NON-ISOMORPHIC EQUIVALENT AZUMAYA ALGEBRAS

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ABSTRACT. We explicitly describe an infinite collection of pairs of Azumaya algebras over the ring of integers of real quadratic number fields $K$ which are maximal orders in the usual quaternion algebra over $K$, hence Brauer equivalent, but are not isomorphic. The result follows from an identification of the groups of norm one units, using the classification of Coxeter.

In [10], R. G. Swan constructed a pair of Azumaya algebras over the ring of integers $R$ of a quartic extension of the rational numbers which were equivalent in the Brauer group of $R$ but were not isomorphic. In this note we describe an infinite collection of such pairs over rings of integers of quadratic fields.

Let $m$ be a rational integer congruent to 3 (mod 4), and $R = \mathbb{Z} [\sqrt{m}]$, the ring of integers of $K = \mathbb{Q} (\sqrt{m})$. Suppose $2R = b^2 R$, the square of a principal ideal. (This will always be the case if $m$ is prime, for then $R$ has odd class number.) Let $\overline{b}$ be the conjugate of $b$. Our examples are both maximal orders over $R$ in the usual quaternion algebra $H(K)$ over $K$, the algebra generated over $K$ by $i$ and $j$ with $i^2 = j^2 = -1$, $ij = j = k$.

The two examples have bases as free $R$-modules as follows:

$$A = \left\{ 1, \frac{1 + i}{\overline{b}}, \frac{1 + j}{b}, \frac{1 + i + j + k}{2} \right\};$$

$$D = \left\{ 1, \frac{\sqrt{m} + i}{2}, j, \frac{\sqrt{mj} + k}{2} \right\}.$$

Both may be seen to be Azumaya $R$-algebras by recognizing them as smash products. Gainst and Hocchsmann [8] have shown that if $S$, $T$ are Galois objects (in the sense of [2]) with respect to a dual pair of Hopf algebras $H$, $H^* = \text{Hom}_R (H, R)$, then the smash product $S \# T$ is an Azumaya $R$-algebra.

Now $D$ is the smash product of a Galois $(RG)^*$-object $S$ and a Galois $RG$-object $T$. Here

$$S = R \left[ \frac{\sqrt{m} + 1}{2} \right]$$

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is the ring of integers of the unramified extension $L = K[i]$ of $K$, hence $S$ is a Galois extension of $R$ with group $G = \text{Gal}(L/K)$ in the sense of [1], hence a Galois $(RG)^*$-object; and

$$T = R[j], \ j^2 + 1 = 0$$

is a $G$-graded $R$-algebra and a Galois $RG$-object.

If we let $H_b$ be the free Hopf $R$-algebra, $H_b = R[x]$ with $x^2 = bx$ and comultiplication $\Delta$, counit $\epsilon$ and antipode $\lambda$ defined by

$$\Delta(x) = x \otimes 1 + 1 \otimes x - \frac{2}{b} (x \otimes x)$$

$$\epsilon(x) = 0$$

$$\lambda(x) = x,$$

then $A = S \# T$ where

$$S = R[w], \ w^2 = \frac{b}{b} w$$

$$T = R[z], \ z^2 = \frac{b}{b} z$$

are Galois $H$-objects for $H = H_b$ and $H_b^* = H^*_b$, respectively. Then $S \# T$ embeds in $H(K)$ by

$$w \# 1 \rightarrow \frac{1 + i}{b}, \ 1 \# z \rightarrow \frac{1 + j}{b}.$$

**Theorem.** The algebras $A$ and $D$ are in the same class in $Br(R)$ but are not isomorphic.

**Proof.** Since the map from $Br(R)$ to $Br(K)$ is $1 - 1$ and $A$ and $D$ are both orders over $R$ in the same $K$-algebra $H(K)$, $A$ and $D$ are in the same class in $Br(R)$. To show $A$ and $D$ are not isomorphic, let $A^*_o, D^*_o$ denote the groups of units of $A, D$, respectively, of norm 1, where

$$n(\alpha) = n(a + bi + cj + dk) = a^2 + b^2 + c^2 + d^2$$

is the usual norm from $H$ to $K$. A result of Eichler (c.f. Swan [10], Remark 2) shows that $A^*_o$ and $D^*_o$ are finite. We show that $A^*_o$ and $D^*_o$ are not isomorphic. In fact, we show that

$$A^*_o = \{ \pm 1, \pm i, \pm j, \pm k, \ (\pm 1 \pm i \pm j \pm k) / 2 \}$$

a group of order 24, in Coxeter’s notation of [6], $A^*_o = \langle 2, 3, 3 \rangle$; while $D^*_o$ is a dicyclic group.

Now $A^*_o, D^*_o$, being finite, are made up of roots of unity in $H(K)$. Since $H(K)$ is a skew field, if $\zeta$ is a primitive $e$ th root of 1 in $H(K)$, then $\mathbb{Q}(\zeta)$ is a commutative subfield of $H(K)$. So $\mathbb{Q}(\zeta): \mathbb{Q} \leq 4$, hence $\phi(e) \leq 4$, where $\phi$ is Euler’s function. Now $\phi(e) = 1$ for $e = 1, 2; \ \phi(e) = 2$ for $e = 3, 4, 6; \ \phi(e) = 4$ for $e = 5, 8, 10, 12,$ and $\phi(e) > 4$ for all other $e$. 

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If \( \phi(e) = 4 \), then \( \mathbb{Q}(\xi) \supset K \). But the only real quadratic subfield of \( \mathbb{Q}(\xi) \) for \( e = 5, 8, 10 \) or 12 is \( \mathbb{Q}(\sqrt{m}) \) where \( m = 5, 2, 5 \) and 3, respectively. So \( H(K) \) has no elements of order 5 or 8, and no element of order 12 unless \( m = 3 \).

The known list of finite groups of real quaternions ([11], p. 17) shows that since \( A^*_\varphi \), \( D^*_\varphi \) contain no elements of order 5 or 8, each must be isomorphic either to \( E_{24} \), the binary tetrahedral group of order 24, or to a dicyclic group of order \( 4n \).

Now
\[
E_{24} = \{ \pm 1, \pm i, \pm j, \pm k, (\pm 1 \pm i \pm j \pm k) / 2 \} \subseteq A^*_\varphi;
\]
hence \( A^*_\varphi \) is not dicyclic, so \( A^*_\varphi = E_{24} \).

To see that \( D^*_\varphi \neq A^*_\varphi \) we show that if \( m > 3 \) \( \varphi \) contains no cube roots of 1, while if \( m = 3 \) \( D^*_\varphi \) contains a 12th root of 1.

Now any element of \( D \) is of the form
\[
\tau = \frac{\alpha}{2} + \frac{\beta}{2} i + \frac{\gamma}{2} j + \frac{\delta}{2} k, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}.
\]
If \( \tau \) is a primitive 3rd or 6th root of unity, then \( \tau^2 \pm \tau + 1 = 0 \). Also \( n(\tau) = (\alpha - \tau) = 1 \), so \( \tau^2 - \alpha \tau + 1 = 0 \), so \( \alpha = 1 \) or \(-1 \). Thus
\[
n(\tau) = \frac{1}{4} + \frac{\beta^2}{4} + \frac{\gamma^2}{4} + \frac{\delta^2}{4} = 1
\]
or
\[(*) \quad \beta^2 + \gamma^2 + \delta^2 = 3
\]
Setting \( \beta = c + d \sqrt{m} \), \( \gamma = e + f \sqrt{m} \), \( \delta = g + h \sqrt{m} \), \( c, d, e, f, g, h \) in \( \mathbb{Z} \), (*) becomes
\[
\begin{cases}
c^2 + md^2 + e^2 + mf^2 + g^2 + mh^2 = 3 \\
\quad cd + ef + gh = 0
\end{cases}
\]
If \( m > 3 \), the only solution of (**) is \( c^2 = e^2 = g^2 = 1 \). But, as is easily seen, \( (\pm 1 \pm i \pm j \pm k) / 2 \) is not in \( D \). Thus \( D^*_\varphi \) has no elements of order 3, hence cannot be isomorphic to \( E_{24} = A^*_\varphi \).

If \( m = 3 \), \( D^*_\varphi \) contains \( \frac{\sqrt{3} + i}{2} \), a primitive 12th root of 1. Since \( E_{24} \) has no elements of order 12, again \( D^*_\varphi \) is dicyclic for \( m = 3 \), and, in particular, not isomorphic to \( A^*_\varphi \). That completes the proof.

A bit more computation (which we omit) shows that, in Coxeter’s notation of [16], \( D^*_\varphi = (2, 2, 2) \) for \( m > 3 \), while for \( m = 3 \), \( D^*_\varphi = (2, 2, 6) \).

REMARKS. The algebras \( A \) and \( D \), being non-isomorphic representatives of the non-trivial class of \( Br(R) \) give further explicit examples of the failure of cancellation of projective modules [10] and of the failure of the Skolem-Noether theorem for Azumaya algebras [3], [4].
Using Eichler’s class number formula [7], Theorem, and a closed form description of the zeta function of $K$ evaluated at 2 found in [9], page 40, one can show (not without some difficulty) that the number $t$ of isomorphism types of maximal orders in the quaternion algebra $H(K)$, $K = \mathbb{Q}(\sqrt{p})$, satisfies $t = 2$ if and only if $p = 3$. Thus the algebras $A$ and $D$ represent all isomorphism types of maximal orders in $H(\mathbb{Q}(\sqrt{p}))$, $p \equiv 3 \pmod{4}$, if and only if $p = 3$. We omit the details, some of which may be found in [5].

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REFERENCES