ac)(bc)$; i.e., $ac\lambda_1 bc$ and λ_1 is compatible on the right. One can show similarly that ρ_1 is compatible on the left. The rest of the proposition is proved by induction like Lemma 2.3, using the definition. ■

In the sequel, we shall use the following result several times (compare Gerhard-Petrich [1987b; 2.2]). Let B be an idempotent semigroup. Then for any $a, b \in B$ it follows that:

$$a\mathcal{L}^0 b \iff xa\mathcal{L}xb \quad \forall x \in B$$
$$a\mathcal{R}^0 b \iff ax\mathcal{R}bx \quad \forall x \in B$$

We give another description of the quasi-orders λ_n and ρ_n.

LEMMA 2.5. *Let B be an idempotent semigroup Then the following holds for any $n \geq 2$:*

$$a\lambda_n b \iff \begin{cases} ax\mathcal{L}_{n-1}abx & \forall x \in B \quad \text{if } n \text{ is odd} \\ xa\mathcal{L}_{n-1}xba & \forall x \in B \quad \text{if } n \text{ is even} \end{cases}$$
$$a\rho_n b \iff \begin{cases} ax\mathcal{R}_{n-1}abx & \forall x \in B \quad \text{if } n \text{ is even} \\ xa\mathcal{R}_{n-1}xba & \forall x \in B \quad \text{if } n \text{ is odd} \end{cases}$$

PROOF. Proceed by induction:
$n = 2$: Let $a\lambda_2 b$; i.e., by definition $\bar{a}\rho_1\bar{b}$ in $B/\mathcal{L}^0 \iff \bar{a} = \bar{b}\bar{a}$ in $B/\mathcal{L}^0 \iff a\mathcal{L}^0 ba \iff xa\mathcal{L}xba \; \forall x \in B$ analogous for ρ_2.

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$n \longrightarrow n+1$: Let $a\lambda_{n+1}b$, and without loss of generality, let n be even; Definition 2.2 yields $\bar{a}\rho_n\bar{b}$ in $B/\mathcal{L}^0$. Now we have

$$\begin{aligned} \bar{a}\rho_n\bar{b} &\iff \bar{a}\bar{x}\mathcal{R}_{n-1}\bar{a}\bar{b}\bar{x} \quad \forall \bar{x} \in B/\mathcal{L}^0 \text{ by induction hypothesis} \\ &\iff \overline{ax}\rho_{n-1}\overline{abx} \text{ and } \overline{abx}\rho_{n-1}\overline{ax} \quad \forall x \in B \text{ by Definition 2.2} \\ &\iff ax\lambda_n abx \text{ and } abx\lambda_n ax \quad \forall x \in B \text{ by Definition 2.2} \\ &\iff ax\mathcal{L}_n abx \quad \forall x \in B \text{ by Definition of } \mathcal{L}_n. \end{aligned}$$

This shows the first assertion of the lemma. The second assertion follows similarly. ∎

We can now show a very interesting connection of the equivalence relations $\mathcal{L}_n$ and $\mathcal{R}_n$ with Green's relation $\mathcal{D}$:

COROLLARY 2.6. *$\mathcal{D}_n \subseteq \mathcal{D}$ $\forall n \in \mathbb{N}$ for every band B.*

PROOF. Is given by induction

$n = 1$: trivial.

$n \longrightarrow n+1$: without loss of generality, let n be odd and $a\mathcal{L}_{n+1}b$; that is $a\lambda_{n+1}b$ and $b\lambda_{n+1}a$. Therefore,

$$\begin{aligned} xa\mathcal{L}_n xba &\quad \forall x \in B \\ \text{and } xb\mathcal{L}_n xab &\quad \forall x \in B \text{ by Lemma 5.} \end{aligned}$$

Induction hypothesis gives: $xa\mathcal{D}xba$ and $xb\mathcal{D}xab$ for all $x \in B$. Let $x = a$, resp. $x = b$, so we get $a\mathcal{D}aba\mathcal{D}bab\mathcal{D}b$. Analogously it may be shown that $\mathcal{R}_{n+1} \subseteq \mathcal{D}$ and therefore $\mathcal{D}_{n+1} = \mathcal{L}_{n+1} \vee \mathcal{R}_{n+1} \subseteq \mathcal{D}$. ∎

To show the main theorem we need the following result, which is proved in Gerhard-Petrich [1987b; 4.5 and 5.3].

THEOREM 2.7. *For every idempotent semigroup B, the following holds:*

(i) B satisfies $G_n = H_n \iff B/\mathcal{L}^0$ satisfies $\begin{cases} \overline{xya = yxa} & \text{for } n = 3 \\ \overline{G_{n-1}} = \overline{H_{n-1}} & \text{for } n > 3. \end{cases}$

(ii) B satisfies $\overline{G_n} = \overline{H_n} \iff B/\mathcal{R}^0$ satisfies $\begin{cases} axy = ayx & \text{for } n = 3 \\ G_{n-1} = H_{n-1} & \text{for } n > 3 \end{cases}$

(iii) B satisfies $G_n = I_n \iff B/\mathcal{L}^0$ satisfies $\overline{G_{n-1}} = \overline{I_{n-1}}$ for $n \geq 3$

(iv) B satisfies $\overline{G_n} = \overline{I_n} \iff B/\mathcal{R}^0$ satisfies $G_{n-1} = I_{n-1}$ for $n \geq 3$. ∎

Now it is possible to describe each variety of bands included in Theorem 2.7 by properties of the compatibility of the quasi-orders λ_n and ρ_n, respectively of the equivalence relations $\mathcal{L}_n$ and $\mathcal{R}_n$.

THEOREM 2.8. *For every idempotent semigroup B and $n \geq 3$, the following holds:*

(i) B satisfies $G_n = H_n \iff \lambda_{n-2}$ is compatible on B

(ii) B satisfies $\overline{G_n} = \overline{H_n} \iff \rho_{n-2}$ is compatible on B

(iii) B satisfies $G_n = I_n \iff \mathcal{L}_{n-2}$ is compatible on B (and therefore $\mathcal{L}_{n-2}$ is a congruence)

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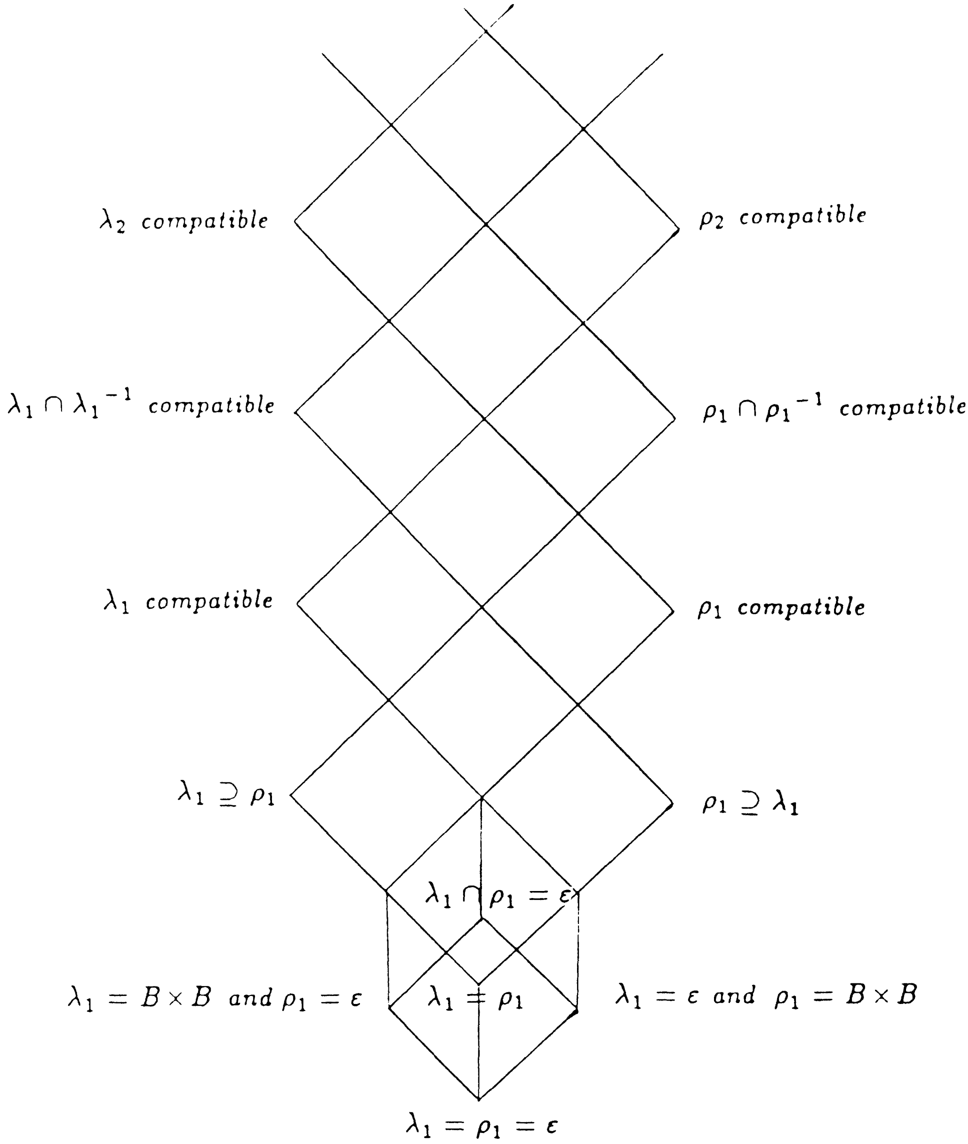

Figure 2

(iv) *B satisfies* $\overline{G_n} = \overline{I_n} \iff \mathcal{R}_{n-2}$ *is compatible on B (and therefore* $\mathcal{R}_{n-2}$ *is a congruence).*

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PROOF. The proof is given by induction on n over the parts (i)–(iv):

Let $n = 3$: ad (i):($\Rightarrow$) Let B satisfy $G_3 = H_3$; i.e. B is leftseminormal. Therefore we have $cab = cabcb$ in B. Let $a, b \in B$ and $a\lambda_1 b$; i.e. $a = ab$. Then we get $(ca)(cb) = cabcb = cab = ca$. This is equivalent with $ca\lambda_1 cb$; which means that λ_1 is left compatible and therefore compatible by Proposition 2.4.

($\Leftarrow$) Let $a, b \in B$ and $a \leq b$; i.e. $a\lambda_1 b$ and $a\rho_1 b$; for ρ_1 is left compatible by Proposition 2.4; the hypothesis yields, that $ca\lambda_1 cb$ and $ca\rho_1 cb$. This is equivalent with $ca \leq cb$; which means that $\leq$ is left compatible and the statement follows directly from Petrich [1977; II.3.15]. The proof of (iii) follows directly from Petrich [1977; II.3.5]. That ones for (ii) and (iv) are similar.

$n \longrightarrow n+1$: ad(i): let B satisfy $G_{n+1} = H_{n+1}$; by Theorem 2.7 we have proved that $B/\mathcal{L}^0$ satisfies $\overline{G_n} = \overline{H_n}$; by induction hypothesis this is equivalent with ρ_{n-2} is compatible on $B/\mathcal{L}^0$; i.e. $\bar{a}\rho_{n-2}\bar{b} \Rightarrow \bar{a}\bar{c}\rho_{n-2}\bar{b}\bar{c}$ and $\bar{c}\bar{a}\rho_{n-2}\bar{c}\bar{b}$ in $B/\mathcal{L}^0$; which means by definition $a\lambda_{n-1}b \Rightarrow ac\lambda_{n-1}bc$ and $ca\lambda_{n-1}cb$ for all $c \in B$. That means that λ_{n-1} is compatible on B. The induction steps for (ii), (iii) and (iv) are similar. ∎

Figure 2 indicates the characterization of band varieties with Theorem 8. The inner points in the diagram can be described as intersections of two peripheral points.

The points forming the bottom of Figure 2 are exceptions. So we have to give the following proof for the class of left regular bands.

LEMMA 2.9. *For an idempotent semigroup B the following are equivalent:*
(i) B is left regular
(ii) $\rho_1 \subseteq \lambda_1$.

PROOF. (i)$\longrightarrow$(ii): As B is left regular, we have $ab = aba$. Now let $a\rho_1 b$, which means $a = ba \Rightarrow a = aba = ab$. This implies $a\lambda_1 b$ showing (ii).

(ii)$\longrightarrow$(i): Let $a\mathcal{D}b$. This is equivalent with $a = (ab)a$ and $b = (ba)b$; which means $a\rho_1 ab$ and $b\rho_1 ba$. Now (ii) gives $a\lambda_1 ab$ and $b\lambda_1 ba$. This is equivalent with $a = ab$ and $b = ba$; which means $a\mathcal{L}b$. Therefore $\mathcal{D} = \mathcal{L}$ on B and B is left regular by Petrich [1977; II.3.12]. ∎

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