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TAUBERIAN- AND CONVEXITY THEOREMS FOR CERTAIN (N,p,q)-MEANS

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ABSTRACT. The summability fields of generalized Nörlund means $(N,p^{*\alpha},p)$, $\alpha\in\mathbb{N}$, are increasing with α and are contained in that of the corresponding power series method (P,p). Particular cases are the Cesàro- and Euler-means with corresponding power series methods of Abel and Borel. In this paper we generalize a convexity theorem, which is well-known for the Cesàro means and which was recently shown for the Euler means to a large class of generalized Nörlund means.

1. **Introduction.** We consider throughout complex sequences (s_n) and discuss the relations of certain summability methods.

We say a sequence (s_n) of complex numbers is *summable* to s by the

(i) *Cesàro-method* of order $\alpha>-1$, briefly $s_n\longrightarrow s(C_\alpha)$, if

$$\frac{1}{\binom{n+\alpha}{n}}\sum_{k=0}^{n}\binom{n-k+\alpha-1}{n-k}s_k\longrightarrow s\quad(n\longrightarrow\infty);$$

(ii) *Euler-method* of order $0<p\le 1$, briefly $s_n\longrightarrow s(E_p)$, if

$$\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}s_k\longrightarrow s\quad(n\longrightarrow\infty);$$

(iii) *Abel-method*, briefly $s_n\longrightarrow s(A)$, if

$$f(t)=(1-t)\sum_{n=0}^{\infty}s_nt^n\quad\text{exists for } 0<t<1 \text{ and } f(t)\longrightarrow s\ (t\longrightarrow 1-);$$

(iv) *Borel-method*, briefly $s_n\longrightarrow s(B)$, if

$$g(t)=e^{-t}\sum_{n=0}^{\infty}\frac{s_n}{n!}t^n\quad\text{exists for } t\in\mathbb{R} \text{ and } g(t)\longrightarrow s\ (t\longrightarrow\infty).$$

The Cesàro- and Abel-method resp. the Euler- and Borel-method are known to be closely related, see [9, 17, 19].

Especially the following Abelian inclusions are well known, see *e.g.* [9; Theorems 43, 55, 118, 128]

$$\text{for } -1<\alpha\le\beta\text{: } s_n\longrightarrow s(C_\alpha)\Rightarrow s_n\longrightarrow s(C_\beta)\Rightarrow s_n\longrightarrow s(A),$$
$$\text{for } 0<p\le q\le 1\text{: } s_n\longrightarrow s(E_q)\Rightarrow s_n\longrightarrow s(E_p)\Rightarrow s_n\longrightarrow s(B).$$

The following converse or Tauberian theorem for the Cesàro-Abel-case goes back to Littlewood [14] $(\alpha,\beta\in\mathbb{N})$, and Anderson [1] $(\alpha,\beta\ge -1)$.

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THEOREM TC 1. *(i) Let* $-1 < \alpha < \beta$ *then* $s_n \longrightarrow s(A)$ *and* $s_n = O(1)(C_\alpha)$ *imply* $s_n \longrightarrow s(C_\beta)$.

(ii) For $-1 < \alpha < \delta \leq \beta$ *we have the so-called* convexity-theorem $s_n \longrightarrow s(C_\beta)$ *and* $s_n = O(1)(C_\alpha)$ *imply* $s_n \longrightarrow s(C_\delta)$.

Quite recently Boos and Tietz [4] proved that the situation is completely analogous for the Euler-Borel-case.

THEOREM TC 2. *(i) Let* $0 < p < q \leq 1$ *then* $s_n \longrightarrow s(B)$ *and* $s_n = O(1)(E_q)$ *imply* $s_n \longrightarrow s(E_p)$.

(ii) For $0 < p \leq r < q \leq 1$ *we have the convexity-theorem* $s_n \longrightarrow s(E_p)$ *and* $s_n = O(1)(E_q)$ *imply* $s_n \longrightarrow s(E_r)$.

Obviously part (ii) is in both cases a trivial consequence of the Abelian inclusion and part (i).

The aim of this paper is to show that the above results are special cases of a more general setting.

For the following assume that (p_n) is a sequence of reals with the following properties:

$$(1.1) \qquad \begin{aligned} &p_0 > 0,\ p_n \geq 0,\ n \in \mathbb{N}, \text{ such that the power series} \\ &p(t) = \sum_{n=0}^{\infty} p_n t^n \text{ has radius of convergence } R > 0. \end{aligned}$$

Since we can use $p_n R^n$ as weights in case $0 < R < \infty$, we only have to deal with the two cases $R = 1$ and $R = \infty$.

Furthermore we define the α-th convolution $p_n^{*\alpha}$ of a sequence (p_n) by

$$p_n^{*1} := p_n, \quad n = 0, 1, 2, \ldots \quad \text{and} \quad p_n^{*(\alpha+1)} := \sum_{k=0}^{n} p_{n-k}^{*\alpha} p_k.$$

We now generalize the summability methods used in Theorems TC1 and TC2. To this end we need a further sequence (q_n) of nonnegative reals, also satisfying (1.1), in general with a different radius of convergence R_q for the associated power series.

We then say, that a sequence (s_n) is summable to s by the

(i) *power series method* of summability (P, p), briefly $s_n \longrightarrow s(P, p)$, if

$$(1.2) \quad p_s(t) = \sum_{n=0}^{\infty} s_n p_n t^n \text{ converges for } |t| < R \text{ and if } \sigma_p(t) = \frac{p_s(t)}{p(t)} \longrightarrow s \text{ as } t \longrightarrow R-.$$

(In case $R = 1$ we have the so-called (J_p)-methods, in case $R = \infty$ the (B_p)-methods).

(ii) *general Nörlund-means* $(N, p^{*\alpha}, q^{*\beta})$; $\alpha, \beta \in \mathbb{N}$, briefly $s_n \longrightarrow s\ (N, p^{*\alpha}, q^{*\beta})$, if

$$(1.3) \qquad \begin{aligned} &\frac{1}{r_n} \sum_{k=0}^{n} p_{n-k}^{*\alpha} q_k^{*\beta} s_k \longrightarrow s\ (n \longrightarrow \infty), \text{ where we suppose that} \\ &r_n := (p^{*\alpha} * q^{*\beta})_n = \sum_{k=0}^{n} p_{n-k}^{*\alpha} q_k^{*\beta} > 0 \quad \text{for } n = 0, 1, \ldots. \end{aligned}$$

We require all methods to be regular. By Theorem 5 in [9], we have regularity for a power series method if and only if

(1.4) (A) $P_n = \sum_{k=0}^{n} p_k \longrightarrow \infty$, $(n \longrightarrow \infty)$, in case $R = 1$, and
(B) $p(t)$ is not a polynomial, *i.e.* $p_n \not\equiv 0$ for infinitely many n in case $R = \infty$.

By Theorem 3 in [9] the general Nörlund mean $(N, p^{*\alpha}, q^{*\beta})$ is regular if and only if

$$\frac{p^{*\alpha}_{n-k}}{r_n} \longrightarrow 0 \quad \text{for any fixed } k. \tag{1.5}$$

REMARK 1. Important special cases are
(i) The Cesàro-Abel-methods:

$$p_n = 1 : (P, p) = (A),\ (N, p^{*\alpha}, p) = (C_\alpha) \quad \alpha \in \mathbb{N}.$$

(ii) The generalized Abel-method ($\delta > 0$):

$$p_n = \binom{n - 1 + \delta}{n} : (P, p) = (A_{\delta-1}),\ (N, p^{*\alpha}, \mathbf{1}) = (C_{\alpha\delta}), \quad \alpha \in \mathbb{N}.$$

(iii) The Euler-Borel-methods:

$$p_n = 1/n! : (P, p) = (B),\ (N, p^{*\alpha}, p) = (E_{\frac{1}{1+\alpha}}), \quad \alpha \in \mathbb{N}.$$

(We use the notation $\mathbf{1}$ for the sequence $(1, 1, \ldots)$).

We now generalize the above results to our general setting, provided some regularity assumptions are satisfied.

2. **Main results.** In [10], Proposition 1, R. Kiesel showed that for $\alpha \leq \beta$, $\alpha, \beta \in \mathbb{N}$ the following inclusions hold true:

$$s_n \longrightarrow s(N, p^{*\alpha}, p) \Rightarrow s_n \longrightarrow s(N, p^{*\beta}, p) \Rightarrow s_n \longrightarrow s(P, p),$$

provided that for all $\gamma \in \mathbb{N}$ the methods $(N, p^{*\gamma}, p)$ are regular (for the second inclusion only the regularity of the (P, p)-method is needed.) This is especially the case, if one of the following conditions is satisfied.

(2.1) (A) $p_n \sim n^\sigma L(n)$, $\sigma \geq 0$, $n^\sigma L(n)$ is nondecreasing and $L(.)$ is slowly varying, see [3] §1.2 for the definition;
(B) $p_n \sim \exp\{-g(n)\}$, where $g \in C_2[0, \infty)$, with $g''(x) \downarrow 0$, $x^2 g''(x) \uparrow \infty$ $(x \longrightarrow \infty)$.

Using the sequence of "maximal weights" (Δ_n) defined by

$$\Delta_n = \inf_{0<t<R} p(t) t^{-n}, \tag{2.2}$$

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we have in the above cases the following relationship

$$\Delta_n = \sqrt{2\pi}\phi(n)p_n(n \longrightarrow \infty), \tag{2.3}$$

where $\phi(.)$ is a suitable, positive function.

For $(x \longrightarrow \infty)$ we have in case (A) that $\sqrt{2\pi}\phi(x) \sim \Gamma(\sigma+1)(\frac{\sigma+1}{e})^{-\sigma-1}x$ and in case (B) that $\phi(x) \sim \big(g''(x)\big)^{-\frac{1}{2}}$.

Following [2, 3 §2.11] we call a function $\psi\colon(0,\infty) \longrightarrow (0,\infty)$ *self-neglecting* if ψ satisfies $\psi(x) = o(x)$ $(x \longrightarrow \infty)$, and if $\psi\big(x+t\psi(x)\big)/\psi(x) \longrightarrow 1$ $(x \longrightarrow \infty)$ locally uniformly in $t \in \mathbb{R}$.

Observe that $g''(x)^{-\frac{1}{2}}$ is self-neglecting because of (2.1) and since for *e.g.* $t \geq 0$

$$\left(\frac{g''\big(x+tg''(x)^{-1/2}\big)}{g''(x)}\right)^{-\frac{1}{2}} \geq 1, \text{ and}$$

$$\left(\frac{g''\big(x+tg''(x)^{-1/2}\big)}{g''(x)}\right)^{-\frac{1}{2}} = \left(\frac{\big(x+tg''(x)^{-1/2}\big)^2 g''\big(x+tg''(x)^{-1/2}\big)}{x^2g''(x)}\right)^{-\frac{1}{2}}\left(1+\frac{t}{\sqrt{x^2g''(x)}}\right)$$

$$\leq 1+\frac{t}{\sqrt{x^2g''(x)}} \longrightarrow 1 \ (x \longrightarrow \infty), \text{ locally uniformly in } t.$$

Because of this locally uniform convergence $\phi(.)$ is self-neglecting, too.

We can now state our main theorem

THEOREM 1. *Let $\alpha,\beta,\gamma,\delta \in \mathbb{N}$ with $\alpha < \delta \leq \beta$ and assume that (p_n) satisfies (2.1). Then*

(i) $s_n \longrightarrow s(P,p^{*\gamma})$ and $s_n = O(1)$ $(N,p^{*\alpha},p^{*\gamma})$ imply $s_n \longrightarrow s(N,p^{*\beta},p^{*\gamma})$.

(ii) $s_n \longrightarrow s(N,p^{*\beta},p^{*\gamma})$ and $s_n = O(1)$ $(N,p^{*\alpha},p^{*\gamma})$ imply $s_n \longrightarrow s(N,p^{*\delta},p^{*\gamma})$.

REMARK 2. In case $p_n \equiv 1, \gamma = 1$ resp. $p_n = 1/n!, \gamma = 1$ Theorem 1 is Theorem TC1 resp. TC2 in the discrete index case.

In our paradigms Abel- and Borel-method we have the following relations of the methods (see [5]):

(i) Abel-case: $(A_{\alpha-1}) = \left(P, \binom{n+\alpha-1}{n}\right) = (P, \mathbf{1}^{*\alpha})$, $\alpha > 0$, then for $\mu > \lambda > -1$:

$$s_n \longrightarrow s(A_\mu) \Rightarrow s_n \longrightarrow s(A_\lambda).$$

(ii) Borel-case: Since $p_n^{*\alpha} = \alpha^n/n!$, we have

$$(B) = (P, 1/n!) \approx \Big(P, \big((\alpha^n)/n!\big)\Big) = \big(P, (1/n!)^{*\alpha}\big).$$

(Where we use $\approx$ to note that two methods are equivalent.)

So the question arises what the relation of $(P,p^{*\alpha})$ and $(P,p^{*\beta})$ resp. $(N,p,p^{*\alpha})$ and $(N,p,p^{*\beta})$ in the general case is. Unfortunately we can only present answers to the question under additional assumptions.

PROPOSITION 1. *Suppose $\alpha, \beta \in \mathbb{N}$ and the sequence (p_n) satisfies (1.1) with $R = 1$ or $R = \infty$ and $p_n^{*\alpha} > 0$. If we have furthermore that $\mu_n = (p_n^{*\beta})/(p_n^{*\alpha})$ is a totally monotone sequence, i.e.*

$$\mu_n = \int_0^R t^n \, d\chi(t) < \infty \tag{2.4}$$

for all $n = 0, 1, \ldots$ with some (bounded) nondecreasing function χ, then we have

$$s_n \longrightarrow s(P, p^{*\alpha}) \text{ implies } s_n \longrightarrow s(P, p^{*\beta}).$$

This result can also be obtained using a theorem of Borwein in [5], but we are able to present a somewhat easier proof. An answer to the question of inclusion in case of the $(N, p, p^{*\alpha})$-means, was already given by Das [8], but again only under restricting additional assumptions.

PROPOSITION 2. *Let $\alpha, \beta \in \mathbb{N}$ and (p_n) a sequence of strictly positive reals. If*

$$\frac{p_{n+1}}{p_n} \uparrow 1 \quad (n \longrightarrow \infty) \tag{2.5}$$

and if additionally either

$$\frac{p_n^{*\beta}}{p_n^{*\alpha}} \geq \frac{p_{n+1}^{*\beta}}{p_{n+1}^{*\alpha}} \quad \text{and} \quad (N, p, p^{*\beta}) \text{ is regular},$$

or

$$\frac{p_n^{*\beta}}{p_n^{*\alpha}} \leq \frac{p_{n+1}^{*\beta}}{p_{n+1}^{*\alpha}} \quad \text{and} \quad \frac{p_n^{*\beta} p_n^{*\alpha+1}}{p_n^{*\beta+1} p_n^{*\alpha}} = O(1) \quad \text{and} \quad (N, p, p^{*\alpha}) \text{ is regular},$$

then $(N, p, p^{*\alpha})$ convergence implies $(N, p, p^{*\beta})$-convergence.

3. **Auxiliary results.** First we discuss the asymptotic properties of the (N, p, q)-means.

LEMMA 1. *Assume that (p_n) satisfies (2.1).*

(i) In case (A), i.e. $p_n = n^\sigma L(n)$, we have

$$p_n^{*2} \sim \begin{cases} n^{2\sigma+1} L^2(n) B(\sigma+1, \sigma+1), & \text{if } \sigma > -1, \\ L^*(n) n^{-1}, & \text{if } \sigma = -1, \end{cases}$$

with $B(.,.)$ denoting the beta-integral and $L^*(.)$ some slowly varying function.

(ii) In case (B), we have for any $\alpha \in \mathbb{N}$

$$p_n^{*\alpha} \sim \sqrt{(2\pi)^{\alpha-1}/\alpha}\, \phi(n/\alpha)^{\alpha-1} \exp\{-\alpha g(n/\alpha)\} \quad (n \longrightarrow \infty), \tag{3.1}$$

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$\phi(.)$ as in (2.3).

PROOF. (i) is a slight generalisation of Theorem 42 in [9] and Theorem 2.3.1 in Chapter 5 of [20]. (ii) For $\alpha = 2$ the result is contained in Proposition 3 of [10]. We use induction on α for the general case. By Definition we have

$$p_n^{*(\alpha+1)} = \sum_{\nu=0}^{n} p_\nu^{*\alpha} p_{n-\nu}.$$

We define a function

$$\varepsilon(x) = x\big(x^2 g''(x)\big)^{-1/4}. \tag{3.2}$$

Then we can show that the essential part of the sum occurs for $\nu \in M(n)$ with

$$M(n) := \Big\{\nu : \Big|\nu - \frac{\alpha n}{\alpha+1}\Big| \leq \varepsilon\Big(\frac{\alpha n}{\alpha+1}\Big)\Big\}.$$

(Use techniques similar to those in the proof of Lemma 2 in [6], see also related calculations in [12, 13].)

By the induction hypotheses we find

$$p_n^{*(\alpha+1)} \sim \sum_{\nu\in M(n)} \sqrt{(2\pi)^{\alpha-1}/\alpha\phi(\nu/\alpha)^{\alpha-1}} \exp\{-\alpha g(\nu/\alpha)\} \exp\{-g(n-\nu)\}.$$

We now use the asymptotics for p_n and the Taylor-expansion ($\theta, \vartheta \in (0,1)$):

$$\begin{aligned} p_n^{*(\alpha+1)} \sim \sum_{\nu\in M(n)} \sqrt{\frac{1}{\alpha}\Big(\frac{2\pi}{g''(\frac{\nu}{\alpha})}\Big)^{\alpha-1}} \exp\Big\{-\alpha\Big(g\Big(\frac{n}{\alpha+1}\Big) + g'\Big(\frac{n}{\alpha+1}\Big)\Big(\frac{\nu}{\alpha} - \frac{n}{\alpha+1}\Big) \\ + \frac{1}{2}g''\Big(\frac{n}{\alpha+1} + \theta\Big(\frac{\nu}{\alpha} - \frac{n}{\alpha+1}\Big)\Big)\Big(\frac{\nu}{\alpha} - \frac{n}{\alpha+1}\Big)^2\Big) \\ - g\Big(\frac{n}{\alpha+1}\Big) - g'\Big(\frac{n}{\alpha+1}\Big)\Big(n-\nu-\frac{n}{\alpha+1}\Big) \\ - \frac{1}{2}g''\Big(\frac{n}{\alpha+1} + \vartheta\Big(n-\nu-\frac{n}{\alpha+1}\Big)\Big)\Big(n-\nu-\frac{n}{\alpha+1}\Big)^2\Big\}. \end{aligned}$$

Now we use the basic inequality (13) in [6], namely

$$\Big|\frac{g''(t)}{g''(x)} - 1\Big| \leq 4\frac{|t-x|}{x} \text{ for all sufficiently large } t, x, \quad \text{if } |t-x| \leq x/4,$$

which is satisfied in our range $M(n)$, and the fact that $\varepsilon(n)/n \longrightarrow 0$ as $n \longrightarrow \infty$ to obtain

$$\begin{aligned} p_n^{*(\alpha+1)} &\sim \sqrt{\frac{1}{\alpha}\Big(\frac{2\pi}{g''(\frac{n}{\alpha+1})}\Big)^{\alpha-1}} \times \exp\Big\{-(\alpha+1)g\Big(\frac{n}{\alpha+1}\Big)\Big\} \\ &\quad \times \sum_{\nu\in M(n)} \exp\Big\{-\frac{(\alpha+1)}{2\alpha}g''\Big(\frac{n}{\alpha+1}\Big)\Big(\nu - \frac{n\alpha}{\alpha+1}\Big)^2\Big\}\big(1+o(1)\big) \\ &\sim \sqrt{\frac{(2\pi)^\alpha}{\alpha+1}\phi\Big(\frac{n}{\alpha+1}\Big)^\alpha} \exp\Big\{-(\alpha+1)g\Big(\frac{n}{\alpha+1}\Big)\Big\}. \end{aligned}$$

For the last step use the approximation of the sum with the integral of a Gaussian density with variance $\alpha/\Big((\alpha+1)g''\big(n/(\alpha+1)\big)\Big)$. ∎

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COROLLARY. *If (p_n) satisfies (2.1(B)) and $\alpha, \beta \in \mathbb{N}$, then we have for the entry $a_{n,k}$ of the $(N, p^{*\alpha}, p^{*\beta})$-matrix the asymptotic relation*

$$a_{n,k} \sim \sqrt{\frac{\alpha+\beta}{2\pi\alpha\beta}\phi\Big(\frac{n}{\alpha+\beta}\Big)^{-1}} \exp\Big\{-\frac{\alpha+\beta}{2\alpha\beta}\Big(\frac{k-\frac{n\beta}{\alpha+\beta}}{\phi(\frac{n}{\alpha+\beta})}\Big)^2\Big\}$$

if $\left|k - \frac{n\beta}{\alpha+\beta}\right| \leq \varepsilon(n)$ *with $\varepsilon(.)$ as in (3.2) and furthermore*

$$\sum_{|k-\frac{n\beta}{\alpha+\beta}|>\varepsilon(n)} a_{n,k} \longrightarrow 0 \quad (n \longrightarrow \infty).$$

PROOF. If $\left|k - \frac{n\beta}{\alpha+\beta}\right| \leq \varepsilon(n)$ then $\big(\theta, \xi \in (0,1)\big)$

$$\begin{aligned}
\frac{p^{*\alpha}_{n-k}p^{*\beta}_k}{p^{*(\alpha+\beta)}_n} &\sim \frac{\exp\{-\alpha g(\frac{n-k}{\alpha}) - \beta g(\frac{k}{\beta})\}}{\exp\{-(\alpha+\beta)g(\frac{n}{\alpha+\beta})\}}\sqrt{\frac{(2\pi)^{\alpha-1}(2\pi)^{\beta-1}g''(\frac{n}{\alpha+\beta})^{\alpha+\beta-1}(\alpha+\beta)}{(2\pi)^{\alpha+\beta-1}g''(\frac{n-k}{\alpha})^{\alpha-1}g''(\frac{k}{\beta})^{\beta-1}\alpha\beta}} \\
&\sim \exp\Big\{-\alpha\Big(g\Big(\frac{n}{\alpha+\beta}\Big) + g'\Big(\frac{n}{\alpha+\beta}\Big)\Big(\frac{n-k}{\alpha} - \frac{n}{\alpha+\beta}\Big) \\
&\qquad + \frac{1}{2}g''\Big(\frac{n}{\alpha+\beta} + \theta\Big(\frac{n-k}{\alpha} - \frac{n}{\alpha+\beta}\Big)\Big)\Big(\frac{n-k}{\alpha} - \frac{n}{\alpha+\beta}\Big)^2\Big) \\
&\qquad - \beta\Big(g\Big(\frac{n}{\alpha+\beta}\Big) + g'\Big(\frac{n}{\alpha+\beta}\Big)\Big(\frac{k}{\beta} - \frac{n}{\alpha+\beta}\Big) \\
&\qquad + \frac{1}{2}g''\Big(\frac{n}{\alpha+\beta} + \xi\Big(\frac{k}{\beta} - \frac{n}{\alpha+\beta}\Big)\Big)\Big(\frac{k}{\beta} - \frac{n}{\alpha+\beta}\Big)^2\Big)\Big\} \\
&\qquad \times \exp\Big\{(\alpha+\beta)g\Big(\frac{n}{\alpha+\beta}\Big)\Big\}\sqrt{\big(1+o(1)\big)\frac{\alpha+\beta}{2\pi\alpha\beta}g''\Big(\frac{n}{\alpha+\beta}\Big)}.
\end{aligned}$$

Now $\left|\frac{k}{\beta} - \frac{n}{\alpha+\beta}\right| \leq \frac{\varepsilon(n)}{\beta}$ and $\left|\frac{n-k}{\alpha} - \frac{n}{\alpha+\beta}\right| \leq \frac{\varepsilon(n)}{\alpha}$. Therefore we obtain the desired result by the same calculations as used in Lemma 1. For the second part observe that

$$\sum_{k=0}^{n} \frac{p^{*\alpha}_{n-k}p^{*\beta}_k}{p^{*(\alpha+\beta)}_n} = 1 \sim \big(1+o(1)\big)\sum_{|k-\frac{n\beta}{\alpha+\beta}|\leq\varepsilon(n)} \exp\{\cdots\}\sqrt{\cdots}. \qquad \blacksquare$$

We now give the asymptotics of the relevant power-series methods and show that for bounded sequences these methods are equivalent to certain generalized Valiron-type means, compare [6, 11].

LEMMA 2. *Assume that (p_n) satisfies (2.1(B)). Then we have as $x \longrightarrow \infty$*

(i)

$$\Big(p\Big(\exp\Big\{g'\Big(\frac{x}{\mu}\Big)\Big\}\Big)\Big)^{\mu} \sim \Big(\sqrt{2\pi\mu}\phi\Big(\frac{x}{\mu}\Big)\Big)^{\mu}\exp\Big\{-\Big(g\Big(\frac{x}{\mu}\Big) - \frac{x}{\mu}g'\Big(\frac{x}{\mu}\Big)\Big)\Big\}.$$

(ii) For bounded sequences (s_n) *the following equivalence holds true*

$$s_n \longrightarrow s(P,p^{*\mu}) \Leftrightarrow \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\mu}\left(\frac{x-t}{\phi(\frac{x}{\mu})}\right)^2\right\} s(t)\frac{dt}{\sqrt{2\pi\mu}\phi(\frac{x}{\mu})} \longrightarrow s,$$

where $s(t) = s_{[t]}$, *for* $t \geq 0$ *and* $s(t) = 0$ *elsewhere.*

PROOF. (i) follows directly from [12], Lemma 5, resp. [13], Lemma 8, see also Lemma 2 in [6].

(ii) In this case the calculations are similar to the calculations used in [6], Lemma 2 and [11], Theorem 2, so we only outline the major steps. We have by using Lemma 1 and part (i) (For the notation see (1.2)).

$$\begin{aligned}
\sigma_{p^{*\mu}}(e^{g'(\frac{x}{\mu})}) &= \frac{\big(1+o(1)\big)}{\sqrt{2\pi\mu}\phi(\frac{x}{\mu})}\sum_{n=0}^{\infty} s_n \exp\left\{-\mu g\Big(\frac{n}{\mu}\Big) + ng'\Big(\frac{x}{\mu}\Big) + \mu g\Big(\frac{x}{\mu}\Big) - xg'\Big(\frac{x}{\mu}\Big)\right\} \\
&= \frac{\big(1+o(1)\big)}{\sqrt{2\pi\mu}\phi(\frac{x}{\mu})}\sum_{n=0}^{\infty} s_n \exp\left\{-\frac{\mu}{2}g''\Big(\frac{x}{\mu}+\theta\Big(\frac{n}{\mu}-\frac{x}{\mu}\Big)\Big)\Big(\frac{n}{\mu}-\frac{x}{\mu}\Big)^2\right\} \\
&= \frac{\big(1+o(1)\big)}{\sqrt{2\pi\mu}\phi(\frac{x}{\mu})}\sum_{n=0}^{\infty} s_n \exp\left\{-\frac{1}{2\mu}g''\Big(\frac{x}{\mu}\Big)\Big(\frac{n}{\mu}-\frac{x}{\mu}\Big)^2\right\} \\
&= \big(1+o(1)\big)\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\mu}\left(\frac{x-t}{\phi(\frac{x}{\mu})}\right)^2\right\} s(t)\frac{dt}{\sqrt{2\pi\mu}\phi(\frac{x}{\mu})}. \qquad \blacksquare
\end{aligned}$$

Next we show that the $(N,p^{*\alpha},p^{*\beta})$-means generalize some important properties of the Euler means.

First we consider the well known product-formula for the Euler-means

$$E_\alpha \circ E_\beta = E_{\alpha+\beta}.$$

This becomes

LEMMA 3. *Assume that* (p_n) *and* (q_n) *satisfy (1.1) (with possibly different radii of convergence) and let* $\alpha,\beta,\gamma \in \mathbb{N}$, $\alpha \leq \beta$.

(i) With $r^{*(\alpha+\beta)} := p^{*\alpha} * q^{*\beta}$, *we have*

$$(N,p^{*\beta},q^{*\gamma}) = (N,p^{*(\beta-\alpha)},r^{*(\alpha+\gamma)}) \circ (N,p^{*\alpha},q^{*\gamma}) \tag{3.3}$$

resp. in case $(p_n) = (q_n)$

$$(N,p^{*\beta},p^{*\gamma}) = (N,p^{*(\beta-\alpha)},p^{*(\alpha+\gamma)}) \circ (N,p^{*\alpha},p^{*\gamma}).$$

(ii) If $(N,p^{*(\beta-\alpha)},r^{*(\alpha+\gamma)})$ *is regular, then* $s_n \longrightarrow s(N,p^{*\alpha},q^{*\gamma})$ *implies* $s_n \longrightarrow s(N,p^{*\beta},q^{*\gamma})$.

PROOF. (ii) is a trivial consequence of (i).

To prove (i) observe that

$$(p^{*(\beta-\alpha)} * r^{*(\alpha+\gamma)})_n = (p^{*\beta} * q^{*\gamma})_n$$

and

$$\sum_{k=0}^{n} p_{n-k}^{*(\beta-\alpha)} r_k^{*(\alpha+\gamma)} \frac{1}{r_k^{*(\alpha+\gamma)}} \sum_{\nu=0}^{k} p_{k-\nu}^{*\alpha} q_\nu^{*\gamma} s_\nu = \sum_{\nu=0}^{n} q_\nu^{*\gamma} s_\nu \sum_{k=0}^{n-\nu} p_{n-\nu-k}^{*(\beta-\alpha)} p_k^{*\alpha} = \sum_{\nu=0}^{n} p_{n-\nu}^{*\beta} q_\nu^{*\gamma} s_\nu .$$

Now $s_n \longrightarrow s\ (N, p^{*(\beta-\alpha)}, r^{*(\alpha+\gamma)}) \circ (N, p^{*\alpha}, q^{*\gamma})$ means that

$$\frac{1}{(p^{*(\beta-\alpha)} * r^{*(\alpha+\gamma)})_n} \sum_{k=0}^{n} p_{n-k}^{*(\beta-\alpha)} r_k^{*(\alpha+\gamma)} \frac{1}{r_k^{*(\alpha+\gamma)}} \sum_{\nu=0}^{k} p_{k-\nu}^{*\alpha} q_\nu^{*\gamma} s_\nu \longrightarrow s \quad (n \longrightarrow \infty),$$

but by the above identities this is the same as

$$\frac{1}{(p^{*\beta} * q^{*\gamma})_n} \sum_{\nu=0}^{n} p_{n-\nu}^{*\beta} q_\nu^{*\gamma} s_\nu \longrightarrow s \quad (n \longrightarrow \infty),$$

which is $(N, p^{*\beta}, q^{*\gamma})$ convergence. ■

A classical result of Knopp [9, Theorem 149] gives a connection between Cesàro convergence with speed and Euler convergence. We generalize this for general (p_n) with an additional condition on the sequence (s_n). (In [10, Theorem 2] this generalization is given with an additional condition on the (p_n), but without conditions on the (s_n).)

LEMMA 4. *Let (p_n) be a sequence of weights satisfying (2.1(B)) and $\phi(.)$ as in (2.3). Furthermore assume that $s_n = O(1)$. Then*

$$\frac{1}{n+1} \sum_{k=0}^{n} (s_k + \varepsilon_k) = s + o\left(\frac{\phi(n)}{n}\right), \quad (n \longrightarrow \infty), \text{ with some nullsequence } (\varepsilon_n)$$

implies $s_n \longrightarrow s(N, p^{*\alpha}, p^{*\beta})$ for every $\alpha, \beta \in \mathbb{N}$.

PROOF. Since $s_n = O(1)$ we can use the asymptotic weights computed in the Corollary to Lemma 1 in the $(N, p^{*\alpha}, p^{*\beta})$ method. By inclusion we have only to show the implication for the $(N, p, p^{*\beta})$ method. Because of regularity and linearity we can suppose $s = 0$ and omit the convergent sequence (ε_k). Thus the hypothesis becomes

$$\sum_{k=0}^{n} s_k = o\big(\phi(n)\big) \quad (n \longrightarrow \infty).$$

For given $\varepsilon > 0$ we can find a $N \in \mathbb{N}$ such that for $n \geq l \geq m \geq N$

$$\left|\sum_{k=m}^{l} s_k\right| \leq \varepsilon\phi(l) \leq \varepsilon\phi(n),$$

using also the monotonicity of $\phi(.)$. By the Corollary to Lemma 1 and since $s_n = O(1)$ we have for the $(N, p, p^{*\beta})$-transform t_n

$$t_n = \sqrt{\frac{\beta+1}{2\pi\beta}}\, \phi\left(\frac{n}{\beta+1}\right)^{-1} \sum_{|k-\frac{n\beta}{\beta+1}| \leq \varepsilon(n)} \exp\left\{-\frac{\beta+1}{2\beta}\left(\frac{k-\frac{n\beta}{\beta+1}}{\phi(\frac{n}{\beta+1})}\right)^2\right\} s_k + o(1),$$

with a function $\varepsilon(.)$ as in (3.2). So the weights are piecewise monotonic and the maximal weight is for $k = \frac{n\beta}{\beta+1}$. We therefore split the sum in two parts, namely

$$t_n = \sum_{\frac{n\beta}{\beta+1}-\varepsilon(n)\leq k<\frac{n\beta}{\beta+1}} \cdots + \sum_{\frac{n\beta}{\beta+1}\leq k\leq \frac{n\beta}{\beta+1}+\varepsilon(n)} \cdots + o(1).$$

Using Abels partial summation and the monotonicity of the weights we find that each of the two sums is bounded by $\varepsilon \frac{\phi(n)}{\phi(n/(\beta+1))}$. Since $\phi(n/\gamma) = O\big(\phi(n)\big)$ for any fixed $\gamma > 0$, we obtain the desired result. ∎

Cesàro-convergence with speed is also connected to the methods of moving-averages by the following

PROPOSITION 3. *The following statements are equivalent for a self-neglecting function $\phi(.)$*

(i) $\frac{1}{n+1}\sum_{k=0}^{n}(s_k+\varepsilon_k) = s + o\big(\frac{\phi(n)}{n}\big)$ $(n \longrightarrow \infty)$ *for some* $\varepsilon_n \longrightarrow 0$.

(ii) $\frac{1}{u\phi(n)}\sum_{n\leq k<n+u\phi(n)} s_k \longrightarrow s$, $\forall u > 0$, $(n \longrightarrow \infty)$.

For the proof see [2], for notation and properties of self-neglecting functions consult [3, §2.11].

In the Euler-Borel case we have the identity $(B)\circ(E_p) \approx (B)$. A similar identity can be obtained in the general case. For a related calculation compare [7].

LEMMA 5. *Assume that (p_n) and (q_n) satisfy (1.1) with the same radius of convergence R and let $\alpha, \beta \in \mathbb{N}$ then*

$$s_n \longrightarrow s(P, q^{*\beta}) \Leftrightarrow s_n \longrightarrow s(P, r^{*(\alpha+\beta)}) \circ (N, p^{*\alpha}, q^{*\beta}).$$

PROOF. $s_n \longrightarrow s(P, q^{*\beta})$ means that $\frac{\sum_{n=0}^{\infty} s_n q_n^{*\beta} x^n}{(q(x))^\beta} \longrightarrow s$, $(x \longrightarrow R)$, and $s_n \longrightarrow s(P, r^{*(\alpha+\beta)})\circ (N, p^{*\alpha}, q^{*\beta})$ means that

$$\frac{\sum_{n=0}^{\infty}\big(\sum_{k=0}^{n} p_{n-k}^{*\alpha} s_k q_k^{*\beta}\big)x^n}{\big(p(x)\big)^\alpha\big(q(x)\big)^\beta} \longrightarrow s \quad (x \longrightarrow R).$$

But

$$\frac{\sum_{n=0}^{\infty} s_n q_n^{*\beta} x^n}{\big(q(x)\big)^\beta} = \frac{\sum_{n=0}^{\infty}\big(\sum_{k=0}^{n} p_{n-k}^{*\alpha} s_k q_k^{*\beta}\big)x^n}{\big(p(x)\big)^\alpha\big(q(x)\big)^\beta},$$

and this proves the proposition. ∎

Using Borwein's Theorem, *i.e.* Proposition 1, we obtain

COROLLARY. *If the assumptions of Lemma 5 hold true and if $\frac{q_n^{*\beta}}{r_n^{*(\alpha+\beta)}}$ is a totally monotone sequence, then*

$$s_n \longrightarrow s\,(P, q^{*\beta}) \Rightarrow s_n \longrightarrow s\,(P, q^{*\beta}) \circ (N, p^{*\alpha}, q^{*\beta}).$$

Generalizing Theorem 1 in [10] slightly we obtain the following Tauberian theorem:

THEOREM 2. *Assume that* (p_n) *satisfies (1.1) and (2.1(B)). Then we have under the Tauberian condition* $s_n = O(1)$ *that for any* $\gamma \in \mathbb{N}$

$$s_n \longrightarrow s(P, p^{*\gamma}) \text{ implies } s_n \longrightarrow s(N, p^{*\alpha}, p^{*\beta})$$

for all $\alpha, \beta \in \mathbb{N}$.

REMARK 3. (i) Under (2.1) $(N, p^{*\alpha}, p^{*\beta})$ is regular for all $\alpha, \beta \in \mathbb{N}$.

(ii) $s_n \longrightarrow s(N, p^{*\alpha}, p^{*\beta})$ implies always $s_n \longrightarrow s(P, p^{*\beta})$, since

$$\sigma_{p^{*\beta}}(t) = \frac{\sum_{n=0}^{\infty} s_n p_n^{*\beta} x^n}{\big(p(x)\big)^{\beta}} = \frac{\sum_{n=0}^{\infty} p_n^{*(\alpha+\beta)} \frac{1}{p_n^{*(\alpha+\beta)}} \big(\sum_{k=0}^{n} p_{n-k}^{*\alpha} p_k^{*\beta} s_k\big) x^n}{\big(p(x)\big)^{\alpha} \big(p(x)\big)^{\beta}},$$

and since $(P, p^{*(\alpha+\beta)})$ is regular, the Abelian conclusion follows.

PROOF. By Lemma 3(ii), it is sufficient to consider $\alpha = 1$. Define $s(u) = s_{[u]}$ if $u \geq 0$ and $s(u) = 0$ if $u < 0$ and $K(x) = 1/\sqrt{2\pi}\exp\{-x^2/2\}$.

Since $s_n = O(1)$ we have by Lemma 2(ii), that $s_n \longrightarrow s(P, p^{*\gamma})$ implies

$$\lim_{x \to \infty} \int_{-\infty}^{\infty} K\left(\frac{x-t}{\sqrt{\gamma}\phi(\frac{x}{\gamma})}\right) s(t) \frac{dt}{\sqrt{\gamma}\phi(x/\gamma)} = s. \tag{3.4}$$

The conditions of Theorem 1 of [15], *i.e.* $K(x) \in L^1(-\infty, \infty)$, the Fourier-transform of K is nonvanishing for any real argument and $\phi(.)$ is self-neglecting, are trivially satisfied.

It follows now from that theorem that if we choose $\varepsilon > 0$ and define

$$H(x) = \begin{cases} \frac{1}{\varepsilon}, & \text{if } x \in (-\varepsilon, 0), \\ 0 & \text{if } x \notin (-\varepsilon, 0), \end{cases}$$

that

$$\lim_{x \to \infty} \int_{-\infty}^{\infty} H\left(\frac{x-t}{\sqrt{\gamma}\phi(x)}\right) s(t) \frac{dt}{\sqrt{\gamma}\phi(x/\gamma)} = \lim_{x \to \infty} \frac{1}{\varepsilon\sqrt{\gamma}\phi(x/\gamma)} \sum_{x \leq k < x+\varepsilon\sqrt{\gamma}\phi(x/\gamma)} s_k = s.$$

Because $\phi(.)$ is self-neglecting and $\phi(x/\gamma) = O\big(\phi(x)\big)$, for any fixed $\gamma > 0$, we obtain by Proposition 3 , that

$$\frac{1}{n+1} \sum_{k=0}^{n} (s_k + \varepsilon_k) = s + o\left(\frac{\phi(n)}{n}\right),$$

which in turn by Lemma 4 implies that $s_n \longrightarrow s(N, p, p^{*\beta})$. ∎

4. **Proofs.**

PROOF OF THEOREM 1. Part (i) by Lemma 5:

$$s_n \longrightarrow s(P,p^{*\gamma}) \Longleftrightarrow s_n \longrightarrow s(P,p^{*(\alpha+\gamma)}) \circ (N,p^{*\alpha},p^{*\gamma}).$$

In case (A): We apply Karamatas' Tauberian theorem (observe Lemma 1) (see [2, Theorem 1.7.6, 18]) and obtain

$$s_n \longrightarrow s(N,\mathbf{1},p^{*(\alpha+\gamma)}) \circ (N,p^{*\alpha},p^{*\gamma}).$$

Since $s_n = O(1)(N,p^{*\alpha},p^{*\gamma})$ we can use the asymptotic weights and assume w.l.o.g that $p_n^{*(\beta-\alpha)}$ is nondecreasing and by Theorem 3 in Das [8] we get

$$s_n \longrightarrow s(N,p^{*(\beta-\alpha)},p^{*(\alpha+\gamma)}) \circ (N,p^{*\alpha},p^{*\gamma}),$$

which by Lemma 3(i) implies our result.

In case (B): Since $s_n = O(1)(N,p^{*\alpha},p^{*\gamma})$ we can use Theorem 2 to obtain directly

$$s_n \longrightarrow s(N,p^{*(\beta-\alpha)},p^{*(\alpha+\gamma)}) \circ (N,p^{*\alpha},p^{*\gamma}).$$

The last step is as above.

Part (ii) is directly implied by part (i) and by the Abelian inclusion. ■

REMARK 4. Boos/Tietz [4] gave an alternative proof of Theorem 1 in the Borel-case. The basic steps are as follows ($\alpha=\gamma=1,\beta=2$)

(i) $s_n \longrightarrow s(P,p) \Rightarrow s_n \longrightarrow s(P,p^{*3})(N,p^{*2},p)$

(ii) $(N,p^{*2},p) = (N,p,p^{*2})(N,p,p)$. Hence if (*) $\big((N,p,p^{*2})x\big)_n - \big((N,p,p^{*2})x\big)_{n-1} = O\big(1/\phi(n)\big)$ for bounded sequences (x_n), one can use the O-Tauberian theorems in [12, 13] to conclude

(iii) $s_n \longrightarrow s(N,p^{*2},p)$.

The statement (*) in (ii) is true for some special cases, like $p_n = 1/n!$, but has not been obtained in general so far.

PROOF OF PROPOSITION 1. Observe that *e.g.* in case $R=\infty$

$$\sigma_{p^{*\beta}}(x) = \frac{\sum_{n=0}^{\infty} s_n \frac{p_n^{*\beta}}{p_n^{*\alpha}} p_n^{*\alpha} x^n}{\big(p(x)\big)^{\beta}} = \int_0^{\infty} \frac{p(xt)^{\alpha}}{p(x)^{\beta}} \sigma_{p^{*\alpha}}(xt)\, d\chi(t) = L\big(\sigma_{p^{*\alpha}}(.\,),x\big).$$

The interchange of integral and sum is allowed because of the absolute convergence for $x>0$. We now follow the arguments in an unpublished paper by A. Jakimovski (oral communication, see also [16] for details.)

$L(f,x)$ is a positive linear operator on a linear space of real functions in $C[0,\infty)$ with the properties:

(i) There exists $e(t)>0$, $e(t)\longrightarrow 1$, $t\longrightarrow\infty$ such that $L\big(e(.\,),x\big)\longrightarrow 1$, $x\longrightarrow\infty$, namely $e(t)=\sigma_{p^{*\alpha}}(t)$ with the sequence (s_n) chosen to be $(1,1,\ldots)$.

(ii) There exists some $e_0(t) > 0$ such that $L\big(e_0(.\,), x\big) \longrightarrow 0$, $x \longrightarrow \infty$, namely $e_0(t) = \sigma_{p^{*\alpha}}(t) = p_0^{*\alpha}/p(t)^\alpha$, with the sequence (s_n) chosen to be $(1, 0, 0, \ldots)$.

From (i) and the assumptions we find

$$|f(t) - se(t)| < \varepsilon/2 \leq \varepsilon e(t), \quad \text{for } t \geq t_0(\varepsilon),$$

and by (ii)

$$|f(t) - se(t)| \leq M \leq \frac{M}{m} e_0(t), \quad t \in [0, t_0(\varepsilon)],$$

with suitable M, m. Hence for $t \geq 0$:

$$|f(t) - se(t)| \leq \varepsilon e(t) + \frac{M}{m} e_0(t).$$

Since L is linear and positive we obtain that $L\big(f(.\,), x\big) \longrightarrow s$ if $f(x) \longrightarrow s$, which yields the desired result. ∎

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