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GROUPS THAT INVOLVE FINITELY MANY PRIMES AND HAVE ALL SUBGROUPS SUBNORMAL II

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Abstract. It is shown that if G is a hypercentral group with all subgroups subnormal, and if the torsion subgroup of G is a π-group for some finite set π of primes, then G is nilpotent. In the case where G is not hypercentral there is a section of G that is much like one of the well-known Heineken-Mohamed groups. It is also shown that if G is a residually nilpotent group with all subgroups subnormal whose torsion subgroup satisfies the above condition then G is nilpotent.

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It was proved in [**12**] that if a group G has all subgroups subnormal and the torsion subgroup of G involves just finitely many primes then G is nilpotent-by-divisible Chernikov. This may be viewed as a generalisation of the result of Casolo in [**2**] that a p-group with all subgroups subnormal is nilpotent-by-divisible Chernikov; however, by means of Roseblade's Theorem [**7**], which states that a group with all subgroups subnormal of bounded defect is nilpotent of bounded class, Casolo's result on p-groups extends easily to the case of arbitrary torsion groups: thus (as proved in [**2**]) every torsion group with all subgroups subnormal is nilpotent-by-divisible Chernikov. The same does not hold for arbitrary groups with all subgroups subnormal, as indicated by examples constructed in [**8**, **11**], though it was shown in [**1**, **9**] that torsion-free groups with all subgroups subnormal are nilpotent. The reader is referred to the article [**12**] for further discussion along these lines.

Now, the examples referred to above are all hypercentral (of length exactly $\omega + 1$), while in [**6**] it was established that periodic hypercentral groups with all subgroups subnormal are nilpotent. Here, we employ the main result of [**12**] (referred to above) to obtain the following.

THEOREM 1. *Let G be a hypercentral group with all subgroups subnormal and suppose that the torsion subgroup of G is a π-group for some finite set π of primes. Then G is nilpotent.*

Next, we know from [**10**] that a group G with all subgroups subnormal that is both locally finite and residually nilpotent is nilpotent. (See also Theorem 3 of [**2**].) By means of an application of [**12**] this too generalises, as follows.

THEOREM 2. *Let G be a residually nilpotent group with all subgroups subnormal and suppose that the torsion subgroup of G is a π-group for some finite set π of primes. Then G is nilpotent.*

Suppose now that G is a non-nilpotent group with all subgroups subnormal and that $\pi(T)$ is finite, where T denotes the torsion subgroup of G. All known examples of such groups appear to be based on those constructed by Heineken and Mohamed in [**4**]. (The reader is referred to Chapter 6 of [**5**] and to the article [**12**] for some explanation of what is meant here). Now the original Heineken-Mohamed groups have all proper subgroups subnormal *and* nilpotent, and are centreless, metabelian p-groups, each of which is an extension of an elementary abelian subgroup by a Prüfer group. For the purposes of this discussion, we introduce the following terminology.

Let p be a prime, G a non-nilpotent group with all subgroups subnormal.

(a) G is an HMp-group if G' is an elementary abelian p-group, $G/G' \cong C_{p^\infty}$, every proper subgroup of G is nilpotent and $Z(G) = 1$.
(b) G is an HMp^*-group if the following conditions hold.

(i) The torsion subgroup T of G is an elementary abelian p-group, $T = G'$, which is the nilpotent residual of G, and $G/T \cong D$, the additive group of p-adic rationals.
(ii) There is an abelian subgroup A of G containing T such that $G/A \cong C_{p^\infty}$.
(iii) The hypercentre of G is $Z(G)$ and is contained in A, and $G/Z(G)$ is an HMp-group.

Observe that the examples constructed by Heineken and Mohamed in [**4**] are HMp-groups and that in each of the above definitions some of the stated conditions could in fact be omitted, since they follow from the remaining properties – however, in the interests of clarity they are included here. The next point to note is that there do indeed exist groups as defined in (b) above. (The construction which follows was suggested by the referee, to whom I am grateful for providing this substantial simplification of my original example.)

Proposition 3. *For each prime p there exists an HMp^*-group.*

Proof. Let H be an HMp-group, let D be a group isomorphic to the additive group of p-adic rationals and set $W = H \times D$. It is easy to see that every subgroup of W is subnormal. Now let $A = H'$ and choose $Z \leq D$ so that $D/Z \cong C_{p^\infty}$; then $W/(A \times Z)$ is a divisible p-group of rank two. Let $G/(A \times Z)$ be the diagonal subgroup of $W/(A \times Z)$. It is routine to verify that G is an HMp^*-group.

Theorem 4. *Let G be a group with all subgroups subnormal and suppose that the torsion subgroup T of G is a π-group for some finite set π of primes. Then G is non-nilpotent if and only if G has a section isomorphic to an HMp-group for some prime p. Furthermore, if the torsion subgroup of G is nilpotent then G is non-nilpotent if and only if G has a section isomorphic to an HMp^*-group.*

The following result is well known.

Lemma 5. *Let A be a periodic abelian group and suppose that G is a divisible group that acts on A. If* $[A, G, G] = 1$ *then* $[A, G] = 1$.

Proof. Assuming the result false, choose $a \in A$ with $[a, G] \neq 1$. The map θ from G into $[A, G]$ given by $\theta(g) = [a, g]$ for all $g \in G$ is a homomorphism with image of finite exponent, and so $ker(\theta) = G$ and we have a contradiction.

COROLLARY 6. *Let G be a locally nilpotent group with torsion subgroup T and suppose that there is a normal nilpotent subgroup K of G containing T such that G/K is divisible and periodic. If G is either hypercentral or residually nilpotent then G is nilpotent.*

Proof. First note that G/T is nilpotent since it is torsion-free and nilpotent-by-periodic. Let $T_i = T \cap Z_i(K)$ for each $i = 1, \dots, c$, where c is the class of K, and suppose first that G is hypercentral. Writing $S_1 = T_1 \cap Z(G)$, $S_2 = T_1 \cap Z_2(G)$, we see that G/K acts on T_1 (by conjugation). Thus, we have from Lemma 5 that $S_2 = S_1$, and it follows easily that $T_1 \leq Z(G)$. An easy induction now gives $T = T_c \leq Z_c(G)$, and the result follows in this case. Now suppose that G is residually nilpotent. Since $[T_1, G] = [T_1, G, G]$, by an application of Lemma 5 to the group $G/[A, G, G]$, it again follows that T_1 is central in G. Furthermore, if $a \in \cap_{j=1}^{\infty}([T,_j G]T_1)$ then $[a, K] \leq \cap_{j=1}^{\infty}([T,_j G] = 1)$, which gives $a \in T_1$ and hence G/T_1 residually nilpotent, and again an easy induction (on c) establishes that G is nilpotent.

Proof of Theorems 1 *and* 2. By the main result of **[12]** we have in each case that G has a normal nilpotent subgroup K such that G/K is divisible Chernikov, and we saw above that a periodic group with all subgroups subnormal is nilpotent if it is either hypercentral or residually nilpotent. Thus we may assume (by Fitting's Theorem) that $T \leq K$, and both results now follow from Corollary 6.

PROPOSITION 7. *Let G be a torsion-free abelian group, A a subgroup of G such that $G/A \cong C_{p^\infty}$ for some prime p. Then there is a subgroup J of G such that $G = JA$ and J/N is isomorphic to the additive group D of p-adic rationals for some subgroup N with NA/A finite.*

Proof. Suppose that G is a counterexample to the statement of the theorem, and assume first that G has finite rank n, say, which we may choose to be minimal for a counterexample. Note that $n > 1$, for if $n = 1$ then G has a subgroup J isomorphic to D; clearly $JA = G$ and we may set $N = 1$ and obtain a contradiction. Choose a free abelian subgroup H of A such that G/H is periodic, and write $G/H = G_0/H \times G_1/H$, where G_0/H, G_1/H are the p, p'-components respectively of G/H. It is easy to see that $G_1 \leq A$ and that $G_0A = G$. Thus, G_0 is also a counterexample and we may assume that G/H is a p-group. If G is p-divisible then it is isomorphic to the direct product of n copies of D, one of which supplements A, and we have a contradiction once more. Thus G is not p-divisible, and since G/H is a p-group we may choose a nontrivial isolated cyclic subgroup $\langle g \rangle$ of G. Thus $G/\langle g \rangle$ is torsion-free. By minimality of n, there is a subgroup J of G that contains g, supplements $A\langle g \rangle$ and has a subgroup N containing g with $J/N \cong D$ and $NA\langle g \rangle/A\langle g \rangle$ finite. Since $A\langle g \rangle/A$ is also finite it follows that NA/A is finite, and we have another contradiction that dispenses with the case where G has finite rank.

Suppose next that there is a subgroup K of G of finite rank that supplements A. By the above, K has a subgroup J such that $K = J(K \cap A)$ and $J/N \cong D$ for some N such that $N(K \cap A)/(K \cap A)$ is finite. But then JA contains KA and hence equals G, and since N is finite mod $K \cap A$ it is finite mod A, a contradiction. Choose now a generating set $S := \{g_i : i \geq 0\}$ for G mod A, where $g_0 \in A$ and, for each $i \geq 0$, $g_{i+1}^p \equiv g_i$ mod A. The subgroup generated by S has infinite rank, and so we may choose an infinite subset $S^* := \{h_0, h_1, ...\}$ of S, where $h_0 = g_0$ and the order of h_i mod A increases with i, such that the subgroup J of G generated by S^* is free abelian of infinite rank. Clearly

$JA = G$. There are positive integers $k_{i+1}, i \geq 0$, such that $h_{i+1}^{p^{k_{i+1}}} \equiv h_i \bmod A$; write $N = \langle h_i^{-1} h_{i+1}^{p^{k_{i+1}}} : i \geq 0 \rangle$. Then $N \leq A$ and it is easy to see that $J/N \cong D$. The result is as follows.

Proposition 8. *Let G be a periodic group with all subgroups subnormal. Then G is non-nilpotent if and only if G has a section G^* that is isomorphic to an HMp-group, for some prime p.*

Proof. Clearly any group that has an HMp-section is non-nilpotent. Suppose that G is periodic and has all subgroups subnormal but that G is not nilpotent. Then there is a prime p such that the p-component of G is non-nilpotent (see, for example, [**2**]), so we may assume that G is a p-group. By [**12**] G has a normal nilpotent subgroup A such that G/A is divisible Chernikov. Now if every Prüfer extension of A is nilpotent then we have the contradiction that G is nilpotent, by Fitting's Theorem; thus we may pass to a suitable subgroup and assume that $G/A \cong C_{p^\infty}$.

By [**3**] G/A' is not nilpotent, so we may further assume that A is abelian. By Lemma 5 we have $[A, G, G] = [A, G]$. If $[A, G]$ is divisible then it is central in G, giving the contradiction that G is nilpotent. Thus $[A^p, G] = [A, G]^p < [A, G]$ and so $G/[A, G]^p$ is non-nilpotent, and we may factor and hence assume that A^p is central in G. Factoring once more we may suppose that $A^p = 1$.

Now $G/[A, G]$ is centre-by-locally cyclic and so it is abelian, and hence $G' = [A, G] = R$, say, the nilpotent residual of G. Furthermore, $G/R = B/R \times G_0/R$ for some B/R of finite exponent and $G_0/R \cong C_{p^\infty}$. Since $B/B \cap A$ is finite, B is abelian-by-finite and hence nilpotent, so G_0 is non-nilpotent. Also, $AG_0 = G$, and so $[A, G_0, G_0] = [A, G_0] = [A, G]$; so we have $G_0/R \cong C^{p^\infty}$ and $[R, G_0] = R = G_0'$, which is the nilpotent residual of G_0. Relabelling, we may assume that $[A, G] = A = G'$ and hence that $G/G' \cong C_{p^\infty}$ and A is the nilpotent residual of G.

Next, let H be a subgroup of G and suppose that $G = AH$. Then $A \cap H \lhd G$ and $G/A \cap H$ is the product of the abelian normal subgroup $A/A \cap H$ and the subnormal abelian subgroup $H/A \cap H$. Thus $G/A \cap H$ is nilpotent and hence abelian, and we deduce that $A = A \cap H$, that is, $A \leq H$ and hence $H = G$. It follows that if H is an arbitrary proper subgroup of G then AH/A is finite and so AH is nilpotent. In particular, every proper subgroup H of G is nilpotent.

Finally, let K be the hypercentre of G. Then K is nilpotent [**6**] and hence a proper subgroup of G, and G/K has trivial centre. (Indeed, as is well known, the properties that we have so far ensure that $K = Z(G)$: see the proof of Lemma 10 below.) Setting $G^* = G/K$ completes the proof of Proposition 8.

Lemma 9. *Let G be a non-nilpotent group with all subgroups subnormal and suppose that the torsion subgroup T of G is a π-group for some finite set π of primes. Suppose too that T is nilpotent, as is every section of G that, for some $p \in \pi$, is an extension of an elementary abelian p-group by a torsion-free abelian group. Then G is nilpotent.*

Proof. Let G be as stated and suppose for a contradiction that G is not nilpotent. For each $p \in \pi$, let V_p denote the p'-component of T. Then some factor G/V_p is non-nilpotent, and we may assume that T is a (nilpotent) p-group. By [**3**], G/T' is not nilpotent, so we may factor and hence assume that T is abelian. Next, by induction on the derived length of the (non-trivial and nilpotent) factor G/T we may assume that the isolator I of TG' in G is nilpotent. By [**12**] there is a normal nilpotent subgroup K of

G with G/K divisible Chernikov, and since KI is nilpotent we may assume that $I \le K$. We may also suppose that G/K is a p-group, since the p'-isolator of K is nilpotent.

Let S/I' be the torsion subgroup of G/I' and let U/I' be the p'-component of S/I'; note that $S \le I$ since G/I is torsion-free. If G/U is nilpotent then so is $G/U \cap T$; but $U \cap T \le I'$ and thus G/I' is nilpotent, which gives the contradiction that G is nilpotent. We may therefore pass to the factor group G/U, and there is nothing lost in re-labelling so that $S/U = T$, I is a normal isolated abelian subgroup of G containing T and K is a normal nilpotent subgroup containing I such that G/K is a divisible Chernikov p-group.

By Theorem 2, the nilpotent residual R of G is $\gamma_m(G)$ for some integer m, and so $R = [T,_n G]$ for some n. Clearly $R \ne 1$, and since K is nilpotent we have $Q := [R, K] < R$. Factoring by Q, we may suppose that R is central in K, so that G acts on R as a divisible p-group. If R is divisible then each element of G acts trivially on R, which gives the contradiction that R is central in G. It follows that $R^p < R$, and once again we may factor. Now we have $1 = R^p = [T^p,_n G]$ and so $T^p \le Z_n(G)$. Thus G/T^p is not nilpotent, and we may factor yet again and assume that T has exponent p. Finally, $I^p \cap T = T^p = 1$, and so G/I^p is not nilpotent, contradicting the hypothesis on such sections of G. This completes the proof of the lemma.

LEMMA 10. *Let G be a group with all subgroups subnormal. Suppose that the torsion subgroup T of G is an elementary abelian p-group for some prime p and that $G/T \cong D$, the additive group of p-adic rationals. Suppose too that there is an abelian normal subgroup A of G containing T such that $G/A \cong C_{p^\infty}$. Then G has a section isomorphic to an HMp^*-group.*

Proof. The factor group $G/[T, G]$ is centre-by-locally cyclic and so it is abelian. Observe that $T \cap G^p \le [T, G]T^p = [T, G]$. Also, $A^p \cap T = 1$; hence $[A^p, G] = 1$ and A^p is central in G. It follows that $G/Z(G)$ is a p-group.

Writing $H = [T, G]G^p$ we have $HT = G$ and $H \cap T = [T, G]$, so $[T, G]$ is the torsion subgroup of the non-nilpotent subgroup H. In addition, $[T, G, H] = [T, G, TH] = [T, G, G] = [T, G]$, by Lemma 5 and, replacing G by H if necessary, we may assume that $[T, G] = T$ and hence that $T = G'$.

If $x \in Z_2(G)$ then the map θ from G to $Z(G) \cap T$ given by $\theta(g) = [x, g]$ for all $g \in G$ is a homomorphism whose image has exponent dividing p and whose kernel contains $G' = T$. Since $G = TG^p$, it follows that $x \in Z(G)$ and hence that $Z := Z(G)$ is the hypercentre of G, which we may therefore assume is contained in A. Clearly $Z = (Z \cap T) \times \langle a \rangle$ for some infinite cyclic subgroup $\langle a \rangle$ of A, and we may factor by $Z \cap T$ and hence assume that $Z = \langle a \rangle$. Note that G/Z has trivial centre and $G/TZ \cong C_{p^\infty}$.

Finally, if $(K/Z)(TZ/Z) = G/Z$ then, writing $(K \cap TZ) = L$, we see that $L \lhd G$ and G/L is the product of the normal abelian subgroup TZ/L and the subnormal subgroup K/L, which is also abelian since $K/(K \cap T)$ is abelian. Hence G/L is nilpotent and hence abelian, and it follows that $T \le L \le K$ and hence that $K = K(TZ) = G$. Thus no proper subgroup of G/Z supplements TZ/Z, and since $G/TZ \cong C_{p^\infty}$ this implies that every proper subgroup of G/Z is finite mod TZ/Z and hence nilpotent. Thus G is now an HMp^*-group, and this concludes the proof of Lemma 10.

Proof of Theorem 4. Clearly any group that has an HMp-section is non-nilpotent. Let G be a non-nilpotent group with all subgroups subnormal and suppose that T is an elementary abelian p-group and G/T is abelian. By Proposition 8 and Lemma 9, all we have to show is that such a group G has a section isomorphic to an HMp^*-group. By [**12**] there is a normal nilpotent subgroup K of G, which we may choose so that $T \leq K$, such that G/K is a divisible Chernikov p-group. Since the product of finitely many nilpotent normal subgroups is nilpotent, there is a non-nilpotent subgroup G_0 of G such that $K \leq G_0$ and $G_0/K \cong C_{p^\infty}$; we may assume that $G_0 = G$. By Proposition 7 there is a subgroup J of G containing T such that $JK = G$ and J has a subgroup N containing T with NK/K finite and $J/N \cong D$, the additive group of p-adic rationals. Since K is nilpotent and J is subnormal in G we see that J is not nilpotent, and from now on we work with the group J and proceed to show that J has a section of the required type. Since $A := NK$ is a normal nilpotent subgroup of J we have J/A' non-nilpotent, and since $A' \leq T$ we may factor and hence assume that A is abelian.

We have a series of subgroups $T \leq N < A < J$ with $J/A \cong C_{p^\infty}$, $J/N \cong D$ and both A and J/T abelian. Now J/N^p is not nilpotent, since J/T is abelian and $N^p \cap T = 1$. Observe that, in the factor group J/N^p, N/N^p is the torsion subgroup, since J/N is torsion-free. Thus, we may factor by N^p and hence assume that $J/T \cong D$. An application of Lemma 10 now gives the desired result.

REFERENCES

1. C. Casolo, Torsion-free groups in which every subgroup is subnormal, *Rend. Circolo Mat. Palermo.* **L** (2001), 321–324.

2. C. Casolo, On the structure of groups with all subgroups subnormal, *J. Group Theory* **5** (2002), 293–300.

3. P. Hall, Some sufficient conditions for a group to be nilpotent, *Illinois J. Math.* **2** (1958), 787–801.

4. H. Heineken and I. J. Mohamed, A group with trivial centre satisfying the normalizer condition, *J. Algebra* **10** (1968), 368–376.

5. J. C. Lennox and S. E. Stonehewer, *Subnormal subgroups of groups* (Clarendon, Oxford, 1987).

6. W. Möhres, Hyperzentrale Gruppen, deren Untergruppen alle subnormal sind, *Illinois J. Math.* **35** (1991), 147–157.

7. J. E. Roseblade, On groups in which every subgroup is subnormal, *J. Algebra* **2** (1965), 402–412.

8. H. Smith, Hypercentral groups with all subgroups subnormal. *Bull. London Math. Soc.* **15** (1983), 229–234.

9. H. Smith, Torsion-free groups with all subgroups subnormal, *Arch. Math.* **76** (2001), 1–6.

10. H. Smith, Residually nilpotent groups with all subgroups subnormal, *J. Algebra.* **244** (2001), 845–850.

11. H. Smith, On non-nilpotent groups with all subgroups subnormal, *Ricerche di Mat.* **L** (2001), 217–221.

12. H. Smith, Groups that involve finitely many primes and have all subgroups subnormal, *J. Algebra.* **347** (2011), 133–142.