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A CONDITION FOR EQUALITY OF CARDINALS OF MINIMAL GENERATORS UNDER CLOSURE OPERATORS

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Let C be an operator on the subsets of a set X with values among the subsets of X. We assume that C is a closure operator in X, i.e. a monotone, idempotent and extensive operator in X (cf., e.g., Birkhoff [3, p. 39], Schmidt [1], [2]). If $A \subseteq X$ and $B \subseteq X$, we say that A and B are *C-equivalent* if $C(A)=C(B)$ (Bleicher–Marczewski [4, p. 210]). If $A \subseteq X$, we say that A is *C-independent* if $C(B) \neq C(A)$ for each proper subset B of A. C is said to have *finite character* in X (cf., e.g., Schmidt [2, p. 236]) if the following condition is satisfied: If $A \subseteq X$ and $x \in C(A)$, then $x \in C(B)$ for some finite subset B of A.

It is known that if C has finite character in X, then any two C-independent C-equivalent subsets of X, one of which is infinite, have the same cardinal number (see Bleicher–Preston [5, p. 210]). The purpose of this note is to introduce property (α) below, to show that it is sufficient for equality of cardinals of C-independent C-equivalent infinite subsets of X and to establish some sufficient conditions for property (α). (We use the symbol $|S|$ to denote the cardinal of a set S.)

Property (α): If E and F are C-equivalent subsets of X and $x \in C(E)$, then $x \in C(A)$ for some subset A of F such that $|A| \leq |E|$.

THEOREM 1. *If C has property (α) and $E \subseteq X$ and $C(E)$ includes a C-independent set F which is C-equivalent to E, then $|F| \leq |E|^2$.*

Proof. We assume the hypothesis of the theorem. We use property (α) and choose a family $\{F_x\}_{x \in E}$ of subsets of F such that $|F_x| \leq |E|$ and $x \in C(F_x)$ if $x \in E$. Since C is a closure operator in X, it follows that

$$E = \bigcup_{x \in E} \{x\} \subseteq \bigcup_{x \in E} C(F_x) \subseteq C\Big(\bigcup_{x \in E} F_x\Big) \subseteq C(F) = C(E).$$

Therefore, $C(F)=C(\bigcup_{x \in E} F_x)$ and, since F is C-independent and $\bigcup_{x \in E} F_x \subseteq F$, it follows that $\bigcup_{x \in E} F_x = F$. Consequently,

$$|F| = |\bigcup_{x \in E} F_x| \leq \sum_{x \in E} |F_x| \leq \sum_{x \in E} |E| = |E|^2.$$

This proves the theorem.

THEOREM 2. *If any two C-independent C-equivalent subsets of X have the same cardinal number and each subset B of X includes a C-independent subset which is C-equivalent to B, then C has property* (α).

Proof. We assume the hypothesis of the theorem. Suppose there are C-equivalent subsets E and F of X and a member x of $C(E)$ such that $x \notin C(A)$ if $A \subseteq F$ and $|A| \leq |E|$. Using the second assertion in our assumption, we choose a C-independent subset E_1 of E which is C-equivalent to E and a C-independent subset F_1 of F which is C-equivalent to F. Then $x \in C(F_1)$ and it follows that $|F_1| > |E| \geq |E_1|$ while F_1 and E_1 are C-independent C-equivalent subsets of X. This is a contradiction of our hypothesis. The theorem follows.

REMARKS. If, in Theorem 1, we take the additional hypothesis that E is C-independent and F is infinite, then it will follow that $|F| \leq |E|^2 = |E| \leq |F|^2 = |F|$. Therefore, if C has property (α), then any two C-independent C-equivalent sets, one of which is infinite, have the same cardinal number. It is obvious that finite character is sufficient for property (α) in the case of infinite C-independent C-equivalent subsets of X. Examples of closure operators which have property (α) but not finite character may be found among infinite dimensional inner product spaces. A particular example is the following: Let X be the complete inner product space of all real-valued sequences f such that $\sum_{i=1}^{\infty} |f(i)| < \infty$. Define $C(E)$ to be the closed linear manifold generated by E if $E \subseteq X$ (see Taylor [6, p. 109, p. 84]). If $i \geq 1$, let $f_i(j)$ be 0 if $j \neq i$ and 1 otherwise. Let $f(i) = i^{-1}$ if $i \geq 1$. Then $f \in C(\{f_i: i \geq 1\})$. It is easy to verify that C is a closure operator in X and that $f \notin C(A)$ if A is a finite subset of $\{f_i : i \geq 1\}$. The hypothesis of Theorem 2 is satisfied (see Taylor [6, pp. 106–118]). Therefore, C has property (α).

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