COMPOSITIO MATHEMATICA

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Compositio Math. **150** (2014), 369–395

doi:10.1112/S0010437X13007458

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY

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Abstract

In this paper we examine different problems regarding complete intersection varieties of high multidegree in a smooth complex projective variety. First we prove an existence theorem for jet differential equations that generalizes a theorem of Diverio. Then we show how one can deduce hyperbolicity for generic complete intersections of high multidegree and high codimension from the known results on hypersurfaces. Finally, motivated by a conjecture of Debarre, we focus on the positivity of the cotangent bundle of complete intersections, and prove some results towards this conjecture; among other things, we prove that a generic complete intersection surface of high multidegree in a projective space of dimension at least four has an ample cotangent bundle.

1. Introduction

Varieties with an ample cotangent bundle satisfy many interesting properties, and are supposed to be abundant. However, until recently, relatively few examples of such varieties were known. For constructions of such examples we refer to [Miy77, Sch86, Som84] and [Deb05]. In [Deb05], Debarre constructed such varieties by proving that the intersection of at least $N/2$ sufficiently ample generic hypersurfaces in an N-dimensional abelian variety has an ample cotangent bundle. Motivated by this result, he also conjectured that the analogous statement holds in projective space, that is to say, *the intersection, in* $\mathbb{P}^N$, *of at least* $N/2$ *generic hypersurfaces of sufficiently high degrees has an ample cotangent bundle.*

It is a well-known fact that varieties with an ample cotangent bundle are hyperbolic in the sense of Kobayashi. In this sense, Debarre's conjecture on complete intersection varieties relates to Kobayashi's conjecture for hypersurfaces which predicts that a generic hypersurface of sufficiently high degree in $\mathbb{P}^N$ is hyperbolic. A strategy towards this conjecture was explained by Siu in [Siu04], and was afterwards further developed by Diverio *et al.* in [DMR10] to get effective algebraic degeneracy for entire curves on generic hypersurfaces of high degree. The fact that varieties with an ample cotangent bundle are hyperbolic is not the only relation between Debarre's conjecture and Kobayashi's conjecture. As pointed out by Diverio and Trapani [DT10], the jet differentials point of view might be the good setting to interpolate between those two conjectures. This is made precise by a conjecture of Diverio and Trapani that generalizes Debarre's conjecture by replacing the cotangent bundle by suitable jet differentials bundles; more precisely, they conjecture that *if* $X \subset \mathbb{P}^N$ *is the intersection of at least* $N/(k+1)$ *generic*

Received 15 June 2012, accepted in final form 13 May 2013, published online 18 November 2013.
2010 Mathematics Subject Classification 14M10, 14F10 (primary).
Keywords: complete intersections, ample cotangent bundle, jet differential equations.

hypersurfaces of sufficiently high degrees, then $E_{k,m}\Omega_X$ is ample for $m \geqslant 1$. We refer to § 2.1 for the definition of the invariant jet differential bundles $E_{k,m}\Omega_X$.

The aim of this paper is to study hyperbolicity-type questions for complete intersection varieties. It is divided into three parts and an appendix, each part being concerned with some aspect of this problematic.

The first part (§ 2) is concerned with the existence of jet differentials on complete intersection varieties. More precisely our result is the following.

Theorem A. *Let M be an N-dimensional smooth complex projective variety and H an ample divisor on M. Fix $a \in \mathbb{N}$, fix $1 \leqslant c \leqslant N-1$, set $n := N-c$ and take $k \geqslant \lceil n/c \rceil$. Take $A_1, \ldots, A_c$ very ample line bundles on M. For $d_1, \ldots, d_c \in \mathbb{N}$ big enough take generic hypersurfaces $H_1 \in |d_1A_1|, \ldots, H_c \in |d_cA_c|$ and let $X = H_1 \cap \cdots \cap H_c$. Then $\mathcal{O}_{X_k}(1) \otimes \pi_k^* H^{-a}$ is big on X_k. In particular for $m \gg 0$*

$$H^0(X, E_{k,m}\Omega_X \otimes H^{-ma}) \neq 0.$$

We refer to § 2.1 for the definition of the spaces X_k. Diverio [Div09] proved this result for hypersurfaces in $\mathbb{P}^N$. Moreover, Diverio [Div08] proved that when $M = \mathbb{P}^N$ and $X \subseteq \mathbb{P}^N$ is an n-dimensional complete intersection, then $E_{k,m}\Omega_X$ has no global sections when $k < \lceil n/c \rceil$. Therefore Theorem A completes the global picture of existence of jet differentials for complete intersection varieties of high multidegree in projective space. This result can also be viewed as a first piece of evidence towards Diverio and Trapani's conjecture.

The second part (§ 3) is concerned with algebraic degeneracy of entire curves in complete intersection varieties. The main result of this section is the following.

Theorem B. *Let M be an N-dimensional smooth complex projective variety. Take $A_1, \ldots, A_c$ ample line bundles on M. For $d_1, \ldots, d_c \in \mathbb{N}$ big enough take generic hypersurfaces $H_1 \in |d_1A_1|, \ldots, H_c \in |d_cA_c|$ and let $X = H_1 \cap \cdots \cap H_c$. Then there exists an algebraic subset $Z \subset X$ of codimension at least $2c$ such that all the entire curves of X lie in Z.*

We will derive Theorem B from the results of Diverio *et al.* [DMR10] and the results of Diverio and Trapani [DT10]. Note that, in particular, Theorem B implies that the complete intersection varieties considered in Debarre's conjecture are hyperbolic.

The last part (§ 4) is dedicated to giving partial results toward Debarre's conjecture. The main result of this section goes as follows.

Theorem C. *Fix $a \in \mathbb{N}$. There exists $B_{N,n,a} \in \mathbb{N}$ such that, if $X \subset \mathbb{P}^N$ is a generic complete intersection of dimension n, codimension c and multidegree $(d_1, \ldots, d_c)$ satisfying $c \geqslant n$ and $d_i \geqslant B_{N,n,a}$ for all $1 \leqslant i \leqslant c$, then $\Omega_X \otimes \mathcal{O}_X(-a)$ is ample modulo an algebraic subset of codimension at least two in X.*

As a corollary we obtain Debarre's conjecture for surfaces.

Corollary. *If $N \geqslant 4$ and $S \subset \mathbb{P}^N$ is a generic complete intersection surface of multidegree $(d_1, \ldots, d_{N-2})$ satisfying $d_i > (8N+2)/(N-3)$, then Ω_S is ample.*

Finally, the appendix is concerned with two more technical issues. First we adapt to our situation the main result of Merker's paper [Mer09]. This result is used in an essential way during the proof of Theorem C.

Afterwards, we compute some bounds on the number $B_{N,n,a}$ in Theorem C. In particular, we compute the bound we announced in the above corollary.

Notation and conventions. In this paper, we work over the field of complex numbers $\mathbb{C}$. Given a smooth projective variety X, we denote by TX the tangent bundle of X, by $\Omega_X = TX^*$ the cotangent bundle of X, and by $K_X = \det \Omega_X$ the canonical line bundle of X. The Néron–Severi group of X is denoted by $N^1(X)$. Given a holomorphic vector bundle on X, we denote its ith Chern class by $c_i(E)$ and its ith Segre class by $s_i(E)$ (see § 2.1). We also denote by $\pi_E : \mathbb{P}(E) \to X$ the projective bundle of one-dimensional *quotients* of E, and $\mathcal{O}_{\mathbb{P}(E)}(1)$ will denote the tautological bundle of $\mathbb{P}(E)$. Given a line bundle L on X we denote by $\mathrm{Bs}(L)$ the stable base locus of L:

$$\mathrm{Bs}(L) := \bigcap_{\substack{m>0 \\ s \in H^0(X, mL)}} (s = 0).$$

We will say that a property holds for a *generic* member of $|L|$, if there exists a non-empty Zariski open subset $U \subseteq |L|$ such that the property holds for each element in U. We will say that a property holds for a *generic* complete intersection of codimension c in $\mathbb{P}^N$ of multidegree $(d_1, \ldots, d_c)$, if there exists a non-empty Zariski open subset $U \subseteq |\mathcal{O}_{\mathbb{P}^N}(d_1)| \times \cdots \times |\mathcal{O}_{\mathbb{P}^N}(d_c)|$ such that the property holds for each element in U.

2. Jet differentials on complete intersection varieties

This part is motivated by two theorems of Diverio (see [Div08] and [Div09]). In [Div09], he proved a non-vanishing theorem for jet differentials on hypersurfaces of high degree in projective space.

Theorem 2.1 [Div09, Theorem 1]. *Fix $n \geqslant 1$, fix $k \geqslant n$ and fix $a > 0$. There exists an integer $d_{n,k}$ such that, if $X \subset \mathbb{P}^{n+1}$ is a smooth projective hypersurface of degree greater than $d_{n,k}$ and if $m \in \mathbb{N}$ is big enough, then*

$$H^0(X, E_{k,m}\Omega_X \otimes \mathcal{O}_X(-a)) \neq 0.$$

He also proved a vanishing theorem for jet differentials on complete intersection varieties in projective space.

Theorem 2.2 [Div08, Theorem 7]. *Let X be a complete intersection variety in $\mathbb{P}^N$, of dimension n and codimension c. For $1 \leqslant k < \lceil n/c \rceil$ and for all $m \geqslant 1$, one has*

$$H^0(X, E_{k,m}\Omega_X) = 0.$$

It seems therefore natural to look for the non-vanishing of $H^0(X, E_{\lceil n/c \rceil, m}\Omega_X)$ when X is a smooth complete intersection of dimension n and codimension c of high multidegree. Moreover, we can look for a similar statement in any ambient smooth projective variety M instead of $\mathbb{P}^N$. This is the content of Theorem 2.7. Note, however, that Diverio's vanishing result, Theorem 2.2, certainly does not hold if we replace $\mathbb{P}^N$ by another projective variety M; consider for example M to be an abelian variety.

2.1 Segre classes and higher order jet spaces

We start by giving the definition of the Segre classes associated to a vector bundle. If E is a rank-r complex vector bundle on X and $\pi_E : \mathbb{P}(E) \to X$ the projection, the Segre classes of E are defined by

$$s_i(E) := (\pi_E)_* c_1(\mathcal{O}_{\mathbb{P}(E)}(1))^{r-1+i} \in A^i(X).$$

Note that they are denoted by $s_i(E^*)$ in [Ful98]. It is straightforward to check that, for any line bundle $L \to X$,

$$s_i(E \otimes L) = \sum_{j=0}^{i} \binom{r-1+i}{i-j} s_j(E) c_1(L)^{i-j}. \tag{1}$$

Recall that the total Segre class is the formal inverse of the total Chern class of the dual bundle: $s(E) = c(E^*)^{-1}$. Therefore total Segre classes satisfy Whitney's formula for exact sequences of vector bundles.

Now we briefly recall the construction of higher order jet spaces and invariant jet differentials bundles. Details can be found in [Dem97] and [Mou12]. We follow the presentation of [Mou12]. Let X be a projective variety of dimension n. For all $k \in \mathbb{N}$ we can construct a variety X_k and a rank-n vector bundle $\mathcal{F}_k$ on X_k. Inductively, let $X_0 := X$ and $\mathcal{F}_0 := \Omega_X$. Let $k \geqslant 0$ and suppose that X_k and $\mathcal{F}_k$ are constructed. Then $X_{k+1} := \mathbb{P}(\mathcal{F}_k) \xrightarrow{\pi_{k,k+1}} X_k$ and $\mathcal{F}_{k+1}$ is the quotient of $\Omega_{X_{k+1}}$ defined by the following diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & S_{k+1} \arrow[r] \arrow[d] & S_{k+1} \arrow[d] & & \\
0 \arrow[r] & \pi_{k,k+1}^* \Omega_{X_k} \arrow[r] \arrow[d] & \Omega_{X_{k+1}} \arrow[r] \arrow[d] & \Omega_{X_{k+1}/X_k} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_{X_{k+1}}(1) \arrow[r] \arrow[d] & \mathcal{F}_{k+1} \arrow[r] \arrow[d] & \Omega_{X_{k+1}/X_k} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

For all $k > j \geqslant 0$ we will set $\pi_{j,k} = \pi_{j,j+1} \circ \cdots \circ \pi_{k-1,k} : X_k \to X_j$, $\pi_k := \pi_{0,k}$ and $E_{k,m}\Omega_X := (\pi_k)_* \mathcal{O}_{X_k}(m)$. The bundle $E_{k,m}\Omega_X$ is called the bundle of *invariant jet differentials* of order k and degree m. These bundles have important applications to hyperbolicity problems [Dem97].

Note that $n_k := \dim(X_k) = n + k(n-1)$. If we have a k-tuple of integers $(a_1, \dots, a_k)$ we will set

$$\mathcal{O}_{X_k}(a_1, \dots, a_k) := \pi_{1,k}^* \mathcal{O}_{X_1}(a_1) \otimes \cdots \otimes \mathcal{O}_{X_k}(a_k).$$

We set $s_{k,i} := s_i(\mathcal{F}_k)$, $s_i := s_i(\Omega_X)$, $u_k := c_1(\mathcal{O}_{X_k}(1))$. Moreover, if H is an ample line bundle on X, we usually set $h := c_1(H)$. We will make intersection computations on the different X_k, these computations will only involve the different tautological classes u_i and the class h, and therefore we introduce $\mathcal{C}_k(X,H) := \mathbb{Z} \cdot u_k \oplus \cdots \oplus \mathbb{Z} \cdot u_1 \oplus \mathbb{Z} \cdot h \subset N^1(X_k)$. To ease our computations we will also adopt the following abuses of notation: if $k > j$ we will write u_j (a class on X_k) instead of $\pi_{j,k}^* u_j$ and similarly $s_{j,i}$ instead of $\pi_{j,k}^* s_{j,i}$. This should not lead to any confusion.

Now, from the horizontal exact sequences in the diagram, the relative Euler exact sequence, Whitney's formula and (1), one easily derives (as in [Mou12]) the recursion formula

$$s_{k,\ell} = \sum_{j=0}^{\ell} M_{\ell,j}^n s_{k-1,j} u_k^{\ell-j} = s_{k-1,\ell} + \sum_{j=1}^{\ell} M_{\ell,j}^n s_{k-1,j} u_k^{\ell-j}, \tag{2}$$

where $M_{\ell,j}^n = \sum_{i=0}^{\ell-j} (-1)^i \binom{n-2+i+j}{i}$, and in particular $M_{\ell,\ell}^n = 1$.

Lemma 2.3. *Let $k \geqslant 0$, $a \geqslant 0$, $\ell \geqslant 0$, take ℓ positive integers $i_1, \ldots, i_\ell$ and m divisor classes $\gamma_1, \ldots, \gamma_m \in \mathcal{C}_k(X, H)$ such that $i_1 + \cdots + i_\ell + m + a = n_k$. Let $\gamma_q =: \alpha_{q,0} h + \sum_i \alpha_{q,i} u_i$. Then*

$$\int_{X_k} s_{k,i_1} \cdots s_{k,i_\ell} \gamma_1 \cdots \gamma_m h^a = \sum_{j_1,\ldots,j_{\ell+k},b} Q_{j_1,\ldots,j_{\ell+k},b} \int_X s_{j_1} \cdots s_{j_{\ell+k}} h^{a+b},$$

where in each term of the sum we have $b \geqslant 0$ and the $Q_{j_1,\ldots,j_{\ell+k},b}$ are polynomials in the $\alpha_{q,i}$ whose coefficients are independent of X. Moreover, up to reordering of the j_p one has $j_1 \leqslant i_1, \ldots, j_\ell \leqslant i_\ell$.

Proof. This is an immediate induction on k. The result is clear for $k = 0$. Now suppose it is true for some $k > 0$ and take m divisors, $\gamma_1, \ldots, \gamma_m \in \mathcal{C}_{k+1}(X)$ on X_{k+1}. Let $\gamma_q := \alpha_{q,0} h + \sum_i \alpha_{q,i} u_i$. Then using recursion formula (2) and expanding, we get

$$\begin{aligned}\int_{X_{k+1}} s_{k+1,i_1} \cdots s_{k+1,i_\ell} \gamma_1 \cdots \gamma_m h^a &= \sum P^{p_1,\ldots,p_{k+1}}_{j_i,\ldots,j_\ell,b} \int_{X_{k+1}} s_{k,j_1} \cdots s_{k,j_\ell} u_{k+1}^{p_{k+1}} \cdots u_1^{p_1} h^{a+b}\\ &= \sum P^{p_1,\ldots,p_{k+1}}_{j_i,\ldots,j_\ell,b} \int_{X_k} s_{k,j_1} \cdots s_{k,j_\ell} s_{k,r} u_k^{p_k} \cdots u_1^{p_1} h^{a+b},\end{aligned}$$

where in each term of the sum, $r = p_{k+1} - (n-1)$ and moreover, thanks to (2), one has $j_1 \leqslant i_1, \ldots, j_\ell \leqslant i_\ell$. Note also that the $P^{p_1,\ldots,p_{k+1}}_{j_i,\ldots,j_\ell,b}$ are polynomials in the $\alpha_{i,j}$ but their coefficients do not depend on X. Now we can finish by using the induction hypothesis. □

2.2 Segre classes for complete intersections

Let us introduce the setting in which we will work for the rest of this section. Fix M a smooth N-dimensional projective variety, and H an ample line bundle on M. Let $c \geqslant 1$ and take c ample line bundles $A_1, \ldots, A_c$ on M. Take $X := H_1 \cap \cdots \cap H_c$, a smooth complete intersection variety, where $H_i \in |d_i A_i|$ for some $d_i \in \mathbb{N}$. We will set $n := \dim X = N - c$. Also let $h := c_1(H)$ and $\alpha_i := c_1(A_1)$ for $1 \leqslant i \leqslant c$. Moreover, let $\kappa := \lceil n/c \rceil$ and take b such that $n = (\kappa - 1)c + b$; observe that $0 < b \leqslant c$. To simplify our formulas we also set $\hat{\imath} := i + n - 1$ so that $\pi_{k-1,k_*} u_k^{\hat{\imath}} = s_{k-1,i}$.

As we will be interested in the asymptotic behavior of polynomials in $\mathbb{Z}[d_1, \ldots, d_c]$ we need some more notation. Suppose $P \in \mathbb{Z}[d_1, \ldots, d_c]$; then $\deg P$ denotes the total degree of the polynomial and P^{dom} the homogeneous part of P of degree $\deg P$. We will write $P = o(d^k)$ if $\deg P < k$ and if $Q \in \mathbb{Z}[d_1, \ldots, d_c]$ is another polynomial we will write $P \sim Q$ if $P^{\mathrm{dom}} = Q^{\mathrm{dom}}$, and $P \gtrsim Q$ if $P^{\mathrm{dom}} \geqslant Q^{\mathrm{dom}}$ (i.e., $\forall (d_1, \ldots, d_c) \in \mathbb{N}^c$, $P^{\mathrm{dom}}(d_1, \ldots, d_c) \geqslant Q^{\mathrm{dom}}(d_1, \ldots, d_c)$).

Moreover, some of our computations will take place in the polynomial ring $A^*(X)[d_1, \ldots, d_c]$ where $A^*(X)$ denotes the Chow ring of X. The notation we will use are the natural ones. If $P \in A^\ell(X)[d_1, \ldots, d_c]$, we will write $\deg P$ for the degree of P as a polynomial in the d_i, and P^{dom} for the homogenous part of degree $\deg P$ of P. First we compute the Segre classes of Ω_X. To fix notation, let

$$s(\Omega_M) = 1 + \tau_1 + \cdots + \tau_N,$$

where $\tau_i \in A^i(M)$. Now, the conormal bundle exact sequence

$$0 \to \bigoplus_{i=1}^{c} A_{i|X}^{-d_i} \to \Omega_{M|X} \to \Omega_X \to 0$$

yields

$$s(\Omega_X) = \frac{s(\Omega_{M|X})}{s(\bigoplus_{i=1}^{c} A_{i|X}^{-d_i})} = s(\Omega_{M|X})c\bigg(\bigoplus_{i=1}^{c} A_{i|X}^{d_i}\bigg)$$
$$= (1+\tau_{1|X}+\cdots+\tau_{n|X})\prod_{i=1}^{c}(1+d_i\alpha_{i|X}).$$

Expanding the right-hand side as a polynomial in $A^*(X)[d_1,\ldots,d_c]$, we see that $\deg(s_\ell) = \min\{\ell, c\}$. When $\ell \leqslant c$ one has, moreover,

$$s_\ell^{\mathrm{dom}}(\Omega_X) = \sum_{j_1<\cdots<j_\ell} d_{j_1}\cdots d_{j_\ell}\alpha_{j_1|X}\cdots\alpha_{j_\ell|X} = c_\ell^{\mathrm{dom}}\bigg(\bigoplus_{i=1}^{c} A_{i|X}^{d_i}\bigg). \tag{3}$$

Remark 2.4. If $n \leqslant c$, (3) holds for all $\ell \in \mathbb{Z}$ (the case $\ell > n$ is obvious since both side of the equality vanish by a dimension argument).

Remark 2.5. The important point is that the α_i and the τ_i are independent of X so that, in particular, the intersection products involving the α_i and the τ_i are independent of X as well. This in turn implies that the intersection products involving the s_i depend only the d_i and of the intersections of the α_i and the τ_i.

With this we can give estimates for some intersection products on X.

Lemma 2.6. (a) *Let $0 \leqslant i_1 \leqslant \cdots \leqslant i_k$, $\ell > 0$ such that $i_1+\cdots+i_k+\ell = n$. Then*

$$\deg\bigg(\int_X s_{i_1}\cdots s_{i_k}h^\ell\bigg) < N.$$

(b) *Let $0 \leqslant i_1 \leqslant \cdots \leqslant i_k$ such that $i_1+\cdots+i_k = n$. Then $\int_X s_{i_1}\cdots s_{i_k}$ is of degree N if and only if $i_k \leqslant c$.*

(c) *Let $0 \leqslant i_1 \leqslant \cdots \leqslant i_\kappa$. If $i_1 < b$ or if $i_1 = b$ and $i_j < c$ for some $j > 1$, then*

$$\deg\bigg(\int_X s_{i_1}\cdots s_{i_\kappa}\bigg) < N.$$

Proof. Let $0 \leqslant \ell \leqslant n$. Observe that

$$\int_X s_{i_1}\cdots s_{i_k}h^\ell = d_1\cdots d_c\int_M \alpha_1\cdots\alpha_c\cdot s_{i_1}\cdots s_{i_k}h^\ell.$$

Therefore,

$$\deg\bigg(\int_X s_{i_1}\cdots s_{i_k}h^\ell\bigg) = \deg s_{i_1}+\cdots+\deg s_{i_k}+c.$$

Recall also that $\deg(s_i) = \min\{i, c\}$. Now, for the first point, let $\ell > 0$. Then,

$$\deg\bigg(\int_X s_{i_1}\cdots s_{i_k}h^\ell\bigg) \leqslant i_1+\cdots+i_k+c < n+c = N.$$

To see the second point, just observe that

$$\deg\bigg(\int_X s_{i_1}\cdots s_{i_k}\bigg) = \sum_{j=1}^{k}\min\{i_j, c\} \leqslant \sum_{j=1}^{k} i_j = N$$

https://doi.org/10.1112/S0010437X13007458 Published online by Cambridge University Press

and equality holds if and only if $i_j \leqslant c$ for all $1 \leqslant j \leqslant k$. The last point is an easy consequence of the second one thanks to the equality $n = (\kappa - 1)c + b$. □

2.3 Non-vanishing for jet differentials

We can now state and prove our non-vanishing theorem; it will show that Diverio's result [Div08, Theorem 7] is optimal in k.

Theorem 2.7. *Let M be a smooth N-dimensional projective variety and H an ample divisor on M. Fix $a \in \mathbb{N}$, fix $1 \leqslant c \leqslant N-1$, set $n := N - c$ and take $k \geqslant \kappa := \lceil n/c \rceil$. Take $A_1, \ldots, A_c$ very ample line bundles on M. For any positive integers $d_1, \ldots, d_c$, take generic hypersurfaces $H_1 \in |d_1 A_1|, \ldots, H_c \in |d_c A_c|$ and let $X = H_1 \cap \cdots \cap H_c$. There exists a constant $\Gamma_{N,n,a} \in \mathbb{N}$ such that if $d_i \geqslant \Gamma_{N,n,a}$ for all $1 \leqslant i \leqslant c$, then $\mathcal{O}_{X_k}(1) \otimes \pi_k^* H^{-a}$ is big on X_k. In particular, when $m \gg 0$,*

$$H^0(X, E_{k,m}\Omega_X \otimes H^{-ma}) \neq 0.$$

Remark 2.8. We will give an explicit bound on $\Gamma_{N,n,a}$ when $M = \mathbb{P}^N$ and $\kappa = 1$ in § A.2.

Take $r_0 \in \mathbb{N}$ such that $\Omega_M \otimes H^{r_0}$ is numerically effective (nef). Then, as in [Div08] and [Dem97], one can show that the line bundle

$$L_k := \mathcal{O}_{X_k}(2 \cdot 3^{k-2}, \ldots, 2 \cdot 3, 2, 1) \otimes \pi_k^* H^{r_0 \cdot 3^{k-1}}$$

is nef.

Remark 2.9. Diverio [Div08, Lemma 3] initially proved that L_k is nef for $M = \mathbb{P}^N$, $X \subset \mathbb{P}^N$ a hypersurface and $H = \mathcal{O}_{\mathbb{P}^N_{|X}}(1)$, in which case $r_0 = 2$ is sufficient. However, his proof works just as well in this more general setting. Moreover, X could be any smooth subvariety of M: we do not need the complete intersection hypothesis here.

We can write the first Chern class of L_k,

$$\ell_k := c_1(L_k) = u_k + \beta_k,$$

where β_k is a class that comes from X_{k-1}. Now we can state the main technical lemma. This is just the combination of Lemmas 2.3 and 2.6.

Lemma 2.10. *With the above notation we have the following estimates.*

(a) *Let $k \geqslant 1$ and $\gamma_1, \ldots, \gamma_{n_k - 1} \in \mathcal{C}_k(X, H)$. Then*

$$\int_{X_k} \gamma_1 \cdots \gamma_m h = o(d^N).$$

(b) *Let $\gamma_1, \ldots, \gamma_p \in \mathcal{C}_k(X, H)$ and $0 \leqslant i_1 \leqslant \cdots \leqslant i_q$ such that $p + \sum i_j = n_k$. If $i_1 < b$, or if $i_1 = b$ and $i_j < c$ for some $j > 1$, then*

$$\int_{X_k} s_{k,i_1} \cdots s_{k,i_q} \gamma_1 \cdots \gamma_p \quad = o(d^N), \tag{4}$$

$$\int_{X_k} s_{k-1,i_1} \cdots s_{k-1,i_q} \gamma_1 \cdots \gamma_p = o(d^N). \tag{5}$$

(c) *Let $0 < k < \kappa$. Then*

$$\int_{X_k} s_{k,b} s_{k,c}^{\kappa-k-1} \ell_k^{\hat{c}} \cdots \ell_1^{\hat{c}} = \int_{X_{k-1}} s_{k-1,b} s_{k-1,c}^{\kappa-k} \ell_{k-1}^{\hat{c}} \cdots \ell_1^{\hat{c}} + o(d^N).$$

Proof. Assertion (a) is an immediate consequence of Lemma 2.6(a) and Lemma 2.3. Similarly for assertion (b): thanks to Lemma 2.3, we write

$$\int_{X_k} s_{k,i_1}\cdots s_{k,i_q}\gamma_1\cdots\gamma_p=\sum_{j_1,\dots,j_{k+q}} Q_{j_1,\dots,j_{k+q}}\int_X s_{j_1}\cdots s_{j_{k+q}}h^a,$$

where $a \geqslant 0$ and, moreover, we know that, because $j_s \leqslant i_s$ for all s, in each term of this sum, either $j_1 < b$ or $j_p < c$ for some $p > 0$. Thus we can apply Lemma 2.6(a) (if $a > 0$) or 2.6(b) (if $a = 0$). To deduce (5), write $\gamma_i = a_i u_i + \beta_i$, where $\beta_i \in \mathcal{C}_{k-1}(X, H)$. Then,

$$\begin{aligned}\int_{X_k} s_{k-1,i_1}\cdots s_{k-1,i_q}\gamma_1\cdots\gamma_p &= \int_{X_k} s_{k-1,i_1}\cdots s_{k-1,i_q}(a_1u_1+\beta_1)\cdots(a_1u_1+\beta_p)\\ &=\sum_{I\subseteq\{1,\dots,p\}}\Big(\prod_{i\in I}a_i\Big)\int_{X_k} s_{k-1,i_1}\cdots s_{k-1,i_q}u_k^{|I|}\prod_{i\notin I}\beta_i\\ &=\sum_{I\subseteq\{1,\dots,p\}}\Big(\prod_{i\in I}a_i\Big)\int_{X_{k-1}} s_{k-1,i_1}\cdots s_{k-1,i_q}\cdot s_{k-1,|I|-n+1}\prod_{i\notin I}\beta_i\end{aligned}$$

and we finish by applying (4).

To see assertion (c), write

$$\begin{aligned}\int_{X_k} s_{k,b}s_{k,c}^{\kappa-k-1}\ell_k^{\hat c}\cdots\ell_1^{\hat c} &= \int_{X_k}\Big(\sum_{i=0}^{b}M_{b,i}^n s_{k-1,i}u_k^{b-i}\Big)\Big(\sum_{i=0}^{c}M_{c,i}^n s_{k-1,i}u_k^{c-i}\Big)^{\kappa-k-1}\ell_k^{\hat c}\cdots\ell_1^{\hat c}\\ &=\int_{X_k} s_{k-1,b}s_{k-1,c}^{\kappa-k-1}\ell_k^{\hat c}\cdots\ell_1^{\hat c}+o(d^N),\end{aligned}$$

obtained by expanding and using (5) in each term of the obtained sum. We finish with the following computation:

$$\begin{aligned}\int_{X_k} s_{k-1,b}s_{k-1,c}^{\kappa-k-1}\ell_k^{\hat c}\cdots\ell_1^{\hat c} &= \int_{X_k} s_{k-1,b}s_{k-1,c}^{\kappa-k-1}(u_k+\beta_k)^{\hat c}\ell_{k-1}^{\hat c}\cdots\ell_1^{\hat c}\\ &=\int_{X_k} s_{k-1,b}s_{k-1,c}^{\kappa-k-1}\sum_{i=0}^{\hat c}\binom{\hat c}{i}u_k^{\hat c-i}\beta_k^i\ell_{k-1}^{\hat c}\cdots\ell_1^{\hat c}\\ &=\int_{X_{k-1}} s_{k-1,b}s_{k-1,c}^{\kappa-k-1}\sum_{i=0}^{c}\binom{\hat c}{i}s_{k-1,i}\beta_k^i\ell_{k-1}^{\hat c}\cdots\ell_1^{\hat c}\\ &=\int_{X_{k-1}} s_{k-1,b}s_{k-1,c}^{\kappa-k-1}s_{k-1,c}\ell_{k-1}^{\hat c}\cdots\ell_1^{\hat c}+o(d^N)\\ &=\int_{X_{k-1}} s_{k-1,b}s_{k-1,c}^{\kappa-(k-1)-1}\ell_{k-1}^{\hat c}\cdots\ell_1^{\hat c}+o(d^N).\end{aligned}$$ □

Recall also the following consequence of Demailly's holomorphic Morse inequalities (see for example [Laz04a]).

Theorem 2.11. *Let Y be a smooth projective variety of dimension n and let F and G be nef divisors on Y. If $F^n > nG\cdot F^{n-1}$, then $F - G$ is big.*

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We are now ready to prove Theorem 2.7.

Proof. First we recall an argument from [Div09] to show that we just have to check that $\mathcal{O}_{X_k}(a_1,\dots,a_k)\otimes\pi_k^*\mathcal{O}_X(-a)$ is big for some suitable a_i. We know [Dem97] that $D_k:=\mathbb{P}(\Omega_{X_{k-1}/X_{k-2}})\subset X_k$ is an effective divisor which satisfies the relation $\pi_{k-1,k}^*\mathcal{O}_{X_{k-1}}(1)=\mathcal{O}_{X_k}(1)\otimes\mathcal{O}_{X_k}(-D_k)$. From this, an immediate induction shows that for any $k>1$ and any k-uple $(a_1,\dots,a_k)$ we have

$$\mathcal{O}_{X_k}(b_{k+1})=\mathcal{O}_{X_k}(a_1,\dots,a_k)\otimes\pi_{2,k}^*\mathcal{O}_{X_2}(b_1D_2)\otimes\cdots\otimes\mathcal{O}_{X_k}(b_{k-1}D_k),$$

where for all $j>0$, $b_j:=a_1+\cdots+a_j$. Thus when $0\leqslant b_j$ for all $0\leqslant j\leqslant k$ then $\pi_{2,k}^*\mathcal{O}_{X_2}(b_1D_2)\otimes\cdots\otimes\mathcal{O}_{X_k}(b_{k-1}D_k)$ is effective; this means that, under this condition, to prove that $\mathcal{O}_{X_k}(1)\otimes\pi_k^*\mathcal{O}_X(-a)$ is big it is sufficient to show that $\mathcal{O}_{X_k}(a_1,\dots,a_k)\otimes\pi_k^*\mathcal{O}_X(-a)$ is big.

Let $D=F-G$ where, as in [Mou12], we set $F:=L_k\otimes\cdots\otimes L_1$, and $G=\pi_k^*H^{m+a}$ where $m\geqslant 0$ is chosen so that $F\otimes\pi_k^*H^{-m}$ has no component coming from X. It is therefore sufficient to show that D is big. To do so, we will apply holomorphic Morse inequalities to F and G (both nef). We need to prove that

$$F^{n_k}>n_kF^{n_k-1}\cdot G.$$

Clearly, the right-hand side has degree strictly less than N in the d_i thanks to Lemma 2.10 and therefore we just have to show that the left-hand side is larger than a positive polynomial of degree N in the d_i. Let $\alpha:=c_1(\pi_k^*(\mathcal{O}_X(a)))$. Then

$$\begin{aligned}F^{n_k}&=\int_{X_k}(\ell_k+\cdots+\ell_1-\alpha)^{n_k}\\&=\int_{X_k}\sum_{i=0}^{n_k}(-1)^i\binom{n_k}{i}(\ell_k+\cdots+\ell_1)^{n_k-i}\alpha^i\\&=\int_{X_k}(\ell_k+\cdots+\ell_1)^{n_k}+o(d^N)\end{aligned}$$

by applying Lemma 2.10. However, since all the ℓ_i are nef we obtain

$$\begin{aligned}\int_{X_k}(\ell_l+\cdots+\ell_1)^{n_k}&\geqslant\int_{X_k}\ell_k^{n-1}\cdots\ell_{\kappa+1}^{n-1}\cdot\ell_\kappa^{\hat b}\cdot\ell_{\kappa-1}^{\hat c}\cdots\ell_1^{\hat c}\\&=\int_{X_\kappa}\ell_\kappa^{\hat b}\cdot\ell_{\kappa-1}^{\hat c}\cdots\ell_1^{\hat c}\\&=\int_{X_{\kappa-1}}s_{\kappa-1,b}\cdot\ell_{\kappa-1}^{\hat c}\cdots\ell_1^{\hat c}+o(d^N).\end{aligned}$$

The last inequality is obtained by using Lemma 2.10(b) Now an immediate induction proves that for all $p<\kappa$ one has

$$F^{n_\kappa}\geqslant\int_{X_p}s_{p,b}s_{p,c}^{\kappa-p-1}l_k^{\hat c}\cdots l_1^{\hat c}+o(d^N).$$

We just proved the case $p=\kappa-1$, and the other part of the induction is exactly the content of Lemma 2.10(c). Therefore,

$$F^{n_\kappa}\geqslant\int_X s_bs_c^{\kappa-1}+o(d^N).$$

Applying Lemma 2.6(b), we see that $\deg \int_X s_b s_c^{\kappa-1} = N$, and thus

$$F^{n_\kappa} \geqslant \int_X (s_b s_c^{\kappa-1})^{\mathrm{dom}} + o(d^N).$$

We just have to check that $\int_X (s_b s_c^{\kappa-1})^{\mathrm{dom}} > 0$. However, this follows from (3) and the fact that all the α_i are ample (recall that $\alpha_i = c_1(A_i)$). $\square$

3. Algebraic degeneracy and hyperbolicity for complete intersections

Here we show that from the work of [DMR10] and [DT10] one can deduce the hyperbolicity of generic complete intersections of high codimension and of high multidegree, just by 'moving' the hypersurfaces we are intersecting. We begin with a definition.

DEFINITION 3.1. Let X be a projective manifold. We define the *algebraic degeneracy locus* to be the Zariski closure of the union of all non-constant entire curves $f : \mathbb{C} \to X$:

$$\mathrm{dl}(X) := \overline{\bigcup f(\mathbb{C})}.$$

Recall the main result proven in [DMR10] and [DT10].

THEOREM 3.2 (Diverio *et al.* [DMR10], Diverio and Trapani [DT10]). *For any integer $N \geqslant 2$ there exists $\delta_N \in \mathbb{N}$ such that if $H \subset \mathbb{P}^N$ is a generic hypersurface of degree $d \geqslant \delta_N$, then there exists a proper algebraic subset $Y \subset H$ of codimension at least two in H such that $\mathrm{dl}(H) \subset Y$.*

More precisely, consider the universal hypersurface of degree d in $\mathbb{P}^N$

$$\mathcal{H}_d = \{(x, t) \in \mathbb{P}^N \times \mathbb{P}^{N_d} / x \in H_{d,t}\},$$

where $\mathbb{P}^{N_d} := \mathbb{P}(H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))^*)$ and $H_{d,t} = (t = 0)$. Denote by π_d the projection on the second factor. Then, for $d \geqslant \delta_N$, there exists an open subset $U_d \subset \mathbb{P}^{N_d}$ and an algebraic subset $\mathcal{Y}_d \subset \mathcal{H}_{d|U_d} \subset U_d \times \mathbb{P}^N$ such that for all $t \in U_d$, the fibre $Y_{d,t}$ has codimension 2 in $H_{d,t}$ and $\mathrm{dl}(H_{d,t}) \subset Y_{d,t}$.

Remark 3.3. A major result of [DMR10] is that δ_N is effective. They prove that $\delta_N \leqslant 2^{(N-1)^5}$ is sufficient.

We are going to use the standard action of $G := \mathrm{Gl}_{N+1}(\mathbb{C})$ on $\mathbb{P}^N$. For any $g \in G$ and any variety $X \subseteq \mathbb{P}^N$ we write $g \cdot X := g^{-1}(X)$.

Remark 3.4. Let $g \in G$ and $X \subset \mathbb{P}^N$ a projective variety. If $f : \mathbb{C} \to g \cdot X$ is a non-constant entire curve, then $g \circ f : \mathbb{C} \to X$ is a non-constant entire curve, therefore $g \circ f(\mathbb{C}) \subseteq \mathrm{dl}(X)$, and thus $f(\mathbb{C}) \subseteq g \cdot \mathrm{dl}(X)$. This proves that $g \cdot \mathrm{dl}(X) = \mathrm{dl}(g \cdot X)$.

Remark 3.5. Note also that if X_1 and X_2 are two projective varieties in $\mathbb{P}^N$, then $\mathrm{dl}(X_1 \cap X_2) \subseteq \mathrm{dl}(X_1) \cap \mathrm{dl}(X_2)$.

We will combine these remarks with the following moving lemma.

LEMMA 3.6. *Let $V \subset \mathbb{P}^N$ and $W \subset \mathbb{P}^N$ be algebraic subsets (not necessarily irreducible) such that $\dim(V) = n$ and $\dim(W) = m$. Take $g \in G$ to be generic.*

(i) *If $n + m \geqslant N$, then $\dim((g \cdot V) \cap W) = n + m - N$.*

(ii) *If $n + m < N$, then $(g \cdot V) \cap W = \varnothing$.*

Proof. The proof is done by a dimension count on suitable incidence varieties. First fix any $w \in \mathbb{P}^N$. Consider $G \times V$ with projections q_1 (respectively q_2) on G (respectively V). Consider the incidence variety

$$I_w := \{(g, v) \in G \times V / g^{-1}(v) = w\} \subseteq G \times V.$$

For any $v \in V$ the fiber $q_{2|I_w}^{-1}(\{v\}) \cong \{g \in G / g^{-1}(v) = w\}$ is easily seen to be of dimension $(N+1)^2 - N$. Therefore $\dim(I_w) = (N+1)^2 - N + n$. Moreover, $q_{1|I_w} : I_w \to G$ is one-to-one, and thus $q_1(I_w) = \{g \in G / w \in g \cdot V\}$ is of dimension $(N+1)^2 - N + n$.

Now consider $G \times W$ with projections p_1 (respectively p_2) on G (respectively W). And consider the incidence variety

$$I := \{(g, w) \in G \times W / w \in g \cdot V\} \subseteq G \times W.$$

For any $w \in W$ the fiber $p_{2|I}^{-1}(\{w\}) \cong \{g \in G / w \in g \cdot V\} = q_1(I_w)$ is of dimension $(N+1) - N + n$. Therefore $\dim(I) = (N+1)^2 - N + n + m$. Now since $p_{1|I}^{-1}(\{g\}) = g \cdot V \cap W$ and $\dim G = (N+1)^2$ the result follows. $\square$

We need another lemma.

Lemma 3.7. *Let X be a smooth projective variety. Suppose that there is an algebraic subset $Z \subseteq X$ of codimension $c \geqslant 0$ such that $\mathrm{dl}(X) \subseteq Z$. Fix an ample line bundle A on X. For $e \in \mathbb{N}$ big enough, take $H \in |eA|$ to be a generic hypersurface. Then there is an algebraic subset $Z' \subset H$ of codimension at least $2 + c$ in H such that $\mathrm{dl}(H) \subseteq Z'$.*

Proof. As we are looking at the situation for e big enough we might as well suppose that A is very ample. We use A to embed $X \subseteq \mathbb{P}^N$. Under this embedding we have the identification $A \cong \mathcal{O}_{\mathbb{P}^N}(1)_{|X}$. Take $d \geqslant \delta_N$ so that we can apply Theorem 3.2 in $\mathbb{P}^N$. Moreover, take d big enough to have

$$H^1(\mathbb{P}^N, \mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}^N}(d)) = 0.$$

Therefore one has a surjection

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d)) \to H^0(X, \mathcal{O}_X(d)) \to 0.$$

We are therefore able to extend all the hypersurfaces we are interested in to hypersurfaces of $\mathbb{P}^N$.

We decompose the rest of the proof into three assertions.

Assertion 1. *A generic hypersurface $D \in |\mathcal{O}_X(d)|$ can be extended to a generic hypersurface $H \in |\mathcal{O}_{\mathbb{P}^N}(d)|$. More precisely, for any non-empty open subset $U \subseteq |\mathcal{O}_{\mathbb{P}^N}(d)|$, there exists an non-empty open subset $U_X \subseteq |\mathcal{O}_X(d)|$, such that for any $D \in U_X$ there exists $H \in U$ such that $D = X \cap H$.*

Assertion 2. *For a generic hypersurface $H \in |\mathcal{O}_{\mathbb{P}^N}(d)|$ and a generic $g_H \in \mathrm{Gl}_{N+1}(\mathbb{C})$ (genericity depending on H) there exists an algebraic subset $Z' \subset X \cap g_H \cdot H$ of codimension at least $2 + c$ such that $\mathrm{dl}(X \cap g_H \cdot H) \subseteq Z'$.*

Assertion 3. *For a generic $H \in |\mathcal{O}_{\mathbb{P}^N}(d)|$ there exists an algebraic subset $Z' \subset X \cap H$ of codimension at least $2 + c$ such that $\mathrm{dl}(X \cap H) \subseteq Z'$.*

The lemma then clearly follows from Assertions 1 and 3. The subtlety of this lemma is the precise meaning of 'generic' at each step.

Proof of Assertion 1. Let $U \subseteq \mathbb{P}^{N_d} = |\mathcal{O}_{\mathbb{P}^N}(d)|$, be a non-empty open subset which contains the genericity assumption (i.e., H is generic if $H \in U$). We have to prove that there is an open

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subset $U_X \subseteq \mathbb{P}^{N_d(X)} := |\mathcal{O}_X(d)|$ such that any $D \in U_X$ can be extended to an element $H \in U$. Let $W := \mathbb{P}(H^0(\mathbb{P}^N, \mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}^N}(d))^*)$. We therefore obtain a surjective map,

$$\mathbb{P}^{N_d} \backslash W \xrightarrow{\pi} \mathbb{P}^{N_d(X)}.$$

Consider the image $\pi(U\backslash W)$; it is constructible and dense, therefore there exists a dense open subset $U_X \subseteq \pi(U\backslash W)$ which satisfies the expected condition. This simple argument, which simplifies our original argument, was pointed out to us by Olivier Debarre.

Proof of Assertion 2. Take a generic $H \in |\mathcal{O}_{\mathbb{P}^N}(d)|$. By Theorem 3.2 we know that there exists $Y \subset H$ such that $\mathrm{dl}(H) \subset Y$. Then applying Lemma 3.6 to Z and Y combined with Remarks 3.5 and 3.4 yields the expected result.

Proof of Assertion 3. We take the notation of Theorem 3.2. Consider the family $\mathcal{Y}'_d := \mathcal{Y}_d \cap pr_1^{-1} Z \subset \mathcal{H}_d \cap pr_1^{-1} X \subset U \times X$. The application $t \mapsto \dim(Y'_{d,t})$ is upper semi-continuous. Therefore, $W := \{t \in U / \dim(Y'_{d,t}) > n - c - 2\}$ is a closed subset. Applying Assertion 2 tells us that the complement is non-empty and therefore, an open dense subset.

This completes the proof of the Lemma. □

From this lemma, Theorem B follows as a straightforward induction.

Corollary 3.8. *Let M be a smooth N-dimensional projective variety. Take $A_1, \dots, A_c$ ample line bundles on M. For $d_1, \dots, d_c \in \mathbb{N}$ big enough take generic hypersurfaces $H_1 \in |d_1 A_1|, \dots, H_c \in |d_c A_c|$ and let $X = H_1 \cap \cdots \cap H_c$. Then, there exists an algebraic subset $Z \subset X$ of codimension at least $2c$ such that $\mathrm{dl}(X) \subseteq Z$. In particular, when $2c \geqslant n$ then X is hyperbolic.*

In particular we obtain the following consequence of Debarre's conjecture.

Corollary 3.9. *The intersection in $\mathbb{P}^N$ of at least $N/2$ generic hypersurfaces of sufficiently high degrees is hyperbolic.*

Remark 3.10. The case $M = \mathbb{P}^N$ is of particular interest to us. In this situation we let $A_1 = \cdots = A_c = \mathcal{O}_{\mathbb{P}^N}(1)$ which is very ample. Doing the proof in this particular setting, we see that it is sufficient to take $d_i \geqslant \delta_N$ to have the conclusions of Theorem A, where δ_N is any bound that holds in the Diverio–Merker–Rousseau theorem, for example $2^{(N-1)^5}$.

4. Positivity of the cotangent bundle

From now on we will focus on the positivity of the cotangent bundle of a complete intersection in $\mathbb{P}^N$. The organization is the following. In § 4.1 we give a geometric interpretation of the ampleness of $\Omega_X \otimes \mathcal{O}_{\mathbb{P}^N}(2)_{|X}$. In § 4.2 we prove that all positivity conditions on the Chern classes that one might expect are indeed satisfied, which proves the numerical side of the conjecture. Then in § 4.3 we state and prove our main algebraic results, admitting, temporarily, a global generation property whose proof is delayed to § A.1.

4.1 Ampleness of $\Omega_X(2)$

The first remark to make is that the cotangent bundle of an N-dimensional abelian variety A is globally generated. Therefore if X is a smooth subvariety of A, the ampleness of Ω_X is equivalent to the finiteness of the induced map $\mathbb{P}(\Omega_X) \to \mathbb{P}(\Omega_{A,0}) = \mathbb{P}^{N-1}$ (see [Deb05] for more details). There is no such interpretation if X is a smooth subvariety of $\mathbb{P}^N$. However, one can observe that

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$\Omega_{\mathbb{P}^N}(2)$ is globally generated, but not ample. This induces a map $\mathbb{P}(\Omega_X(2)) \to \mathbb{P}^r$ for some $r \in \mathbb{N}$, and the ampleness of $\Omega_X(2)$ is equivalent to the finiteness of that map. As the ampleness of $\Omega_X(2)$ would follow directly from the ampleness of Ω_X, it seems natural to determine for which varieties this bundle is ample. It turns out that this has a simple geometric interpretation, which is that $\Omega_X(2)$ is ample if and only if there are no lines in X, as we shall now see.

Fix an $(N+1)$-dimensional complex vector space V. Denote by $\mathbb{P}^N = \mathbb{P}(V^*)$ the projectivized space of *lines* in V, by $p: V \to \mathbb{P}(V^*)$ the projection, and by $\mathrm{Gr}(2, V) = \mathrm{Gr}(1, \mathbb{P}^N)$ the space of vector planes in V which is also the space of lines in $\mathbb{P}^N$. We will also consider the projection $\pi : \mathbb{P}(\Omega_{\mathbb{P}^N}) \to \mathbb{P}^N$. The key point is the following lemma, which was pointed out to us by Frederic Han.

Lemma 4.1. *There is a map $\varphi : \mathbb{P}(\Omega_{\mathbb{P}^N}) \to \mathbb{P}(\Lambda^2 V^*)$ such that $\varphi^* \mathcal{O}_{\mathbb{P}(\Lambda^2 V^*)}(1) = \mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^N})}(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}^N}(2)$. Moreover, this application factors through the Plücker embedding $\mathrm{Gr}(2, \mathbb{P}^N) = \mathrm{Gr}(2, V) \hookrightarrow \mathbb{P}(\Lambda^2 V^*)$. More precisely, an element $(x, [\xi]) \in \mathbb{P}(\Omega_{\mathbb{P}^N})$ with $x \in \mathbb{P}^N$ and $\xi \in T_x\mathbb{P}^N$, gets mapped to the unique line Δ in $\mathbb{P}^N$ satisfying $\xi \in T_x\Delta \subseteq T_x\mathbb{P}^N$.*

Proof. Take the Euler exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^N} \to V \otimes \mathcal{O}_{\mathbb{P}^N}(1) \to T\mathbb{P}^N \to 0$$

and apply Λ^{N-1} to it in order to get the quotient

$$\Lambda^{N-1} V \otimes \mathcal{O}_{\mathbb{P}^N}(N-1) \to \Lambda^{N-1} T\mathbb{P}^N \to 0.$$

Now using the well-known dualities, $\Lambda^{N-1} V = \Lambda^2 V^* \otimes \det V$ and $\Lambda^{N-1} T\mathbb{P}^N = \Omega_{\mathbb{P}(V^*)} \otimes K^*_{\mathbb{P}(V^*)} = \Omega_{\mathbb{P}(V^*)} \otimes \mathcal{O}_{\mathbb{P}(V^*)}(N+1) \otimes \det V$, and tensoring everything by $\mathcal{O}_{\mathbb{P}^N}(1-N) \otimes \det V^*$, we get

$$\Lambda^2 V^* \to \Omega_{\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^N}(2) \to 0.$$

This yields the map $\varphi : \mathbb{P}(\Omega_{\mathbb{P}^N}) = \mathbb{P}(\Omega_{\mathbb{P}^N}(2)) \hookrightarrow \mathbb{P}^N \times \mathbb{P}(\Lambda^2 V^*) \to \mathbb{P}(\Lambda^2 V^*)$ such that

$$\varphi^* \mathcal{O}_{\mathbb{P}(\Lambda^2 V^*)}(1) = \mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^N}(2))}(1) = \mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^N})}(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}^N}(2).$$

To see the geometric interpretation of this map, it suffices to backtrack through the previous maps. Take a point $x \in \mathbb{P}^N$ and a vector $0 \neq \xi \in T_x\mathbb{P}^N$ and fix a basis $(\xi_0, \dots, \xi_{N-1})$ of $T_x\mathbb{P}^N$ such that $\xi_0 = \xi$. Now take $v \in V$ such that $p(v) = x$ and a basis $(e_0, \dots, e_N)$ of $T_vV = V$ such that $d_vp(e_N) = 0$ and $d_vp(e_i) = \xi_i$ for $i < N$. We just have to check that $(x, [\xi])$ is mapped to the announced line Δ which, with our notation, corresponds to the point $[e_0 \wedge e_N] \in \mathbb{P}(\Lambda^2 V^*)$. This is easily verified as the above maps can be described explicitly, as follows:

$$\begin{array}{ccccccc} \mathbb{P}(\Omega_{\mathbb{P}^N,x}) & \to & \mathbb{P}(\Lambda^{N-1} T_x\mathbb{P}^N) & \to & \mathbb{P}(\Lambda^{N-1} V) & \to & \mathbb{P}(\Lambda^2 V^*) \\ [\xi_0] & \mapsto & [\xi_1^* \wedge \cdots \wedge \xi_{N-1}^*] & \mapsto & [e_1^* \wedge \cdots \wedge e_{N-1}^*] & \mapsto & [e_0 \wedge e_N]. \end{array}$$

□

With this we can prove our proposition,

Proposition 4.2. *Let $X \subseteq \mathbb{P}^N$ be a smooth variety. Then $\Omega_X(2)$ is ample if and only if X does not contain any line.*

Proof. By Lemma 4.1, we know that $\Omega_X(2)$ is ample if and only if the restriction $\varphi_X : \mathbb{P}(\Omega_X) \subseteq \mathbb{P}(\Omega_{\mathbb{P}^N}) \to \mathrm{Gr}(2, \mathbb{P}^N)$ of φ is finite.

Now, if X contains a line Δ then φ_X is not finite since the curve $\mathbb{P}(K_\Delta) \subseteq \mathbb{P}(\Omega_X)$ gets mapped to the point in $\mathrm{Gr}(2, \mathbb{P}^N)$ representing Δ.

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If φ_X is not finite then there is a curve $C \subseteq \mathbb{P}(\Omega_X)$ which gets mapped to a point in $\mathrm{Gr}(2, \mathbb{P}^N)$ corresponding to a line Δ in $\mathbb{P}^N$. Let $\Gamma = \pi(C)$. Lemma 4.1 tells us that the embedded tangent space $\mathbb{T}_x\Gamma$ equals Δ for all $x \in \Gamma$ and therefore $\Delta \subseteq X$. $\square$

Note that by a dimension count on the incidence variety of lines on the universal hypersurface one can see that a generic hypersurface $H \subset \mathbb{P}^N$ of degree $d > 2N - 3$ contains no line and therefore $\Omega_H \otimes \mathcal{O}_{\mathbb{P}^N}(2)$ is ample. In particular, this holds for the complete intersections considered in Debarre's conjecture, which would directly follow from the ampleness of the cotangent bundle. Therefore, this provides a weak piece of evidence towards Debarre's conjecture.

4.2 Numerical positivity of the cotangent bundle

As another piece of evidence towards this conjecture we will now check that all the positivity conditions on the Chern classes that one might expect (according to a theorem of Fulton and Lazarsfeld [FL83]) are indeed satisfied. We start by a preliminary section on numerical positivity before turning to the proof of our result.

4.2.1 *Numerical positivity.* Following Fulton [Ful98] we recall definitions concerning Schur polynomials. Let $c_1, c_2, c_3, \dots$ be a sequence of formal variables. Let ℓ be a positive integer and let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be a partition of ℓ. We define the Schur polynomial associated to $c = (c_i)_{i \in \mathbb{N}}$ and λ to be

$$\Delta_\lambda(c) := \det[(c_{\lambda_i + j - i})_{1 \leqslant i,j \leqslant \ell}].$$

For this to make sense we set $c_0 = 1$ and $c_i = 0$ for $i < 0$; this always holds in our applications. For example, $\Delta_{(1)}(c) = c_1$, $\Delta_{(2,0)}(c) = c_2$ and $\Delta_{(1,1)}(c) = c_1^2 - c_2$.

Now consider two sequences of formal variables, $c_1, c_2, c_3, \dots$ and $s_1, s_2, s_3, \dots$ satisfying the relation

$$(1 + c_1 t + c_2 t^2 + \cdots) \cdot (1 - s_1 t + s_2 t^2 - \cdots) = 1. \tag{6}$$

Note that (6) is satisfied when $c_i = c_i(E)$ are the Chern classes of a vector bundle E over a variety X and $s_i = s_i(E)$ are its Segre classes.

The proof of the following crucial combinatorial result can be found in [Ful98].

Lemma. *Use the same notation. Let $\bar{\lambda}$ be the conjugate partition of λ; then $\Delta_\lambda(c) = \Delta_{\bar{\lambda}}(s)$.*

Let E be a vector bundle of rank r over a projective variety X of dimension n.

Definition 4.3. We will say that E is numerically positive if for any irreducible subvariety $Y \subseteq X$ and for any partition λ of $\ell = \dim(Y)$ one has $\int_Y \Delta_\lambda(c(E)) > 0$.

This definition is motivated by a theorem of Fulton and Lazarsfeld [FL83] which gives numerical consequences of ampleness.

Theorem 4.4 (Fulton–Lazarsfeld [FL83]). *If E is ample then E is numerically positive. Moreover, the Schur polynomials are exactly the relevant polynomials to test ampleness numerically.*

We refer to [FL83] and to [Laz04b] for further details. Note that the converse of this theorem is false; for example, the bundle $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ over $\mathbb{P}^1$ is numerically positive but not ample (we only have to check that $c_1 > 0$ because the conditions are empty for all the other Schur polynomials). See [Ful76] for a more interesting example.

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4.2.2 *Segre classes for complete intersection in* $\mathbb{P}^N$. To prove our result, we need to compute the Segre classes of the twisted cotangent bundle to a complete intersection variety in $\mathbb{P}^N$. In this case one can be more precise that what we did previously. Let us introduce our notation. From now on, we will take $d_1, \dots, d_c \in \mathbb{N}$, and for each $1 \leqslant i \leqslant c$ we take $\sigma_i \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d_i))$ such that $X := H_1 \cap \cdots \cap H_c$ is a smooth complete intersection, where $H_i := (\sigma_i = 0)$. We also set $h_{\mathbb{P}^N} := c_1(\mathcal{O}_{\mathbb{P}^N}(1))$ and $h := c_1(\mathcal{O}_X(1))$. Our computations will take place in the Chow ring $A^*(X)$, so we introduce some more notation. If P is a polynomial in $\mathbb{Z}[d_1, \dots, d_c, h]$, homogeneous in h and of degree k in h, we will write $\tilde{P}$ for the unique polynomial in $\mathbb{Z}[d_1, \dots, d_c]$ satisfying $P = \tilde{P} h^k$.

Let us detail our computations. Let $m \in \mathbb{Z}$. The twisted Euler exact sequence

$$0 \to \Omega_{\mathbb{P}^N}(-m) \to \mathcal{O}_{\mathbb{P}^N}^{\oplus N+1}(-m-1) \to \mathcal{O}_{\mathbb{P}^N}(-m) \to 0$$

yields

$$s(\Omega_{\mathbb{P}^N}(-m)) = \frac{s(\mathcal{O}_{\mathbb{P}^N}^{\oplus N+1}(-m-1))}{s(\mathcal{O}_{\mathbb{P}^N}(-m))} = \frac{(1 + m h_{\mathbb{P}^N})}{(1 + (1+m) h_{\mathbb{P}^N})^{N+1}}.$$

The twisted conormal bundle exact sequence

$$0 \to \bigoplus_{i=1}^{c} \mathcal{O}_X(-d_i - m) \to \Omega_{\mathbb{P}^N_{|X}}(-m) \to \Omega_X(-m) \to 0$$

yields

$$\begin{aligned} s(\Omega_X(-m)) &= \frac{s(\Omega_{\mathbb{P}^N_{|X}}(-m))}{s(\bigoplus_{i=1}^{c} \mathcal{O}_X(-d_i - m))} = \frac{(1 + mh) \prod_{i=1}^{c} (1 + (d_i + m)h)}{(1 + (1+m)h)^{N+1}} \\ &= (1 - (1+m)h + (1+m)^2 h^2 - \cdots)^{N+1} (1 + mh) \prod_{i=1}^{c} (1 + (d_i + m)h). \end{aligned}$$

Expanding the right-hand side as a polynomial in $\mathbb{Z}[d_1, \dots, d_c, h]$, we see that for $\ell \geqslant c$ we have $\deg(\tilde{s_\ell}) = c$ and that for $\ell \leqslant c$ we have

$$s_\ell^{\mathrm{dom}}(\Omega_X(-m)) = \sum_{j_1 < \cdots < j_\ell} d_{j_1} \cdots d_{j_\ell} h^\ell = c_\ell^{\mathrm{dom}}\bigg(\bigoplus_{i=1}^{c} \mathcal{O}_X(d_i + m)\bigg) = c_\ell\bigg(\bigoplus_{i=1}^{c} \mathcal{O}_X(d_i)\bigg). \tag{7}$$

4.2.3 *Numerical positivity of the cotangent bundle.* We can now prove that the cotangent bundle of a complete intersection variety is numerically positive if the codimension is greater than the dimension and if the multidegree is big enough. We will use the notation of § 4.2.2.

THEOREM 4.5. *Fix $a \in \mathbb{Z}$. There exists $D_{N,n,a} \in \mathbb{N}$ such that if $X \subset \mathbb{P}^N$ is a smooth complete intersection of dimension n, of codimension c, and multidegree $(d_1, \dots, d_c)$ such that $c \geqslant n$ and $d_i > D_{N,n,a}$ for all i, then $\Omega_X(-a)$ is numerically positive.*

Proof. By the lemma in § 4.2.1 we have to check that for any subvariety $Y \subseteq X$ of dimension ℓ and for any partition λ of ℓ one has $\int_Y \Delta_{\bar{\lambda}}(s(\Omega_X(-a))) > 0$. Moreover, $\int_Y \Delta_{\bar{\lambda}}(s(\Omega_X(-a))) = \tilde{\Delta}_{\bar{\lambda}}(s(\Omega_X(-a))) \int_Y h^\ell$, and thus we just have to check that $\tilde{\Delta}_{\bar{\lambda}}(s(\Omega_X(-a))) > 0$ when the d_i are large enough, which is equivalent to $\tilde{\Delta}_{\bar{\lambda}}^{\mathrm{dom}}(s(\Omega_X(-a))) > 0$. Now the equality

$$\tilde{\Delta}_{\bar{\lambda}}^{\mathrm{dom}}(s(\Omega_X(-a))) = \det(\tilde{s}_{\lambda_i + j - i}^{\mathrm{dom}}(\Omega_X(-a)))_{1 \leqslant i,j \leqslant \ell} \tag{8}$$

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holds if one can prove that the right-hand side is non-zero. However, by (7) we find

$$\det(\tilde{s}^{\mathrm{dom}}_{\lambda_i+j-i}(\Omega_X(-a)))_{1\leqslant i,j\leqslant \ell} = \det\bigg(\tilde{c}_{\lambda_i+j-i}\bigg(\bigoplus_{j=1}^{k}\mathcal{O}(d_j)\bigg)\bigg)_{1\leqslant i,j\leqslant \ell} = \tilde{\Delta}_{\bar{\lambda}}\bigg(c\bigg(\bigoplus_{j=1}^{k}\mathcal{O}(d_j)\bigg)\bigg).$$

By applying the theorem of Fulton and Lazarsfeld to $\bigoplus_{j=1}^{k}\mathcal{O}_X(d_j)$ (which is ample if $d_i > 0$) we find that this is positive. This yields equality in (8), and we get the desired result. □

Remark 4.6. Note that there are no assumptions on the genericity of our complete intersection.

Remark 4.7. We mention that it is easy to obtain an explicit bound $D_{N,2,0}$. We state it without proof: $D_{4,2,0} = 9$, $D_{5,2,0} = 5$, $D_{6,2,0} = D_{6,2,0} = 4$, $D_{N,2,0} = 3$ for $8 \leqslant N \leqslant 12$ and $D_{N,2,0} = 2$ for $N \geqslant 13$.

4.3 Almost everywhere ampleness of the cotangent bundle

We will now state our main algebraic results.

Theorem 4.8. *Fix $a \in \mathbb{N}$. There exists $\delta_{N,n,a} \in \mathbb{N}$ such that, if $X \subset \mathbb{P}^N$ is a generic complete intersection of dimension n, codimension c and multidegree $(d_1, \dots, d_c)$ satisfying $c \geqslant n$ and $d_i \geqslant \delta_{N,n,a}$ for all $1 \leqslant i \leqslant c$, then $\mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi^*_{\Omega_X}\mathcal{O}_X(-a)$ is big and there exists a subset $Y \subset X$ of codimension at least two such that*

$$\pi_{\Omega_X}(\mathrm{Bs}(\mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi^*_{\Omega_X}\mathcal{O}_X(-a))) \subseteq Y.$$

Remark 4.9. We will give an effective bound for $\delta_{N,n,a}$ in § A.2.

We postpone the proof to point out some noteworthy conclusions. First, we consider almost everywhere ampleness in the sense of Miyaoka [Miy83]. Let us recall the definition.

Definition 4.10. Let E be a vector bundle on a variety X and H an ample line bundle on X. Denote the projection by $\pi_E : \mathbb{P}(E) \to X$. Take $T \subset X$. We say that E is ample modulo T if for a sufficiently small $\epsilon > 0$, any irreducible curve $C \subset \mathbb{P}(E)$ such that $\int_C c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) < \epsilon C \cdot \pi^*_E H$ satisfies $\pi_E(C) \subseteq T$. We say that E is almost everywhere ample if it is ample modulo a proper closed algebraic subset of X.

Remark 4.11. If a rank-e vector bundle E on X is ample modulo a finite number of points, then E is ample.

With this definition we can state the following.

Corollary 4.12. *Fix $a \in \mathbb{N}$. If $X \subseteq \mathbb{P}^N$ is a generic complete intersection variety of dimension n, codimension c and multidegree $(d_1, \dots, d_c)$ satisfying $c \geqslant n$ and $d_i \geqslant \delta_{N,n,a+1}$, then $\Omega_X \otimes \mathcal{O}_X(-a)$ is ample modulo an algebraic subset of codimension at least two in X.*

Proof. Applying the theorem, we find that $\mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi^*_{\Omega_X}\mathcal{O}_X(-a-1)$ is big and that there exists an algebraic subset $Y \subset X$ of codimension 2 in X such that $\pi_{\Omega_X}(\mathrm{Bs}(\mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi^*_{\Omega_X}\mathcal{O}_X(-a-1))) \subseteq Y$. Now take any $\epsilon \leqslant 1$. Take an irreducible curve $C \subset \mathbb{P}(\Omega_X(-a)) = \mathbb{P}(\Omega_X)$ such that

$$\begin{aligned}\int_C c_1(\mathcal{O}_{\mathbb{P}(\Omega_X(-a))}(1)) &= \int_C c_1(\mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi^*_{\Omega_X}\mathcal{O}_X(-a)) \\ &< \epsilon \int_C c_1(\pi^*_{\Omega_X}\mathcal{O}_X(1)) \leqslant \int_C c_1(\pi^*_{\Omega_X}\mathcal{O}_X(1)).\end{aligned}$$

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This implies that $\int_C c_1(\mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi^*_{\Omega_X}\mathcal{O}_X(-a-1)) < 0$. Thus, in particular, we get $C \subseteq \mathrm{Bs}(\mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi^*_{\Omega_X}\mathcal{O}_X(-a-1))$ and thus $\pi_{\Omega_X}(C) \subseteq Y$. □

In view of Remark 4.11, we also obtain a positive answer to Debarre's conjecture for surfaces (the bound given here will be discussed in § A.2.4).

COROLLARY 4.13. *If $N \geqslant 4$ and $S \subset \mathbb{P}^N$ a generic complete intersection surface of multidegree $(d_1, \ldots, d_{N-2})$ satisfying $d_i > (8N+2)/(N-3)$, then Ω_S is ample.*

In particular, in $\mathbb{P}^4$, the intersection of two generic hypersurfaces of degree greater than 35 has an ample cotangent bundle. To our knowledge this is the first example of surfaces with an ample cotangent bundle in $\mathbb{P}^4$, a question that was already raised by Schneider in [Sch92].

4.3.1 *Proof of Theorem 4.8.* The proof is based on the ideas of Siu [Siu04], further developed by Diverio *et al.* [DMR10].

First let us introduce some notation. Let $\mathbf{P} := \mathbb{P}^{N_{d_1}} \times \cdots \times \mathbb{P}^{N_{d_c}}$ where $\mathbb{P}^{N_{d_i}} := \mathbb{P}(H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d_i))^*)$ be the parameter space, and $\mathcal{X} := \{(x, (t_1, \ldots, t_c)) \in \mathbb{P}^N \times \mathbf{P} / t_i(x) = 0\ \forall i\}$ be the universal complete intersection. We will also denote by $\rho_1 : \mathcal{X} \to \mathbb{P}^N$ the projection onto the first factor and $\rho_2 : \mathcal{X} \to \mathbf{P}$. We will use the standard notation $\mathcal{O}_{\mathbf{P}}(a_1, \ldots, a_c)$ to denote line bundles on $\mathbf{P}$. Also, write $\pi : \mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}}) \to \mathcal{X}$ for the standard projection. As in [DMR10] the proof of Theorem 4.8 is based on Theorem 2.7 and on a global generation property that we will prove in § A.1.

THEOREM A.1. *The bundle*

$$T\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}}) \otimes \pi^*\rho_1^*\mathcal{O}_{\mathbb{P}^N}(N) \otimes \pi^*\rho_2^*\mathcal{O}_{\mathbf{P}}(1, \ldots, 1)$$

is globally generated on $\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}})$.

Observe also the following simple remark.

Remark 4.14. Let $X \subset \mathbb{P}^N$ be any projective variety. If $q \geqslant 0$ then

$$\mathrm{Bs}(\mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi^*_{\Omega_X}\mathcal{O}_X(-a)) \subseteq \mathrm{Bs}(\mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi^*_{\Omega_X}\mathcal{O}_X(-a-q)).$$

We are now in position to give the proof of Theorem 4.8. We will prove that $\delta_{N,n,a} := \Gamma_{N,n,a+N}$ (with the notation of Theorem 2.7) will suffice. Fix a multidegree $(d_1, \ldots, d_c)$ such that $d_i \geqslant \delta_{N,n,a} = \Gamma_{N,n,a+N}$ for all $0 \leqslant i \leqslant c$. We start by applying Theorem 2.7 to find some $k \in \mathbb{N}$ such that $H^0(Y, S^k\Omega_Y \otimes \mathcal{O}_Y(-ka-kN)) \neq 0$ for all smooth complete intersection Y of multidegree $(d_1, \ldots, d_c)$. Using the semi-continuity theorem ([Har77, Theorem 12.8, p. 288]) we find a non-empty open subset $U \subset \mathbf{P}$ such that the restriction map

$$H^0(\mathcal{X}_U, S^k\Omega_{\mathcal{X}/\mathbf{P}} \otimes \rho_1^*\mathcal{O}_{\mathbb{P}^N}(-ka-kN)) \to H^0(X_t, S^k\Omega_{X_t} \otimes \mathcal{O}_{X_t}(-ka-kN))$$

is surjective for all $t \in U$, where $\mathcal{X}_U := \rho_2^{-1}(U)$ and $X_t := \rho_2^{-1}(t)$. Fix $t_0 \in U$ such that the corresponding complete intersection X_{t_0} is smooth, take a non-zero section

$$\sigma_0 \in H^0(X_{t_0}, S^k\Omega_{X_{t_0}} \otimes \mathcal{O}_{X_{t_0}}(-ka-kN)),$$

and extend it to a section

$$\sigma \in H^0(\mathcal{X}_U, S^k\Omega_{\mathcal{X}/\mathbf{P}} \otimes \rho_1^*\mathcal{O}_{\mathbb{P}^N}(-ka-kN)).$$

Let $\mathcal{Y} := (\sigma = 0) \subset \mathcal{X}_U$. We will prove that

$$\pi_{t_0}(\mathrm{Bs}(\mathcal{O}_{\mathbb{P}(\Omega_{X_{t_0}})}(1) \otimes \pi^*_{\Omega_{X_{t_0}}}\mathcal{O}_{X_{t_0}}(-a-q))) \subset Y_{t_0},$$

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where $Y_t := X_t \cap \mathcal{Y}$. Denote by $\tilde{\sigma}$ the section in $H^0(\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}})_{|U}, \mathcal{O}_{\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}})}(k) \otimes \pi^*\rho_1^*\mathcal{O}(-ka - kN))$ corresponding to σ under the canonical isomorphism

$$H^0(\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}})_{|U}, \mathcal{O}_{\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}})}(k) \otimes \pi^*\rho_1^*\mathcal{O}_{\mathbb{P}^N}(-ka - kN)) \simeq H^0(\mathcal{X}_U, S^k\Omega_{\mathcal{X}/\mathbf{P}} \otimes \rho_1^*\mathcal{O}_{\mathbb{P}^N}(-ka - kN)).$$

Let $x \in \mathrm{Bs}(\mathcal{O}_{\mathbb{P}(\Omega_{X_{t_0}})}(1) \otimes \pi^*_{\Omega_{X_{t_0}}}\mathcal{O}_{X_{t_0}}(-a))$ so that in particular, thanks to Remark 4.14, $x \in (\tilde{\sigma}_{t_0} = 0)$. We will now show that $\pi_{t_0}(x) \in Y_{t_0}$. Take coordinates around x of the form $(t, z_i, [z'_i])$ such that $(t_0, 0, [1 : 0 : \cdots : 0]) = x$. In those coordinates, we write

$$\sigma = \sum_{i_1+\cdots+i_n=k} q_{i_1,\ldots,i_n}(t, z){z'_1}^{i_1} \cdots {z'_n}^{i_n}.$$

Therefore,

$$(\sigma = 0) = \{(t, z)/\forall(i_1, \ldots, i_n)\ q_{i_1,\ldots,i_n}(t, z) = 0\}.$$

Fix any $(i_1, \ldots, i_n) \in \mathbb{N}^n$ such that $i_1 + \cdots + i_n = k$. We have to show that $q_{i_1,\ldots,i_n}(t_0, 0) = 0$. To do so, we apply Theorem A.1 to construct, for each $1 \leqslant j \leqslant n$,

$$V_j \in H^0(\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}}), T\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}}) \otimes \pi^*\rho_1^*\mathcal{O}_{\mathbb{P}^N}(N) \otimes \pi^*\rho_2^*\mathcal{O}_{\mathbf{P}}(1, \ldots, 1))$$

such that, in our coordinates, $V_j(t_0, 0, [1 : 0 : \cdots : 0]) = \partial/\partial z'_j$. By differentiating i_j times with respect to V_j for each $1 \leqslant j \leqslant n$ we get a new section

$$L_{V_1} \cdots L_{V_1} L_{V_2} \cdots L_{V_n}\tilde{\sigma} \in H^0(\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}})_{|U}, \mathcal{O}_{\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}})}(k) \otimes \pi^*\rho_1^*\mathcal{O}_{\mathbb{P}^N}(-ka)).$$

A local computation gives

$$L_{V_1} \cdots L_{V_1} L_{V_2} \cdots L_{V_n}\tilde{\sigma}(t_0, 0, [1 : 0 : \cdots : 0]) = i_1! \cdots i_n! q_{i_1,\ldots,i_n}(t_0, 0).$$

However, since, by hypothesis, $x \in \mathrm{Bs}(\mathcal{O}_{\mathbb{P}(\Omega_{X_{t_0}})}(1) \otimes \pi^*_{\Omega_{X_{t_0}}}\mathcal{O}_{X_{t_0}}(-a))$, we know that

$$L_{V_1} \cdots L_{V_1} L_{V_2} \cdots L_{V_n}\tilde{\sigma}(t_0, 0, [1 : 0 : \cdots : 0]) = 0.$$

Therefore we see that $q_{i_1,\ldots,i_n}(t_0, 0) = 0$, proving our claim.

To complete the proof we just have to show, as in [DT10], the codimension-two refinement. Suppose that Y_{t_0} has a divisorial component E. Since E is effective and $\mathrm{Pic}(X) = \mathbb{Z}$ we can deduce that E is ample. Therefore there is an $m \in \mathbb{N}$ such that mE is very ample. Now take $\sigma_{t_0}^m \in H^0(X, S^{km}\Omega_{X_{t_0}} \otimes \mathcal{O}_{X_{t_0}}(-mka - mkN))$. The divisorial component of the zero locus of $\sigma_{t_0}^m$ is mE. Now, for any $D \in |mE|$ we get a new section $\sigma_{t_0}^m \otimes D \otimes mE^{-1} \in H^0(X, S^{km}\Omega_{X_{t_0}} \otimes \mathcal{O}_{X_{t_0}}(-mka - mkN))$. By applying the same argument as above we know that the image of the base locus $\mathrm{Bs}(\mathcal{O}_{\mathbb{P}(\Omega_{X_{t_0}})}(1) \otimes \pi^*_{\Omega_{X_{t_0}}}\mathcal{O}_{X_{t_0}}(-a))$ lies in the zero locus of this new section $\sigma_{t_0}^m \otimes D \otimes mE^{-1}$ whose divisorial component is D. Thus, since $|mE|$ is base-point free, we know that the image cannot lie in the divisorial part of Y_{t_0}. This completes the proof.

Acknowledgements

We are highly indebted to C. Mourougane, without his help this work could never have been achieved, nor started. We thank him very warmly for his help, encouragement, and all the discussions we had. We also would like to thank S. Diverio for his many advices, comments and very interesting discussions. We thank F. Han for pointing out to us Lemma 4.1, and O. Debarre for pointing out to us a simplification in the proof of Lemma 3.7 as well as many typographical errors. We thank the Japan Society for the Promotion of Science for supporting part of this work.

Appendix. Vector fields and effective bounds

This appendix contains two sections. We start by explaining Theorem A.1. Then we compute an explicit bound for Theorem 4.8.

A.1 Vector fields

As announced during the proof of Theorem 4.8 we are now going to prove the global generation property we used.

THEOREM A.1. *The bundle*

$$T\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}}) \otimes \pi^*\rho_1^*\mathcal{O}_{\mathbb{P}^N}(N) \otimes \pi^*\rho_2^*\mathcal{O}_{\mathbf{P}}(1,\dots,1)$$

is globally generated on $\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}})$.

The use of meromorphic vector fields on the universal complete intersection was introduced in the pioneering work of Voisin [Voi96] as a part of the so-called variational method, this strategy was further generalized to jet differential equations by Siu [Siu04]. Such global generation results were obtained in small dimensions by Păun [Pău08] and Rousseau [Rou07, Rou09]. Afterwards, it was proved in full generality by Merker [Mer09] for the universal hypersurface.

The proof of Theorem A.1 is almost the same as the proof of the main Theorem of [Mer09], so there is nothing here that was not already in Merker's paper. However, since our situation is slightly different we still show how one can adapt Merker's computations here, and in particular we will point out the small differences, and where we are able to gain the better bound on the order of the poles of the meromorphic vector fields one has to allow to get global generation. This improvement in the bound is due to the fact that in the situation of [Mer09] the constructed vector fields have to satisfy many equations (as many as the dimension plus one) to be tangent to the higher order jet spaces, whereas in our situation we only need to go up to jets of order one, thus the constructed vector fields just have to satisfy two equations. Also, for the reader's convenience, we adopt the notation of [Mer09].

A.1.1 *Notation and coordinates.* We fix homogeneous coordinates on $\mathbb{P}^N$ and $\mathbb{P}^{N_{d_i}}$ for any $1 \leqslant i \leqslant c$:

$$[Z] = [Z_0 : \cdots : Z_N] \in \mathbb{P}^N,$$
$$[A^i] = [(A^i_\alpha)_{\alpha\in\mathbb{N}^{N+1},|\alpha|=d_i}] \in \mathbb{P}^{N_{d_i}}.$$

In those coordinates $\mathcal{X} = (F_1 = 0) \cap \cdots \cap (F_c = 0)$ where

$$F_i = \sum_{\substack{\alpha\in\mathbb{N}^{N+1}\\|\alpha|=d_i}} A^i_\alpha Z^\alpha.$$

To construct vector fields explicitly it will be convenient to work with inhomogeneous coordinates. So from now on we suppose $Z_0 \neq 0$ and $A^i_{(0,d_i,0,\dots,0)} \neq 0$ and we introduce the corresponding coordinates on $\mathbb{C}^N$ and on $\mathbb{C}^{N_{d_i}}$ by setting $z_i := Z_i/Z_0$ and $a^i_{(\alpha_1,\dots,\alpha_N)} = A^i_{(\alpha_0,\dots,\alpha_N)}/A^i_{(0,d_i,0,\dots,0)}$, where $\alpha_0 = d_i - \alpha_1 - \cdots - \alpha_N$. Now in those coordinates the restriction $\mathcal{X}_0$ of $\mathcal{X}$ to the open subset $\mathbb{C}^N \times \mathbf{P}^0 \subset \mathbb{P}^N \times \mathbf{P}$, where $\mathbf{P}^0 = \mathbb{C}^{N_{d_1}} \times \cdots \times \mathbb{C}^{N_{d_c}}$ is defined by

$$\mathcal{X}_0 = (f_1 = 0) \cap \cdots \cap (f_c = 0),$$

https://doi.org/10.1112/S0010437X13007458 Published online by Cambridge University Press

where

$$f_i = \sum_{\substack{\alpha\in\mathbb{N}^N \\ |\alpha|\leqslant d_i}} a^i_\alpha z^\alpha.$$

On $\mathbb{C}^N \times \mathbf{P}^0 \times \mathbb{C}^N$ we will use the coordinates $(z_i, a^1_\alpha, \dots, a^c_\alpha, z'_k)$. Now the equations defining the relative tangent bundle $T_{\mathcal{X}/\mathbf{P}^0} \subset \mathbb{C}^N \times \mathbf{P}^0 \times \mathbb{C}^N$ are

$$f_i = \sum_{\substack{\alpha\in\mathbb{N}^N \\ |\alpha|\leqslant d_i}} a^i_\alpha z^\alpha = 0 \quad \text{and} \quad f'_i = \sum_{\substack{\alpha\in\mathbb{N}^N \\ |\alpha|\leqslant d_i}} \sum_{k=1}^N a^i_\alpha \frac{\partial z^\alpha}{\partial z_k} z'_k = 0.$$

A.1.2 *Vector fields on $T_{\mathcal{X}^0/\mathbf{P}^0}$.* Let $\Sigma = \{(z_i, a^1_\alpha, \dots, a^c_\alpha, z'_k)/(z'_1, \dots, z'_N) \neq 0\}$. Following [Mer09] we are going to construct explicit vector fields on $T_{\mathcal{X}^0/\mathbf{P}^0}$ (outside Σ) with prescribed pole order when we look at them as meromorphic vector fields on $T_{\mathcal{X}/\mathbb{P}}$. It will also be clear that the constructed vector fields can actually be viewed as vector fields on $\mathbb{P}(\Omega_{\mathcal{X}/\mathbf{P}})$.

A global vector field on $\mathbb{C}^N \times \mathbf{P}^0 \times \mathbb{C}^N$ is of the form

$$T = \sum_{j=1}^N Z_j \frac{\partial}{\partial z_j} + \sum_{\substack{\alpha\in\mathbb{N}^N \\ |\alpha|\leqslant d_1}} A^1_\alpha \frac{\partial}{\partial a^1_\alpha} + \cdots + \sum_{\substack{\alpha\in\mathbb{N}^N \\ |\alpha|\leqslant d_c}} A^c_\alpha \frac{\partial}{\partial a^c_\alpha} + \sum_{k=1}^N Z'_k \frac{\partial}{\partial z'_k}.$$

Such a vector field T is tangent to $T_{\mathcal{X}^0/\mathbf{P}^0}$ if for all $1 \leqslant i \leqslant c$

$$\begin{cases} T(f_i) = 0, \\ T(f'_i) = 0. \end{cases}$$

First we construct for each $0 \leqslant i \leqslant c$ vector fields of the form

$$\sum_{\substack{\alpha\in\mathbb{N}^N \\ |\alpha|\leqslant N}} A^i_\alpha \frac{\partial}{\partial a^i_\alpha}.$$

Such a vector field is tangent to $T_{\mathcal{X}^0/\mathbf{P}^0}$ if

$$T(f_i) = \sum_{\substack{\alpha\in\mathbb{N}^N \\ |\alpha|\leqslant N}} A^i_\alpha z^\alpha = 0 \quad \text{and} \quad T(f'_i) = \sum_{\substack{\alpha\in\mathbb{N}^N \\ |\alpha|\leqslant N}} \sum_{k=1}^N A^i_\alpha \frac{\partial z^\alpha}{\partial z_k} z'_k = 0.$$

As we are working outside Σ we may as well suppose $z'_1 \neq 0$. Set

$$\mathcal{R}_0(z, A) = \sum_{\substack{\alpha\in\mathbb{N}^N \\ |\alpha|\leqslant N \\ \alpha\neq(0,\dots,0) \\ \alpha\neq(1,0,\dots,0)}} A^i_\alpha z^\alpha \quad \text{and} \quad \mathcal{R}_1(z, A) = \sum_{\substack{\alpha\in\mathbb{N}^N \\ |\alpha|\leqslant N \\ \alpha\neq(0,\dots,0) \\ \alpha\neq(1,0,\dots,0)}} \sum_{k=1}^N A^i_\alpha \frac{\partial z^\alpha}{\partial z_k} z'_k.$$

With those notation the tangency property is equivalent to solve the system

$$\begin{cases} A^i_{(0,\dots,0)} + A^i_{(1,0,\dots,0)} z_1 + \mathcal{R}_0(z, A) = 0, \\ A^i_{(1,0,\dots,0)} z'_1 + \mathcal{R}_1(z, A) = 0, \end{cases}$$

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and, as $z'_1 \neq 0$, one can solve this in the straightforward way:

$$\begin{cases} A^i_{(1,0,\dots,0)} = \dfrac{-1}{z'_1}\mathcal{R}_1(z,A), \\ A^i_{(0,\dots,0)} = \dfrac{-z_1}{z'_1}\mathcal{R}_1(z,A) - \mathcal{R}_0(z,A). \end{cases}$$

We see that the pole order of vector fields obtained this way is less than N in the z_i. This is where we get the improvement on the pole order. Now in order to span all the other directions, we can take the vector fields constructed by Merker, and the pole order of those fields will be less than N. For the reader's convenience we recall them here, without proof, and refer to [Mer09] for the details.

First we recall how to construct vector fields of higher length in the $\partial/\partial a^i_\alpha$. For any $\alpha \in \mathbb{N}^N$ such that $|\alpha| \leqslant d_i$ and any $\ell \in \mathbb{N}^N$ such that $|\ell| \leqslant N$, set

$$T^\ell_\alpha = \sum_{\substack{\ell'+\ell''=\ell \\ \ell',\ell'' \in \mathbb{N}^N}} \frac{\ell!}{\ell'!\ell''!} z^{\ell''} \frac{\partial}{\partial a_{\alpha-\ell}}.$$

Those vector fields are of order N in the z_i variables. They are also tangent to $T_{\mathcal{X}^0/\mathbf{P}^0}$ and, with the vector fields constructed before, they will span all the $\partial/\partial a^i_\alpha$ directions.

To span the $\partial/\partial z_j$ directions, for all $1 \leqslant j \leqslant N$ we set

$$T_j = \frac{\partial}{\partial z_j} - \sum_{|\alpha| \leqslant d_1 - 1} a^1_{\alpha+e_j}(\alpha_j+1)\frac{\partial}{\partial a^1_\alpha} - \cdots - \sum_{|\alpha| \leqslant d_c - 1} a^c_{\alpha+e_j}(\alpha_j+1)\frac{\partial}{\partial a^c_\alpha},$$

where $e_j = (0,\dots,0,1,0,\dots,0)$ is the N-tuple where the only non-zero term is in slot j. It is now straightforward to check that those vector fields are tangent to $T_{\mathcal{X}^0/\mathbf{P}^0}$. Moreover, they are of order 1 in each of the a^i_α.

We recall now how to span the $\partial/\partial z'_k$ direction. For any $(\Lambda^\ell_k)^{1 \leqslant \ell \leqslant N}_{1 \leqslant k \leqslant N} \in \mathrm{GL}_N(\mathbb{C})$ we look for vector fields of the form

$$T_\Lambda = \sum_{k=1}^N \left(\sum_{\ell=1}^N \Lambda^\ell_k z'_\ell\right)\frac{\partial}{\partial z'_k} + \sum_{|\alpha| \leqslant d_1} A^1_\alpha(z,a,\Lambda)\frac{\partial}{\partial a^1_\alpha} + \cdots + \sum_{|\alpha| \leqslant d_c} A^c_\alpha(z,a,\Lambda)\frac{\partial}{\partial a^c_\alpha}.$$

For each such vector field and for any $1 \leqslant i \leqslant c$ set

$$T^i_\Lambda = \sum_{k=1}^N \left(\sum_{\ell=1}^N \Lambda^\ell_k z'_\ell\right)\frac{\partial}{\partial z'_k} + \sum_{|\alpha| \leqslant d_i} A^i_\alpha(z,a,\Lambda)\frac{\partial}{\partial a^i_\alpha}.$$

We can easily check that we have

$$\begin{cases} T_\Lambda(f_i) = T^i_\Lambda(f_i), \\ T_\Lambda(f'_i) = T^i_\Lambda(f'_i). \end{cases}$$

Therefore, to construct vector fields tangent to $T_{\mathcal{X}^0/\mathbf{P}^0}$, it is enough to solve those equations for each $1 \leqslant i \leqslant c$ independently, but this can be done using Merker's result on this type of vector fields. By doing so we find for each $1 \leqslant i \leqslant c$ solutions of the type

$$A^i_\alpha(z,a,\Lambda) = \sum_{|\beta|<N} \mathcal{L}^\beta_{\alpha,i}(a,\Lambda) z^\beta,$$

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where $\mathcal{L}^{\beta}_{\alpha,i}$ is bilinear in (a, Λ). Therefore the constructed fields will have order less than N in the z_j and of order 1 in each of the a^i_α. We refer again to [Mer09] for a proof of those facts. This leads to the desired result.

A.2 Effective existence of symmetric differentials

For us, the most interesting case of Theorem 2.7 is when $M = \mathbb{P}^N$, $H = \mathcal{O}_{\mathbb{P}^N}(1)$ and $\kappa = 1$, because in this case we are in the setting of Debarre's conjecture. Therefore we will give an effective bound on the degree in this case. First we rewrite this theorem in the present situation.

Theorem A.2. *Fix $a \in \mathbb{N}$. Then there exists a constant $\Gamma_{N,n,a}$ such that if $X \subset \mathbb{P}^N$ is a smooth complete intersection of dimension n, codimension c and multidegree $(d_1, \dots, d_c)$ satisfying $n \leqslant c$ and $d_i \geqslant \Gamma_{N,n,a}$, then $\mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi^*_{\Omega_X}\mathcal{O}_X(-a)$ is big. In particular, when $m \gg 0$,*

$$H^0(X, S^m\Omega_X \otimes \mathcal{O}_X(-am)) \neq 0.$$

We give a rough bound on $\Gamma_{N,n,a}$ that works for any N, n, a and afterwards we give a better bound when $n = 2$.

Remark A.3. We would like to mention that Debarre proved in [Deb05], using Riemann–Roch computations, that $\mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi^*_{\Omega_X}\mathcal{O}_X(1)$ is big under the assumptions of Theorem A.2.

A.2.1 *Segre classes.* We start by giving the detailed expression of the Segre classes of the cotangent bundle of a complete intersection in $\mathbb{P}^N$. Recall from § 4.2.2 that we had

$$s(\Omega_X) = (1 - h + h^2 - \cdots)^{N+1} \prod_{i=1}^{c}(1+d_i h).$$

We introduce another notation: for $0 \leqslant i \leqslant c$, let

$$\epsilon_i(d_1, \dots, d_c) := \sum_{1\leqslant j_1<\cdots<j_i\leqslant c} d_{j_1}\cdots d_{j_i}$$

and, also, $\epsilon_i(d_1, \dots, d_c) = 0$ if $i > c$. Now, by expending, and since $h^k = 0$ if $k > n$, we get

$$\begin{aligned} s(\Omega_X) &= \left(\sum_{k=0}^{n} \binom{N+k}{k}(-1)^k h^k\right)\left(\sum_{i=0}^{c} \epsilon_i(d_1, \dots, d_c)h^i\right) \\ &= \sum_{k=0}^{n}\sum_{i=0}^{n} \binom{N+k}{k}(-1)^k \epsilon_i(d_1, \dots, d_c)h^{i+k} \\ &= \sum_{j=0}^{n}\sum_{k=0}^{j} \binom{N+k}{k}(-1)^k \epsilon_{j-k}(d_1, \dots, d_c)h^{j}. \end{aligned}$$

From this we deduce the explicit form of the Segre classes that will be used afterwards:

$$s_j(\Omega_X) = \sum_{k=0}^{j} \binom{N+k}{N}(-1)^k \epsilon_{j-k}(d_1, \dots, d_c)h^j. \tag{A.1}$$

A.2.2 *Explicit intersection computations.* Here we will do explicitly, in this context, the intersection computations we did during the proof of Theorem 2.7. We take the same notation, under the assumption $n \leqslant c$.

$$
\begin{aligned}
F^{2n-1} &- (2n-1)F^{2n-2} \cdot G \\
&= \int_{\mathbb{P}(\Omega_X)} (u+2h)^{2n-1} - (2n-1)(u+2h)^{2n-2}(2+a)h \\
&= \int_{\mathbb{P}(\Omega_X)} \sum_{i=0}^{2n-1} \binom{2n-1}{i} u^{2n-1-i}(2h)^i - (2n-1)(2+a) \sum_{j=0}^{2n-2} \binom{2n-2}{j} u^{2n-2-j}(2h)^j h \\
&= \int_{\mathbb{P}(\Omega_X)} u^{2n-1} + \sum_{i=1}^{2n-1} \left(2^i \binom{2n-1}{i} - (2n-1)(2+a)2^{i-1} \binom{2n-2}{i-1} \right) u^{2n-1-i} h^i \\
&= \int_{\mathbb{P}(\Omega_X)} u^{2n-1} + \sum_{i=1}^{2n-1} 2^{i-1}(2-i(2+a)) \binom{2n-1}{i} u^{2n-1-i} h^i \\
&= \int_X \sum_{i=0}^{n} 2^{i-1}(2-i(2+a)) \binom{2n-1}{i} s_{n-i} h^i.
\end{aligned}
$$

Now we will use (A.1) to see how this intersection product depend on the multidegree $(d_1, \ldots, d_c)$. To ease the notation we will also just write ϵ_i instead of $\epsilon_i(d_1, \ldots, d_c)$. We have

$$
\begin{aligned}
F^{2n-1} &- (2n-1)F^{2n-2} \cdot G \\
&= \sum_{i=0}^{n} \int_X 2^{i-1}(2-i(2+a)) \binom{2n-1}{i} s_{n-i} h^i \\
&= \sum_{i=0}^{n} 2^{i-1}(2-i(2+a)) \binom{2n-1}{i} \left(\sum_{k=0}^{n-i} \binom{N+k}{N} (-1)^k \epsilon_{n-i-k} \right) \\
&= \sum_{i=0}^{n} \sum_{k=0}^{n-i} (-1)^k 2^{i-1}(2-i(2+a)) \binom{2n-1}{i} \binom{N+k}{N} \epsilon_{n-i-k} \\
&= \sum_{j=0}^{n} \sum_{i=0}^{n-j} (-1)^{n-i-j} 2^{i-1}(2-i(2+a)) \binom{2n-1}{i} \binom{N+n-i-j}{N} \epsilon_j \\
&= \sum_{j=0}^{n} \mathrm{D}_a^{N,n,j}\, \epsilon_j(d_1, \ldots, d_c),
\end{aligned}
$$

where

$$
\mathrm{D}_a^{N,n,j} = (-1)^{n-j} \sum_{i=0}^{n-j} (-1)^i 2^{i-1}(2-i(2+a)) \binom{2n-1}{i} \binom{N+n-i-j}{N}.
$$

A.2.3 *Rough bound.* Now we will be able to give a straightforward rough bound for $\Gamma_{N,n,a}$ for any N, n, a. Let us recall a basic fact to estimate the zero locus of a polynomial in one real variable.

Lemma A.4. *Let* $P(x) := x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$. *If* $x \geqslant 1 + \max_i |a_i|$, *then* $P(x) > 0$.

Now we will see how to apply this in our situation. As previously, we write

$$\epsilon_i(x_1, \ldots, x_c) := \sum_{1 \leqslant j_1 < \cdots < j_i \leqslant c} x_{j_1} \cdots x_{j_i}.$$

Take $k \leqslant c$ and

$$P(x_1, \ldots, x_c) = \epsilon_k(x_1, \ldots, x_c) + a_{k-1}\epsilon_{k-1}(x_1, \ldots, x_c) + \cdots + a_1\epsilon_1(x_1, \ldots, x_c) + a_0.$$

We want to find $r \in \mathbb{R}$ such that if $x_i \geqslant r$ for all $1 \leqslant i \leqslant c$ then $P(x_1, \ldots, x_c) > 0$. We will be done if we are able to find $r \in \mathbb{R}$ satisfying $P(r, \ldots, r) > 0$ and $(\partial P/\partial x_i)(x_1, \ldots, x_c) > 0$ for all $1 \leqslant i \leqslant c$ as soon as $x_j \geqslant r$ for all $1 \leqslant i \leqslant c$. Now observe that because of the symmetries of the ϵ_i, we have to check the positivity of just one partial derivative, say $\partial P/\partial x_c$. By induction, we are left to find $r \in \mathbb{R}$ satisfying $P(r, \ldots, r) > 0$, $(\partial P/\partial x_c)(r, \ldots, r) > 0$, $(\partial/\partial x_{c-1})(\partial P/\partial x_c)(r, \ldots, r) > 0$, $\ldots$, $\partial/\partial x_1 \cdots (\partial/\partial x_{c-1})(\partial P/\partial x_c)(r) > 0$. However, we also have

$$\frac{\partial \epsilon_j}{\partial x_c}(x_1, \ldots, x_c) = \epsilon_{j-1}(x_1, \ldots, x_{c-1}),$$

and thus

$$\frac{\partial P}{\partial x_c} = \frac{\partial}{\partial x_c} \sum_{i=0}^{k} a_i \epsilon_i(x_1, \ldots, x_c) = \sum_{i=0}^{k-1} a_{i+1}\epsilon_i(x_1, \ldots, x_{c-1}),$$

where $a_k = 1$. Similarly

$$\frac{\partial}{\partial x_{c-j}} \cdots \frac{\partial}{\partial x_{c-1}} \frac{\partial P}{\partial x_c} = \sum_{i=0}^{k-j} a_{i+j}\epsilon_i(x_1, \ldots, x_{c-j}).$$

Evaluating in $(r, \ldots, r)$ yields for any $0 \leqslant j \leqslant c$

$$\frac{\partial}{\partial x_{c-j}} \cdots \frac{\partial}{\partial x_{c-1}} \frac{\partial P}{\partial x_c}(r, \ldots, r) = \sum_{i=0}^{k-j} a_{i+j} \binom{c-j}{i} r^i.$$

Now we apply Lemma A.4 to each of those polynomials and therefore we just have to give a bound for

$$\max_{\substack{0 \leqslant j \leqslant k-1 \\ j \leqslant i \leqslant k-1}} \left| a_i \binom{c-j}{i-j} \middle/ \binom{c-j}{k-j} \right|.$$

Since

$$\binom{c-j}{i-j} \middle/ \binom{c-j}{k-j} \leqslant \binom{c}{i} \middle/ \binom{c}{k},$$

we just have to give a bound for

$$\max_{0 \leqslant i \leqslant k-1} \left| a_i \binom{c}{i} \middle/ \binom{c}{k} \right|.$$

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This is what we will do now in our intersection product computation, that is when $k=n$, $c=N-n$ and $a_j=\mathrm{D}_a^{N,n,j}$.

$$
\begin{aligned}
&\left|\mathrm{D}_a^{N,n,j}\left(\binom{c}{j}\Big/\binom{c}{n}\right)\right|\\
&\quad=\left|\sum_{i=0}^{n-j}(-1)^{n-i-j}2^{i-1}(2-i(2+a))\binom{2n-1}{i}\binom{N+n-i-j}{N}\left(\binom{c}{j}\Big/\binom{c}{n}\right)\right|\\
&\quad\leqslant\binom{N+n-j}{N}\left(\binom{c}{j}\Big/\binom{c}{n}\right)\\
&\qquad+\sum_{i=1}^{n-j}2^{i-1}(i(2+a)-2)\binom{2n-1}{i}\binom{N+n-i-j}{N}\left(\binom{c}{j}\Big/\binom{c}{n}\right)\\
&\quad\leqslant\binom{N+n-j}{N}\left(\binom{c}{j}\Big/\binom{c}{n}\right)+2^{n-j-1}((n-j)(2+a)-2)\\
&\qquad\times\sum_{i=1}^{n-j}\binom{2n-1}{i}\binom{N+n-i-j}{N}\left(\binom{c}{j}\Big/\binom{c}{n}\right)\\
&\quad\leqslant\binom{N+n-j}{N}\left(\binom{c}{j}\Big/\binom{c}{n}\right)+2^{n-j-1}((n-j)(2+a)-2)\\
&\qquad\times(n-j)\binom{N+n-j-1}{N}\left(\binom{c}{j}\Big/\binom{c}{n}\right)\binom{2n-1}{n-j}\\
&\quad\leqslant\left(2^{n-1}(n(2+a)-2)\frac{n^2}{(N+1)}\binom{2n-1}{n}+1\right)\binom{n}{\lfloor n/2\rfloor}\frac{(N+n)!(N-2n)!}{N!(N-n)!}.
\end{aligned}
$$

This gives the desired (rough) bound for $\Gamma_{N,n,a}$:

$$
\Gamma_{N,n,a}\leqslant\left(2^{n-1}(n(2+a)-2)\frac{n^2}{(N+1)}\binom{2n-1}{n}+1\right)\binom{n}{\lfloor n/2\rfloor}\frac{(N+n)!(N-2n)!}{N!(N-n)!}.
$$

Obviously this is far from optimal, but we will not go into more details for the general case. However, even with such an estimate, we can make a noteworthy remark. In our main theorem we use $\Gamma_{N,n,a+N}$. Now if we fix n and if we let go N to infinity (that is when the codimension becomes bigger and bigger) we get

$$
\lim_{N\to+\infty}\Gamma_{N,n,N+a}\leqslant 2^{n-1}n^3\binom{2n-1}{n}\binom{n}{\lfloor n/2\rfloor}.
$$

That is to say that this bound decreases as c gets bigger, and it will have a limit depending only on n. This corresponds to the intuition that as the codimension increases the situation is more favorable and the multidegree can be taken smaller. This feature should be a part of any bound on Γ.

A.2.4 *Bound in dimension two.* Here we give a better bound in the case of surfaces. This is of particular interest to us since it is the case where we will have the strongest conclusion. Take the notation of the previous section, and let $n=2$ so that $c=N-2$. Fix $a\in\mathbb{N}$. We want to estimate $F^3 - 3F^2 \cdot G = \sum_{j=0}^{2} \mathrm{D}_a^{N,2,j}\, \epsilon_j(d_1, \ldots, d_c)$, where

$$\begin{aligned}
\mathrm{D}_a^{N,2,2} &= 1, \\
\mathrm{D}_a^{N,2,1} &= -(N+1) - 3a, \\
\mathrm{D}_a^{N,2,0} &= \binom{N+2}{N} + 3a(N+1) - 12(a+1).
\end{aligned}$$

Observe that $\mathrm{D}_a^{N,2,0} \geqslant 0$ when $N \geqslant 4$, thus $F^3 - 3F^2 \cdot G \leqslant \epsilon_2(d_1, \ldots, d_c) - \mathrm{D}_a^{N,2,1}\, \epsilon_1(d_1, \ldots, d_c)$, and therefore we just have to bound one term,

$$\left| \mathrm{D}_a^{N,2,1} \left(\binom{N-2}{1} \Big/ \binom{N-2}{2} \right) \right| = 2\frac{N+1+3a}{N-3}.$$

Thus $\Gamma_{N,2,a} \leqslant 2(N+1+3a)/(N-3)$, and in particular $\delta_{N,2,\varepsilon} = \Gamma_{N,2,N+\varepsilon} \leqslant 2(4N+1+3\varepsilon)/(N-3)$. Note that everything still works if one allows a to be a rational number (up to taking tensor powers). Therefore we can take $\varepsilon \in \mathbb{Q}^+$ small enough to get the bound in Corollary 4.13.

Remark A.5. In the case of surfaces one can actually do even better. One can use an argument of Bogomolov [Bog78] and Riemann–Roch computations to get better bounds. Bogomolov's argument shows that, in our situation, to prove that $h^0(X, S^m\Omega_X \otimes \mathcal{O}_X(-ma)) > 0$ it suffices to prove that $c_1(\Omega_X(-a))^2 - c_2(\Omega_X(-a)) > 0$. A straightforward computation shows that, for $\varepsilon \in \mathbb{Q}^+$ small enough, one gets the following bounds:

$$\delta_{4,2,\varepsilon} \leqslant 30, \quad \delta_{5,2,\varepsilon} \leqslant 18, \quad \delta_{6,2,\varepsilon} \leqslant 14, \quad \delta_{7,2,\varepsilon} \leqslant 12, \quad \delta_{8,2,\varepsilon} \leqslant 11, \quad \delta_{9,2,\varepsilon} \leqslant 10,$$
$$\delta_{N,2,\varepsilon} \leqslant 9 \quad \text{for } 10 \leqslant N \leqslant 12, \quad \delta_{N,2,\varepsilon} \leqslant 8 \quad \text{for } 13 \leqslant N \leqslant 20$$
$$\delta_{N,2,\varepsilon} \leqslant 7 \quad \text{for } 21 \leqslant N \leqslant 84, \quad \delta_{N,2,\varepsilon} \leqslant 6 \quad \text{for } 84 \leqslant N.$$

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