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KOSTANT'S PROBLEM AND PARABOLIC SUBGROUPS

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Abstract. Let $\mathfrak{g}$ be a finite dimensional complex semi-simple Lie algebra with Weyl group W and simple reflections S. For $I \subseteq S$ let $\mathfrak{g}_I$ be the corresponding semi-simple subalgebra of $\mathfrak{g}$. Denote by W_I the Weyl group of $\mathfrak{g}_I$ and let $w_\circ$ and $w_\circ^I$ be the longest elements of W and W_I, respectively. In this paper we show that the answer to Kostant's problem, i.e. whether the universal enveloping algebra surjects onto the space of all ad-finite linear transformations of a given module, is the same for the simple highest weight $\mathfrak{g}_I$-module $L_I(x)$ of highest weight $x \cdot 0$, $x \in W_I$, as the answer for the simple highest weight $\mathfrak{g}$-module $L(xw_\circ^I w_\circ)$ of highest weight $xw_\circ^I w_\circ \cdot 0$. We also give a new description of the unique quasi-simple quotient of the Verma module $\Delta(e)$ with the same annihilator as $L(y)$, $y \in W$.

1. Introduction. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be a finite dimensional complex semi-simple Lie algebra with a chosen triangular decomposition, and let $\mathcal{U}(\mathfrak{g})$ be its universal enveloping algebra. For two $\mathfrak{g}$-modules M and N, the space $\mathrm{Hom}_{\mathbb{C}}(M, N)$ of linear maps from M to N has a $\mathcal{U}(\mathfrak{g})$-bimodule structure in the natural way (see for example [**12**, Kapitel 6]), and hence a $\mathfrak{g}$-module structure via the adjoint action. The $\mathfrak{g}$-submodule of $\mathrm{Hom}_{\mathbb{C}}(M, N)$ consisting of all locally finite elements is in fact a $\mathcal{U}(\mathfrak{g})$-sub-bimodule, which we denote by $\mathcal{L}(M, N)$. As $\mathcal{U}(\mathfrak{g})$ itself is locally finite under the adjoint action, we have a natural homomorphism of $\mathcal{U}(\mathfrak{g})$ into $\mathcal{L}(M, M)$ for every $\mathfrak{g}$-module M, whose kernel is the annihilator $\mathrm{Ann}\, M$ of M in $\mathcal{U}(\mathfrak{g})$. The question raised by Kostant (see for example [**4**, 6.10], [**15**]) is: for which $\mathfrak{g}$-modules M is the natural inclusion

$$\mathcal{U}(\mathfrak{g})/\,\mathrm{Ann}\, M \hookrightarrow \mathcal{L}(M, M)$$

a *surjection*.

This is in general a difficult question, and the answer is not even known for simple highest weight modules. It is known to have a positive answer for Verma modules ([**4**, 6.9] for simple Verma modules, generalised in [**15**, 6.4] for the general case) and for all quotients of dominant Verma modules [**12**, 6.9]. For semi-simple Lie algebras having roots of different length, examples of simple highest weight modules where the answer is negative were found early (see for example [**5**, 6.5], [**15**, 9.5]). More recently, many examples have also been found in type A (see [**19**, **28**]). The answer to Kostant's problem is a valuable tool for example when determining Goldie rank ratios (see [**16**–**18**]), and in the study of generalised Verma modules (see [**21**, **27**, **29**]).

In this note, we investigate how the answer to this question for certain simple highest weight $\mathfrak{g}$-modules relates to the answer for modules of semi-simple subalgebras of $\mathfrak{g}$. More precisely, let W be the Weyl group of $\mathfrak{g}$, with simple reflections S, determined by the triangular decomposition. For a subset $I \subseteq S$, let W_I denote the parabolic subgroup of W generated by I, denote by $\mathfrak{g}_I$ the corresponding semi-simple subalgebra of $\mathfrak{g}$, and let $w_\circ$ and $w_\circ^I$ denote the longest elements of W and W_I. For $x \in W$, let $L(x)$ denote the simple highest weight $\mathfrak{g}$-module with highest weight $x \cdot 0$ (see next section for precise definition), and similarly, for $x \in W_I$, let $L_I(x)$ denote the simple highest weight $\mathfrak{g}_I$-module with highest weight $x \cdot 0$. The main result of this paper is the following theorem.

THEOREM 1.1. *Let $x \in W_I$. Then Kostant's problem has a positive answer for $L_I(x)$ if and only if Kostant's problem has a positive answer for $L(xw_\circ^I w_\circ)$.*

Since Kostant's problem trivially has a positive answer for $L_I(e)$, as well as for $L_I(x)$ for any simple reflection x (see [**26**, Proposition 7] or [**19**, Corollary 18]), this generalises previous results by Conze-Berline and Duflo [**5**, 2.12 and 6.3], later generalised by Gabber and Joseph [**8**, 4.4] (the case when $x = e$), and Mazorchuk [**26**, Theorem 1] (the case when x is a simple reflection).

The idea of the proof is as follows. For each $x \in W_I$, there is a unique quotient D of the dominant Verma module $\Delta_I(e)$ satisfying $\operatorname{Ann} D = \operatorname{Ann} L_I(x)$. Since Kostant's problem has a positive answer for D, as it is a quotient of a dominant Verma module, we see that Kostant's problem has a positive answer for $L_I(x)$ if and only if

$$\mathcal{L}_I(D, D) \cong \mathcal{L}_I(L_I(x), L_I(x)) \tag{1}$$

(where the index I is used to emphasise that objects are defined with respect to $\mathfrak{g}_I$ as opposed to $\mathfrak{g}$). We show that we can 'lift' this situation by parabolic induction, i.e. there exists a $\mathfrak{g}$-module D' for which the answer to Kostant's problem is positive, and such that

$$\mathcal{L}(D', D') \cong \mathcal{L}(L(xw_\circ^I w_\circ), L(xw_\circ^I w_\circ))$$

holds if and only if (1) holds.

In Section 5, we give an alternative description of the so-called *quasi-simple* quotients the dominant Verma module, originally described in [**14**, Section 5], which are used as an important tool in the proof of Theorem 1.1. Finally, in Section 6 we apply Theorem 1.1 to get some new answers to Kostant's problem for the Lie algebra $\mathfrak{sl}_6$.

2. Notation and preliminaries. The subset I of S determines a parabolic subalgebra $\mathfrak{p}_I$ of $\mathfrak{g}$, containing $\mathfrak{g}_I$. The triangular decomposition of $\mathfrak{g}$ induces a triangular decomposition $\mathfrak{g}_I = \mathfrak{n}_I^- \oplus \mathfrak{h}_I \oplus \mathfrak{n}_I$. Let $\mathfrak{u}_I$ be the nilradical of $\mathfrak{p}_I$, and let $\mathfrak{z}_I$ be the orthogonal complement of $\mathfrak{h}_I$ in $\mathfrak{h}$ with respect to the Killing form. We thus have the following decompositions:

$$\mathfrak{h} = \mathfrak{h}_I \oplus \mathfrak{z}_I, \text{ and } \mathfrak{p}_I = \mathfrak{g}_I \oplus \mathfrak{z}_I \oplus \mathfrak{u}_I.$$

The Weyl group W of $\mathfrak{g}$ acts on $\mathfrak{h}^*$ in the natural way $w\lambda$, but in this setting it is more convenient to consider the so-called 'dot action', given by

$$w \cdot \lambda := w(\lambda + \rho) - \rho,$$

where ρ is the half sum of the positive roots. Similarly, we have both the standard action and dot action of W_I on $\mathfrak{h}_I^*$.

Let $\mathcal{O}$ denote the Bernstein-Gelfand-Gelfand (BGG) category (see for example [**3**, **10**]), and let $\mathcal{O}_0$ denote the principal block of $\mathcal{O}$, i.e. the full subcategory of $\mathcal{O}$ consisting of modules that are annihilated by some power of the maximal ideal of the centre of $\mathcal{U}(\mathfrak{g})$ which annihilates the trivial module. The simple modules of $\mathcal{O}_0$ are the simple highest weight modules $L(w)$ of highest weight $w \cdot 0$, where w runs over W. We denote the Verma module with simple head $L(w)$ by $\Delta(w)$, and the projective cover of $L(w)$ by $P(w)$. Finally, for $w \in W$ we denote by θ_w the indecomposable projective functor on $\mathcal{O}_0$ (see [**2**]) satisfying

$$\theta_w \Delta(e) = P(w).$$

The corresponding objects for $\mathfrak{g}_I$ are denoted $\mathcal{O}^I$, $L_I(w)$, $\mathcal{L}_I$, etc.

For a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ (here $\mathfrak{a}$ will be either $\mathfrak{h}_I$ or $\mathfrak{z}_I$), a module $M \in \mathcal{O}$, and $\lambda \in \mathfrak{a}^*$, let

$$M_\lambda := \{\, m \in M \mid xm = \lambda(x)m \text{ for all } x \in \mathfrak{a} \,\},$$

and define the support of M with respect to $\mathfrak{a}$ as

$$\operatorname{Supp}_{\mathfrak{a}} M := \{\, \lambda \in \mathfrak{a}^* \mid M_\lambda \neq 0 \,\}.$$

3. Parabolic induction. For $\lambda \in \mathfrak{z}_I^*$, we define the *induction functor* from $\mathcal{O}^I$ to $\mathcal{O}$ by

$$\operatorname{Ind}_\lambda M := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}_I)} M^\lambda$$

for $M \in \mathcal{O}^I$, where M^λ is the $\mathfrak{p}_I$-module obtained from M by letting $\mathfrak{z}_I$ act by λ, and $\mathfrak{u}_I$ act by 0. We also define the *restriction functor* from $\mathcal{O}$ to $\mathcal{O}^I$ by

$$\operatorname{Res}_\lambda N := N_\lambda$$

for $N \in \mathcal{O}$, where the action is restricted to $\mathfrak{g}_I$.

LEMMA 3.1. *If* $\operatorname{Ann}_{\mathcal{U}(\mathfrak{g}_I)} M = \operatorname{Ann}_{\mathcal{U}(\mathfrak{g}_I)} N$ *for two* $\mathfrak{g}_I$*-modules* M *and* N*, then* $\operatorname{Ann}_{\mathcal{U}(\mathfrak{g})} \operatorname{Ind}_\lambda M = \operatorname{Ann}_{\mathcal{U}(\mathfrak{g})} \operatorname{Ind}_\lambda N$ *for any* $\lambda \in \mathfrak{z}_I^*$.

Proof. We have

$$\begin{aligned}\operatorname{Ann}_{\mathcal{U}(\mathfrak{p}_I)} M^\lambda = \big(\operatorname{Ann}_{\mathcal{U}(\mathfrak{g}_I)} M\big) \otimes \mathcal{U}(\mathfrak{z}_I) \otimes \mathcal{U}(\mathfrak{u}_I) + \mathcal{U}(\mathfrak{g}_I) \otimes \ker \lambda \otimes \mathcal{U}(\mathfrak{u}_I)\\ + \mathcal{U}(\mathfrak{g}_I) \otimes \mathcal{U}(\mathfrak{z}_I) \otimes \mathcal{U}(\mathfrak{u}_I)_{>0},\end{aligned}$$

where $\mathcal{U}(\mathfrak{u}_I)_{>0}$ denotes the elements of $\mathcal{U}(\mathfrak{u}_I)$ of degree at least 1. Hence $\operatorname{Ann}_{\mathcal{U}(\mathfrak{p}_I)} M^\lambda = \operatorname{Ann}_{\mathcal{U}(\mathfrak{p}_I)} N^\lambda$, so the result follows from [**6**, Proposition 5.1.7(ii)]. □

Let R_I be the simple roots corresponding to I. The fundamental weights of $\mathfrak{h}^*$ orthogonal to R_I define a basis B_I of $\mathfrak{z}_I^*$, which in turn define a partial order on $\mathfrak{z}_I^*$ by $\nu \leqslant \lambda$ for $\nu, \lambda \in \mathfrak{z}_I^*$ if $\lambda - \nu$ is in the non-negative span of B_I. For $\lambda \in \mathfrak{z}_I^*$ and $M \in \mathcal{O}$, let $M_{\not\leqslant\lambda}$ be the submodule of M generated by all M_ν, $\nu \not\leqslant \lambda$, and define

$$M^{\leqslant\lambda} := M / M_{\not\leqslant\lambda}.$$

Generalising the situation when tensoring Verma modules with finite dimensional modules, we get the following.

LEMMA 3.2. *For a finite dimensional* $\mathfrak{g}$*-module* V, $M \in \mathcal{O}^I$, *and* $\lambda \in \mathfrak{z}_I^*$, *the module* $V \otimes \operatorname{Ind}_\lambda M$ *has a filtration*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = V \otimes \operatorname{Ind}_\lambda M$$

with

$$M_i/M_{i-1} \cong \operatorname{Ind}_{\lambda+\mu_i}((\operatorname{Res}_{\mu_i} V) \otimes M),$$

where $\mu_1 > \mu_2 > \cdots > \mu_k \in \mathfrak{z}_I^*$ *and* $\operatorname{Supp}_{\mathfrak{z}_I} V = \{\mu_1, \ldots, \mu_k\}$.

Proof. Let $\mu_1, \ldots, \mu_k \in \mathfrak{z}_I^*$ be as in the lemma, let $B_1, \ldots, B_k$ be bases of $\operatorname{Res}_{\mu_1} V, \ldots, \operatorname{Res}_{\mu_k} V$, and let B be a basis of M. Now define

$$M_i := \sum_{1 \leq j \leq i} \mathcal{U}(\mathfrak{g})(B_j \otimes (1 \otimes_{\mathcal{U}(\mathfrak{p}_I)} B)).$$

As in the 'standard' case (se for instance [**12**, Satz 2.2]) we find that each M_i is $\mathcal{U}(\mathfrak{u}_I^-)$-free over

$$\bigcup_{1 \leq j \leq i} B_j \otimes (1 \otimes_{\mathcal{U}(\mathfrak{p}_I)} B).$$

In particular, as $\mathcal{U}(\mathfrak{u}_I^-)$-modules we have that

$$M_i/M_{i-1} \cong \mathcal{U}(\mathfrak{u}_I^-)(B_i \otimes (1 \otimes_{\mathcal{U}(\mathfrak{p}_I)} B)).$$

Furthermore, it is straightforward to see that, as $\mathcal{U}(\mathfrak{g}_I)$-modules,

$$\mathcal{U}(\mathfrak{g}_I)(B_i \otimes (1 \otimes_{\mathcal{U}(\mathfrak{p}_I)} B)) \cong (\operatorname{Res}_{\mu_i} V) \otimes M,$$

from which the statement follows. □

COROLLARY 3.3. *For any* $\lambda, \mu \in \mathfrak{z}_I^*$*, finite dimensional* $\mathfrak{g}$*-module* V, *and* $M \in \mathcal{O}^I$, *we have*

$$\operatorname{Res}_\mu(V \otimes \operatorname{Ind}_\lambda M)^{\leqslant \mu} \cong (\operatorname{Res}_{\mu-\lambda} V) \otimes M.$$

Proof. If $\mu - \lambda \notin \operatorname{Supp}_{\mathfrak{z}_I} V$ the result is immediate as both modules are zero. On the other hand, if $\mu - \lambda \in \operatorname{Supp}_{\mathfrak{z}_I} V$, then by Lemma 3.2 the module $(V \otimes \operatorname{Ind}_\lambda M)^{\leqslant \mu}$ has a submodule M' isomorphic to

$$\operatorname{Ind}_\mu((\operatorname{Res}_{\mu-\lambda} V) \otimes M),$$

and

$$\operatorname{Supp}_{\mathfrak{z}_I}((V \otimes \operatorname{Ind}_\lambda M)^{\leqslant \mu}/M') < \mu,$$

from which the statement follows. □

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We now fix $\xi \in \mathfrak{z}_I^*$ to be the restriction of $w_\circ \cdot 0$ to $\mathfrak{z}_I$, and let $\mathcal{O}^\xi$ be the full subcategory of $\mathcal{O}$ of modules satisfying $\operatorname{Supp}_{\mathfrak{z}_I} M \leqslant \xi$. By [**26**, Proposition 11], Ind_ξ and Res_ξ induce mutually inverse equivalences between $\mathcal{O}_0^\xi$ and $\mathcal{O}_0^I$, identifying $L_I(x)$ with $L(xw_\circ^I w_\circ)$ and $\Delta_I(x)$ with $\Delta(xw_\circ^I w_\circ)$. Let pr_0 and pr_0^I denote the projection functors from $\mathcal{O}$ to $\mathcal{O}_0$ and $\mathcal{O}^I$ to $\mathcal{O}_0^I$, respectively.

LEMMA 3.4. *For any $M \in \mathcal{O}^\xi$ we have*

$$\operatorname{Res}_\xi \circ \operatorname{pr}_0(M) \cong \operatorname{pr}_0^I \circ \operatorname{Res}_\xi(M).$$

Proof. Let $\lambda \in \mathfrak{h}^*$ with $\lambda|_{\mathfrak{z}_I} \leq \xi$. If $\lambda|_{\mathfrak{z}_I} < \xi$ then

$$\operatorname{Res}_\xi \circ \operatorname{pr}_0(L(\lambda)) = \operatorname{pr}_0^I \circ \operatorname{Res}_\xi(L(\lambda)) = 0,$$

so assume $\lambda|_{\mathfrak{z}_I} = \xi$. We then have that

$$\operatorname{Res}_\xi L(\lambda) \cong L_I(\lambda|_{\mathfrak{h}_I}).$$

Furthermore, since $\lambda|_{\mathfrak{z}_I} = (w_\circ \cdot 0)|_{\mathfrak{z}_I}$, we have that

$$\operatorname{pr}_0 L(\lambda) \cong \begin{cases} L(\lambda) & \text{if } \lambda \in W_I w_\circ \cdot 0, \text{ or equivalently, } \lambda|_{\mathfrak{h}_I} \in W_I \cdot 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the statement follows for simple modules since

$$\operatorname{pr}_0^I L_I(\lambda|_{\mathfrak{h}_I}) = \begin{cases} L_I(\lambda|_{\mathfrak{h}_I}) & \text{if } \lambda|_{\mathfrak{h}_I} \in W_I \cdot 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $M \in \mathcal{O}^\xi$, let $M_0 = \operatorname{pr}_0 M$ and let $M_1 \in \mathcal{O}^\xi$ be such that

$$M \cong M_0 \oplus M_1.$$

By definition, we have

$$\operatorname{Res}_\xi \circ \operatorname{pr}_0 M \cong \operatorname{Res}_\xi M_0. \tag{2}$$

Let $L(\lambda)$ be a composition factor of M_1. If $\lambda|_{\mathfrak{z}_I} < \xi$ then $\operatorname{Res}_\xi L(\lambda) = 0$, and if $\lambda|_{\mathfrak{z}_I} = \xi$ we must have $\lambda|_{\mathfrak{z}_I} \notin W_I \cdot 0$, so $\operatorname{pr}_0^I \circ \operatorname{Res}_\xi L(\lambda) = 0$. Since both restriction and projection are exact it follows that

$$\operatorname{pr}_0^I \circ \operatorname{Res}_\xi M_1 = 0.$$

On the other hand, since $M_0 \in \mathcal{O}_0^\xi$ we have $\operatorname{Res}_\xi M_0 \in \mathcal{O}_0^I$, so

$$\operatorname{pr}_0^I \circ \operatorname{Res}_\xi M_0 \cong \operatorname{Res}_\xi M_0.$$

Since both restriction and projection are additive, it follows that

$$\operatorname{pr}_0^I \circ \operatorname{Res}_\xi M \cong \operatorname{Res}_\xi M_0.$$

Comparing with equation (2) yields the result. □

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4. Proof of Theorem 1.1. We start by proving the building blocks used in the proof of Theorem 1.1. We retain the notation from Section 3. In particular, we have fixed $\xi \in \mathfrak{z}_I^*$ to be the restriction of $w_\circ \cdot 0$ to $\mathfrak{z}_I$.

PROPOSITION 4.1. *For each finite dimensional $\mathfrak{g}$-module V and $M, N \in \mathcal{O}_0^I$, we have*

$$\mathrm{Hom}_{\mathfrak{g}}(V \otimes \mathrm{Ind}_\xi M, \mathrm{Ind}_\xi N) \cong \mathrm{Hom}_{\mathfrak{g}_I}(\mathrm{Res}_0 V \otimes M, N).$$

Proof. We have that

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{g}}(V \otimes \mathrm{Ind}_\xi M, \mathrm{Ind}_\xi N) &\cong \mathrm{Hom}_{\mathfrak{g}}(\mathrm{pr}_0(V \otimes \mathrm{Ind}_\xi M)^{\leqslant \xi}, \mathrm{Ind}_\xi N)\\
&\cong \mathrm{Hom}_{\mathfrak{g}_I}(\mathrm{Res}_\xi \circ \mathrm{pr}_0(V \otimes \mathrm{Ind}_\xi M)^{\leqslant \xi}, N)\\
&\cong \mathrm{Hom}_{\mathfrak{g}_I}(\mathrm{pr}_0^I \circ \mathrm{Res}_\xi(V \otimes \mathrm{Ind}_\xi M)^{\leqslant \xi}, N)\\
&\cong \mathrm{Hom}_{\mathfrak{g}_I}(\mathrm{Res}_\xi(V \otimes \mathrm{Ind}_\xi M)^{\leqslant \xi}, N)\\
&\cong \mathrm{Hom}_{\mathfrak{g}_I}(\mathrm{Res}_0 V \otimes M, N),
\end{aligned}$$

where the first isomorphism follows from the fact that $\mathrm{Ind}_\xi N \in \mathcal{O}_0^\xi$, the second by the adjointness of Res_ξ and Ind_ξ, the third by Lemma 3.4, the fourth by the fact that $N \in \mathcal{O}_0^I$, and the fifth by Corollary 3.3. □

COROLLARY 4.2. *For $M, N \in \mathcal{O}_0^I$ we have*

$$\mathrm{Hom}_{\mathfrak{g}_I}(V, \mathcal{L}_I(M, M)) \cong \mathrm{Hom}_{\mathfrak{g}_I}(V, \mathcal{L}_I(N, N))$$

for all finite dimensional $\mathfrak{g}_I$-modules V if and only if

$$\mathrm{Hom}_{\mathfrak{g}}(V', \mathcal{L}(\mathrm{Ind}_\xi M, \mathrm{Ind}_\xi M)) \cong \mathrm{Hom}_{\mathfrak{g}}(V', \mathcal{L}(\mathrm{Ind}_\xi N, \mathrm{Ind}_\xi N))$$

for all finite dimensional $\mathfrak{g}$-modules V'.

Proof. For the 'only if' part, by Proposition 4.1 and [**12**, 6.8 (3)] we have

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{g}}(V', \mathcal{L}(\mathrm{Ind}_\xi M, \mathrm{Ind}_\xi M)) &\cong \mathrm{Hom}_{\mathfrak{g}}(V' \otimes \mathrm{Ind}_\xi M, \mathrm{Ind}_\xi M)\\
&\cong \mathrm{Hom}_{\mathfrak{g}_I}(\mathrm{Res}_0 V' \otimes M, M)\\
&\cong \mathrm{Hom}_{\mathfrak{g}_I}(\mathrm{Res}_0 V', \mathcal{L}(M, M))\\
&\cong \mathrm{Hom}_{\mathfrak{g}_I}(\mathrm{Res}_0 V', \mathcal{L}(N, N))\\
&\cong \mathrm{Hom}_{\mathfrak{g}_I}(\mathrm{Res}_0 V' \otimes N, N)\\
&\cong \mathrm{Hom}_{\mathfrak{g}}(V' \otimes \mathrm{Ind}_\xi N, \mathrm{Ind}_\xi N)\\
&\cong \mathrm{Hom}_{\mathfrak{g}}(V', \mathcal{L}(\mathrm{Ind}_\xi N, \mathrm{Ind}_\xi N))
\end{aligned}$$

for all finite dimensional $\mathfrak{g}$-modules V'. Similarly, for the 'if' part, we find that

$$\mathrm{Hom}_{\mathfrak{g}_I}(\mathrm{Res}_0 V', \mathcal{L}_I(M, M)) \cong \mathrm{Hom}_{\mathfrak{g}_I}(\mathrm{Res}_0 V', \mathcal{L}_I(N, N))$$

for all finite dimensional $\mathfrak{g}$-modules V'. We need to show that this covers all relevant finite dimensional $\mathfrak{g}_I$-modules. We first note that

$$\mathrm{Hom}_{\mathfrak{g}_I}(V \otimes M, M) \neq 0$$

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only if $V_0 \neq \{0\}$, where V_0 denotes the $\mathfrak{h}_I$-invariant subspace of V. This follows from the fact that

$$\mathrm{Supp}_{\mathfrak{h}_I}(V \otimes M) \subseteq \mathrm{Supp}_{\mathfrak{h}_I} V + \mathrm{Supp}_{\mathfrak{h}_I} M$$

and, since $M \in \mathcal{O}_I$,

$$\mathrm{Supp}_{\mathfrak{h}_I} M \subset \mathbb{Z}R_I,$$

while, if $V_0 = \{0\}$,

$$\mathrm{Supp}_{\mathfrak{h}_I} V \cap \mathbb{Z}R_I = \emptyset.$$

On the other hand, extending the highest weights of V from $\mathfrak{g}_I$ to $\mathfrak{g}$ and using the classification of finite dimensional $\mathfrak{g}$-modules (see [**6**, Theorem 7.2.6]) we have that if $V_0 \neq \{0\}$ then there is a finite dimensional $\mathfrak{g}$-module V' such that V is a direct summand of $\mathrm{Res}_0 V'$. Now the result follows by induction on the dimension of V. □

We need the following crucial observation made by V. Mazorchuk.

PROPOSITION 4.3. *Kostant's problem has a positive answer for any quotient of* $\Delta(w_\circ^I w_\circ)$.

Proof. Consider a short exact sequence

$$0 \to X \to \Delta\big(w_\circ^I w_\circ\big) \to Y \to 0.$$

By [**26**, Proposition 5], we need to show that

$$\mathrm{Ext}^1_{\mathcal{O}}\left(\Delta\big(w_\circ^I w_\circ\big), \theta_x X\right) = 0$$

for all $x \in W$. Let C_x and T_x denote the completion functor and the twisting functor associated with $x \in W$, respectively, and let $\mathcal{R}C_x$ and $\mathcal{L}T_x$ denote the corresponding right and left derived functors. They satisfy

$$C_x\Delta(w_\circ) \cong \Delta(x^{-1}w_\circ), \text{ and } T_x\Delta(x^{-1}w_\circ) \cong \Delta(w_\circ),$$

they form mutually inverse equivalences of the bounded derived category $\mathcal{D}^b(\mathcal{O})$, and they commute with projective functors (all this can be found in [**1**, **22**]). Hence we have

$$\begin{aligned}\mathrm{Ext}^1_{\mathcal{O}}\left(\Delta\big(w_\circ^I w_\circ\big), \theta_x X\right) &\cong \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}\left(\Delta\big(w_\circ^I w_\circ\big)[-1], \theta_x X\right)\\ &\cong \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}\left(\mathcal{R}C_{w_\circ^I}\Delta(w_\circ)[-1], \theta_x X\right)\\ &\cong \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}\left(\theta_{x^{-1}}\Delta(w_\circ)[-1], \mathcal{L}T_{w_\circ^I} X\right).\end{aligned}$$

To study $\mathcal{L}T_{w_\circ^I} X$, we note that $X \in \mathcal{O}_0^\xi$, and take a projective resolution

$$P^\bullet \twoheadrightarrow X : \qquad 0 \to P_k \to \cdots \to P_1 \to P_0 \to X \to 0,$$

of X in this category.

In $\mathcal{D}^b(\mathcal{O})$ we now have $X \cong P^\bullet$. Since $\mathcal{O}_0^\xi$ is equivalent to $\mathcal{O}_0^I$, since all projective modules in $\mathcal{O}_0^I$ have Verma flags, and since the equivalence maps Verma modules to

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Verma modules, the modules in $P^\bullet$ have Verma flags. Since $T_{w_\circ^I}$ is acyclic on such modules we have $\mathcal{L}T_{w_\circ^I}P^\bullet = T_{w_\circ^I}P^\bullet$, and hence we have

$$\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}\big(\theta_{x^{-1}}\Delta(w_\circ)[-1], \mathcal{L}T_{w_\circ^I}X\big) \cong \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}\big(\theta_{x^{-1}}\Delta(w_\circ)[-1], T_{w_\circ^I}P^\bullet\big).$$

For $x \in W_I$, let $\tilde{P}(xw_\circ^I w_\circ)$ denote the projective cover of the simple module $L(xw_\circ^I w_\circ)$ in $\mathcal{O}_0^{\leqslant\lambda}$. We have that

$$\Delta\big(w_\circ^I w_\circ\big) = \tilde{P}\big(w_\circ^I w_\circ\big),$$

and, analogous to $\mathcal{O}_0$, for each $x \in W_I$ there is a projective functor $\tilde{\theta}_x$ such that

$$\tilde{P}\big(xw_\circ^I w_\circ\big) \cong \tilde{\theta}_x\Delta\big(w_\circ^I w_\circ\big).$$

Since twisting functors commute with projective functors we have

$$T_{w_\circ^I}\tilde{P}\big(xw_\circ^I w_\circ\big) \cong T_{w_\circ^I}\tilde{\theta}_x\Delta\big(w_\circ^I w_\circ\big) \cong \tilde{\theta}_x T_{w_\circ^I}\Delta\big(w_\circ^I w_\circ\big) \cong \tilde{\theta}_x\Delta(w_\circ).$$

Since $\Delta(w_\circ)$ is a tilting module, and projective functors take tilting modules to tilting modules, we have that $T_{w_\circ^I}\tilde{P}(xw_\circ^I w_\circ)$ is a tilting module for all $x \in W_I$. In particular, $T_{w_\circ^I}P^\bullet$ is a complex of tilting modules. Similarly, $\theta_{x^{-1}}\Delta(w_\circ)$ is a tilting module, and hence we have

$$\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}\big(\theta_{x^{-1}}\Delta(w_\circ)[-1], T_{w_\circ^I}P^\bullet\big) \cong \mathrm{Hom}_{\mathcal{K}^b(\mathcal{O})}\big(\theta_{x^{-1}}\Delta(w_\circ)[-1], T_{w_\circ^I}P^\bullet\big),$$

by [**9**, Chapter III(2), Lemma 2.1], where $\mathcal{K}^b(\mathcal{O})$ is the bounded homotopy category. Since $\theta_{x^{-1}}\Delta(w_\circ)[-1]$ is concentrated in position 1, and $\mathcal{L}T_{w_\circ^I}P^\bullet$ lies between position 0 and $-k$, this last Hom-space must be zero. □

We can now put the above results together to prove Theorem 1.1.

Proof of Theorem 1.1. By [**16**, Lemma 3.3], there is a (unique) quotient D of $\Delta_I(e)$ satisfying $\mathrm{Ann}\, L_I(x) = \mathrm{Ann}\, D$, and Kostant's problem has a positive answer for D, since D is a quotient of the dominant Verma module (see for example [**11**, 6.9]). Hence we have

$$\mathcal{L}_I(D, D) \cong \mathcal{U}(\mathfrak{g}_I)/\,\mathrm{Ann}\, D \cong \mathcal{U}(\mathfrak{g}_I)/\,\mathrm{Ann}\, L_I(x) \hookrightarrow \mathcal{L}_I(L_I(x), L_I(x)). \quad (3)$$

Furthermore, since $L(xw_\circ^I w_\circ) \cong \mathrm{Ind}_\xi\, L_I(x)$ we have

$$\mathrm{Ann}\, L\big(xw_\circ^I w_\circ\big) = \mathrm{Ann}\,\mathrm{Ind}_\xi\, D$$

by Lemma 3.1. Since $\mathrm{Ind}_\xi\, D$ is a quotient of $\mathrm{Ind}_\xi\,\Delta_I(w_\circ^I) \cong \Delta(w_\circ^I w_\circ)$, Kostant's problem has a positive answer for $\mathrm{Ind}_\xi\, D$ by Proposition 4.3. As above, we have

$$\mathcal{L}(\mathrm{Ind}_\xi\, D, \mathrm{Ind}_\xi\, D) \hookrightarrow \mathcal{L}\big(L\big(xw_\circ^I w_\circ\big), L\big(xw_\circ^I w_\circ\big)\big). \quad (4)$$

If Kostant's problem has a positive answer for $L(x)$ then the injection (3) is a bijection, so by Corollary 4.2 we have

$$\mathrm{Hom}_{\mathfrak{g}}(V, \mathcal{L}(\mathrm{Ind}_\xi\, D, \mathrm{Ind}_\xi\, D)) \cong \mathrm{Hom}_{\mathfrak{g}}\left(V, \mathcal{L}\big(L\big(xw_\circ^I w_\circ\big), L\big(xw_\circ^I w_\circ\big)\big)\right)$$

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for all finite dimensional $\mathfrak{g}$-modules V. Hence the injection (4) is a bijection, and Kostant's problem has the positive answer for $L(xw_\circ^I w_\circ)$. The proof of the converse is completely analogous. □

5. Alternative description of D. The module D used in the proof of Theorem 1.1 can be described as follows. If we set $J = \operatorname{Ann} L(x)$, then by [**16**, Lemma 3.3], $J\Delta(e)$ is the unique submodule of $\Delta(e)$ satisfying

$$\operatorname{Ann}(\Delta(e)/J\Delta(e)) = \operatorname{Ann} L(x).$$

In particular, $D := \Delta(e)/J\Delta(e)$ is the unique quotient of $\Delta(e)$ satisfying $\operatorname{Ann} D = \operatorname{Ann} L(x)$. Furthermore, the category $\mathcal{O}$ is equivalent to a certain category of Harish-Chandra bimodules, and under this equivalence the module D corresponds to the bimodule

$$\mathcal{U}(\mathfrak{g})/\operatorname{Ann} L(x),$$

called a *primitive quotient* of $\mathcal{U}(\mathfrak{g})$. A classical problem, which has not yet been solved completely, is to determine the simple composition factors (with multiplicities) of primitive quotients.

When beginning this work, the author used a more direct approach to find the module D, inspired by ideas in [**19**]. Although not necessary for the current exposition, the following proposition is interesting in its own right. It has close resemblance to the description of D given in [**7**], which in our setting gives D as the image of a map

$$\Delta(e) \to C_{x^{-1}}\nabla(x). \tag{5}$$

Proposition 5.1. *Let $x \in W$. The unique quotient D of $\Delta(e)$ satisfying* $\operatorname{Ann} D = \operatorname{Ann} L(x)$ *is isomorphic to the image of a non-zero homomorphism*

$$\Delta(e) \to \theta_x L(x^{-1}).$$

We first note that this image is uniquely defined, since

$$\begin{aligned}\dim \operatorname{Hom}_{\mathfrak{g}}(\Delta(e), \theta_x L(x^{-1})) &= \dim \operatorname{Hom}_{\mathfrak{g}}(\theta_{x^{-1}}\Delta(e), L(x^{-1}))\\ &= \dim \operatorname{Hom}_{\mathfrak{g}}(P(x^{-1}), L(x^{-1}))\\ &= 1.\end{aligned}$$

To prove Proposition 5.1 we need to recall some further theory.

The category $\mathcal{O}_0$ has a $\mathbb{Z}$-graded version $\mathcal{O}_0^{\mathbb{Z}}$, in which the modules $L(x)$, $\Delta(x)$ and $P(x)$, for $x \in W$, all have standard graded lifts (where their heads are concentrated in degree zero). Furthermore, the projective functors θ_x, $x \in W$, also have graded lifts (see [**30**]). For $M \in \mathcal{O}_0^{\mathbb{Z}}$ and $i \in \mathbb{Z}$, let $M\langle i\rangle$ denote the graded module defined by $M\langle i\rangle_j := M_{j-i}$.

The Grothendieck group of $\mathcal{O}_0^{\mathbb{Z}}$ is isomorphic to the Hecke algebra $\mathcal{H}$ of W, i.e. the free $\mathbb{Z}[v, v^{-1}]$-module over the basis $\{H_x \mid x \in W\}$, where multiplication is given by $H_x H_y = H_{xy}$ if $\ell(xy) = \ell(x) + \ell(y)$, and $H_s H_s = H_e + (v^{-1} - v)H_s$ for simple reflections $s \in S$. The Kazhdan–Lusztig basis is a basis of the Hecke algebra, whose elements we denote by $\underline{H}_x$, which are self dual under the duality $H \mapsto \overline{H}$ on $\mathcal{H}$ given

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by $\overline{H}_x = (H_{x^{-1}})^{-1}$ and $\overline{v} = v^{-1}$. We also have the dual Kazhdan–Lusztig basis, whose elements we denote by $\underline{\hat{H}}_x$, which is dual to the Kazhdan–Lusztig basis with respect to the symmetrising trace. We then have

$$\begin{aligned}
[\Delta(x)] &= H_x,\\
[P(x)] &= \underline{H}_x,\\
[L(x)] &= \underline{\hat{H}}_x,\\
[\theta_x __] &= \text{right multiplication by } \underline{H}_x, \text{ and}\\
[__\langle i\rangle] &= \text{multiplication by } v^{-i}.
\end{aligned}$$

For a review of this theory, see [**27**], in particular Section 3.

For $x, y \in W$ and $H \in \mathcal{H}$ let $k^H_{x,y} \in \mathbb{Z}[v, v^{-1}]$ be such that

$$\underline{H}_x H = \sum_{y\in W} k^H_{x,y}\underline{H}_y.$$

The *right preorder* on W is defined by $x \leqslant_R y$ if there exists an $H \in \mathcal{H}$ with $k^H_{x,y} \neq 0$. Dually, if $\hat{k}^H_{x,y} \in \mathbb{Z}[v, v^{-1}]$ is such that

$$\underline{\hat{H}}_x H = \sum_{y\in W} \hat{k}^H_{x,y}\underline{\hat{H}}_y,$$

then $x \geqslant_R y$ if and only if there exists a $H \in \mathcal{H}$ with $\hat{k}^H_{x,y} \neq 0$ (see [**23**, 5.1.16]). The *left preorder* is defined by $x \leqslant_L y$ if and only if $x^{-1} \leqslant_R y^{-1}$. By [**13**, **20**] we have the important fact that

$$x \leqslant_L y \text{ if and only if } \operatorname{Ann} L(x) \supseteq \operatorname{Ann} L(y).$$

The equivalence classes of $\leqslant_R$ and $\leqslant_L$ are called right and left cells, respectively.

For $x, y \in W$, let $h_{x,y} \in \mathbb{Z}[v, v^{-1}]$ with

$$\underline{H}_y = \sum_{x\in W} h_{x,y}H_x,$$

and for $x, y, z \in W$, let $k_{x,y,z} \in \mathbb{Z}[v, v^{-1}]$ with

$$\underline{H}_x\underline{H}_y = \sum_{z\in W} k_{x,y,z}\underline{H}_z.$$

Note in particular that $\overline{k}_{x,y,z} = k_{x,y,z}$. Now Lusztig's **a**-function on W (see [**24**]) can be defined as

$$\mathbf{a}(x) := \max_{y,z\in W} \deg k_{y,z,x}.$$

It is constant on right cells, and in general we have (see [**25**, 1.3(1)])

$$\mathbf{a}(x) \leq \operatorname{mindeg} h_{e,x},$$

where, for $f \in \mathbb{Z}[v, v^{-1}]$, mindeg$f$ is the *minimal degree* of f, i.e. the minimal element $i \in \mathbb{Z}$ such that the coefficient of v^i in f is non-zero. The *Duflo set* $\mathcal{D}$ (sometimes called

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the set of distinguished involutions) is defined as the set of elements $d \in W$ satisfying

$$\mathbf{a}(d) = \operatorname{mindeg} h_{e,d}.$$

By [**25**, Proposition 1.4, Theorem 1.10], each right cell contains precisely one Duflo involution. Note that, by the BGG reciprocity, we have

$$[\Delta(e)] = \sum_{x \in W} h_{e,x}[L(x)].$$

Hence, given a right cell R of W, all composition factors in the form $L(x)$, $x \in R$ of $\Delta(e)$ occur in degree at least $\mathbf{a}(x)$, and there is precisely one such factor which occur in degree $\mathbf{a}(x)$, namely the one corresponding to the Duflo involution in R.

Proof of Proposition 5.1. Fix $x \in W$ and denote the image of a non-zero homomorphism from $\Delta(e)$ to $\theta_x L(x^{-1})$ by $\bar{D}$. Since θ_x is exact, applying it to

$$P(x^{-1}) \twoheadrightarrow L(x^{-1})$$

gives

$$\theta_x P(x^{-1}) \twoheadrightarrow \theta_x L(x^{-1}). \tag{6}$$

First, we have, for some $\hat{k}_{x^{-1},x,z} \in \mathbb{Z}[v, v^{-1}]$,

$$[\theta_x L(x^{-1})] = \underline{\hat{H}}_{x^{-1}} \underline{H}_x = \sum_{z \in W} \hat{k}_{x^{-1},x,z} \underline{\hat{H}}_z = \sum_{z \in W} \hat{k}_{x^{-1},x,z}[L(z)],$$

and $\hat{k}_{x^{-1},x,z} \neq 0$ implies $z \leqslant_R x^{-1}$ so all composition factors of $\theta_x L(x^{-1})$ are in the form $L(y)$, where $y \leqslant_R x^{-1}$. On the other hand, we have

$$[\theta_x P(x^{-1})] = \underline{H}_{x^{-1}} \underline{H}_x = \sum_{z \in W} k_{x^{-1},x,z} \underline{H}_z = \sum_{z \in W} k_{x^{-1},x,z}[P(z)],$$

and $k_{x^{-1},x,z} \neq 0$ implies $z \geqslant_R x^{-1}$. Hence the head of $\theta_x P(x^{-1})$ has only simple factors in the form $L(y)$, $y \geqslant_R x^{-1}$. From equation (6) it follows that $\theta_x L(x^{-1})$ has minimal degree greater than or equal to $-\mathbf{a}(x^{-1})$, and that the head of $\theta_x L(x^{-1})$ has only simple factors in the form $L(y)$, $y \sim_R x^{-1}$. Furthermore, since $\theta_x L(x^{-1})$ is self-dual, $\theta_x L(x^{-1})$ has maximal degree smaller than or equal to $\mathbf{a}(x^{-1})$, and all its simple submodules are in the form $L(y)$, $y \sim_R x^{-1}$.

In particular, the maximal degree of $\bar{D}$ is bounded by $\mathbf{a}(x^{-1})$, and all simple submodules of $\bar{D}$ are in the form $L(y)$, $y \sim_R x^{-1}$. But the only such submodule occurring on degree $\mathbf{a}(x^{-1})$ or smaller in $\Delta(e)$ is $L(d)$, where d is the unique Duflo involution in the same right cell as x^{-1}, occurring precisely once in degree $\mathbf{a}(x^{-1})$. Hence $\bar{D}$ has $L(d)$ as its unique simple submodule, and all other simple composition factors are in the form $L(y)$, $y <_R d$. By [**15**, Proposition 6.2 (ii)] it follows that $\operatorname{Ann} \bar{D} = \operatorname{Ann} L(d)$, and $\operatorname{Ann} L(d) = \operatorname{Ann} L(x)$ as $d \sim_L x$. Since D is the unique quotient of $\Delta(e)$ with this property, we must have $\bar{D} = D$. □

In [**31**] Stroppel shows that the map (5) has a graded lift, which allows her to give an upper bound on the multiplicities of simple composition factors of the primitive

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quotient $\mathcal{U}(\mathfrak{g})/\operatorname{Ann} L(x)$ as (here written in the setting of $\mathcal{O}_0$)

$$[D\colon L(y)] \leq \sum_{i\in\mathbb{Z}} \max\left\{[\Delta(e)\colon L(y)\langle i\rangle], [C_{x^{-1}}\nabla(x)\colon L(y)\langle i\rangle]\right\}. \tag{7}$$

Similarly, from Proposition 5.1 we obtain the following upper bound:

$$[D\colon L(y)] \leq \sum_{i\in\mathbb{Z}} \max\left\{[\Delta(e)\colon L(y)\langle i\rangle], [\theta_x L(x^{-1})\colon L(y)\langle i\rangle]\right\}. \tag{8}$$

For Lie algebras of small rank, the bound (8) is finer than the bound (7), but we have not been able to show this in general. However, in [**31**] Stroppel greatly refines the bound (7) by using the the Duflo–Zhelobenko four step exact sequence

$$0 \to C_{x^{-1}}\nabla(sx) \to C_{(sx)^{-1}}\nabla(sx) \xrightarrow{f_{sx,x}} C_{x^{-1}}\nabla(x) \to C_{(sx)^{-1}}\nabla(x) \to 0.$$

It does not seem to be possible to find a similar refinement of our bound.

6. Kostant's problem for $\mathfrak{sl}_6$. In [**19**], the answer to Kostant's problem was given for all simple modules in $\mathcal{O}_0$ for $\mathfrak{sl}_n$, $n \leq 5$, and partial results were obtained for $\mathfrak{sl}_6$. In type A the answer to Kostant's problem is a left cell invariant by [**27**, Theorem 60]. Furthermore, since in type A there is a unique involution in each left cell, it suffices to solve Kostant's problem for involutions. The Weyl group for $\mathfrak{sl}_6$ is S_6, which contains 76 involutions. For 45 of these Kostant's problem were shown to have a positive answer, for 17 the answer was negative, and for 11 it remained unknown.

We expected that Theorem 1.1 would answer many of these 11 unknown cases, but it actually turned out to answer only two. The involution $s_1s_2s_1s_5$ is in the same left cell as the element

$$s_1s_4 \cdot w_\circ^I w_\circ,$$

where $I = \{s_1, s_2, s_3, s_4\}$. By [**19**, Corollary 21], Kostant's problem has a positive answer for the $\mathfrak{sl}_5$-module $L(s_1s_4)$, and hence by Theorem 1.1 Kostant's problem has a positive answer for the $\mathfrak{sl}_6$-module $L(s_1s_2s_1s_5)$. By symmetry of the Coxeter diagram, Kostant's problem also has a positive answer for $L(s_1s_4s_5s_4)$. Hence answer to Kostant's problem is still open for the modules

$$\begin{array}{lll} L(s_2s_3s_4s_3s_2), & L(s_2s_1s_4s_3s_2s_5s_4), & L(s_1s_3s_2s_4s_3s_2s_1s_5s_4s_3),\\ L(s_2s_1s_3s_4s_3s_2), & L(s_1s_2s_3s_2s_4s_3s_2s_1), & L(s_2s_1s_3s_2s_1s_4s_5s_4s_3s_2),\\ L(s_2s_4s_3s_2s_5s_4), & L(s_2s_3s_2s_4s_5s_4s_3s_2), & L(s_2s_1s_3s_2s_4s_3s_2s_1s_5s_4s_3s_2). \end{array}$$

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