TAUBERIAN- AND CONVEXITY THEOREMS FOR CERTAIN \((N,p,q)\)-MEANS

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ABSTRACT. The summability fields of generalized Nörlund means \((N,p^{\alpha},p)\), \(\alpha \in \mathbb{N}\), are increasing with \(\alpha\) and are contained in that of the corresponding power series method \((P,p)\). Particular cases are the Cesàro- and Euler-means with corresponding power series methods of Abel and Borel. In this paper we generalize a convexity theorem, which is well-known for the Cesàro means and which was recently shown for the Euler means to a large class of generalized Nörlund means.

1. Introduction. We consider throughout complex sequences \((s_n)\) and discuss the relations of certain summability methods.

We say a sequence \((s_n)\) of complex numbers is summable to \(s\) by the

(i) Cesàro-method of order \(\alpha > -1\), briefly \(s_n \to s(C_{\alpha})\), if

\[
\frac{1}{n^{\alpha+1}} \sum_{k=0}^{n} \left( \frac{n-k+\alpha-1}{n-k} \right) s_k \to s \quad (n \to \infty);
\]

(ii) Euler-method of order \(0 < p \leq 1\), briefly \(s_n \to s(E_p)\), if

\[
\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k-1} s_k \to s \quad (n \to \infty);
\]

(iii) Abel-method, briefly \(s_n \to s(A)\), if

\[
f(t) = (1-t) \sum_{n=0}^{\infty} s_n t^n \quad \text{exists for } 0 < t < 1 \text{ and } f(t) \to s \quad (t \to 1-);
\]

(iv) Borel-method, briefly \(s_n \to s(B)\), if

\[
g(t) = e^{-t} \sum_{n=0}^{\infty} \frac{s_n}{n!} t^n \quad \text{exists for } t \in \mathbb{R} \text{ and } g(t) \to s \quad (t \to \infty).
\]

The Cesàro- and Abel-method resp. the Euler- and Borel-method are known to be closely related, see \([9, 17, 19]\).

Especially the following Abelian inclusions are well known, see \(e.g.\) \([9; \text{Theorems } 43, 55, 118, 128]\)

\[
\text{for } -1 < \alpha \leq \beta: s_n \to s(C_{\alpha}) \Rightarrow s_n \to s(C_{\beta}) \Rightarrow s_n \to s(A),
\]

\[
\text{for } 0 < p \leq q \leq 1: s_n \to s(E_q) \Rightarrow s_n \to s(E_p) \Rightarrow s_n \to s(B).
\]

The following converse or Tauberian theorem for the Cesàro-Abel-case goes back to Littlewood \([14]\) \((\alpha, \beta \in \mathbb{N})\), and Anderson \([1]\) \((\alpha, \beta \geq -1)\).
THEOREM TC 1. (i) Let \(-1 < \alpha < \beta\) then \(s_n \to s(A)\) and \(s_n = O(1)(C_{\alpha})\) imply \(s_n \to s(C_{\beta})\).

(ii) For \(-1 < \alpha < \delta < \beta\) we have the so-called convexity-theorem \(s_n \to s(C_{\alpha})\) and \(s_n = O(1)(C_{\alpha})\) imply \(s_n \to s(C_{\delta})\).

Quite recently Boos and Tietz [4] proved that the situation is completely analogous for the Euler-Borel-case.

THEOREM TC 2. (i) Let \(0 < p < q < 1\) then \(s_n \to s(B)\) and \(s_n = O(1)(E_{q})\) imply \(s_n \to s(E_{p})\).

(ii) For \(0 < p < r < q < 1\) we have the convexity-theorem \(s_n \to s(E_{p})\) and \(s_n = O(1)(E_{q})\) imply \(s_n \to s(E_{r})\).

Obviously part (ii) is in both cases a trivial consequence of the Abelian inclusion and part (i).

The aim of this paper is to show that the above results are special cases of a more general setting.

For the following assume that \((p_n)\) is a sequence of reals with the following properties:

\[ p_0 > 0, \quad p_n \geq 0, \quad n \in \mathbb{N}, \text{ such that the power series} \]

\[ p(t) = \sum_{n=0}^{\infty} p_n t^n \text{ has radius of convergence } R > 0. \]  

Since we can use \(p_n R^n\) as weights in case \(0 < R < \infty\), we only have to deal with the two cases \(R = 1\) and \(R = \infty\).

Furthermore we define the \(\alpha\)-th convolution \(p_n^*\) of a sequence \((p_n)\) by

\[ p_n^1 := p_n, \quad n = 0, 1, 2, \ldots \quad \text{and} \quad p_n^{*(\alpha+1)} := \sum_{k=0}^{n} p_n^* q_{n-k} p_k. \]

We now generalize the summability methods used in Theorems TC1 and TC2. To this end we need a further sequence \((q_n)\) of nonnegative reals, also satisfying (1.1), in general with a different radius of convergence \(R_q\) for the associated power series.

We then say, that a sequence \((s_n)\) is summable to \(s\) by the

(i) **power series method** of summability \((P, p)\), briefly \(s_n \to s(P, p)\), if

\[ p_s(t) = \sum_{n=0}^{\infty} s_n p_n t^n \text{ converges for } |t| < R \text{ and if } \sigma_p(t) = \frac{p_s(t)}{p(t)} \to s \text{ as } t \to R-. \]  

(In case \(R = 1\) we have the so-called \((J_p)\)-methods, in case \(R = \infty\) the \((B_p)\)-methods).

(ii) **general Norlund-means** \((N, p^*\alpha, q^*\beta); \alpha, \beta \in \mathbb{N}\), briefly \(s_n \to s(N, p^*\alpha, q^*\beta)\), if

\[ \frac{1}{r_n} \sum_{k=0}^{n} p_n^* q_{n-k}^* s_k \to s \quad (n \to \infty), \text{ where we suppose that} \]

\[ r_n := (p^*\alpha \ast q^*\beta)_n = \sum_{k=0}^{n} p_n^* q_{n-k}^* > 0 \quad \text{for } n = 0, 1, \ldots. \]
We require all methods to be regular. By Theorem 5 in [9], we have regularity for a power series method if and only if

\[(A) \quad P_n = \sum_{k=0}^{\infty} p_k \to \infty, \quad (n \to \infty), \text{ in case } R = 1, \text{ and } \\
(B) \quad p(t) \text{ is not a polynomial, i.e. } p_n \neq 0 \text{ for infinitely many } n \text{ in case } R = \infty.\]

By Theorem 3 in [9] the general Nörlund mean \((N, p^{*\alpha}, q^{*\beta})\) is regular if and only if

\[(1.5) \quad \frac{p_{n-k}}{r_n} \to 0 \quad \text{for any fixed } k.\]

**REMARK 1.** Important special cases are

(i) The Cesàro-Abel-methods:

\[p_n = 1 : (P, p) = (A), \quad (N, p^{*\alpha}, p) = (C_\alpha) \quad \alpha \in \mathbb{N}.\]

(ii) The generalized Abel-method \((\delta > 0)\):

\[p_n = \left(\frac{n - 1 + \delta}{n}\right) : (P, p) = (A_{\delta-1}), \quad (N, p^{*\alpha}, 1) = (C_{\alpha\delta}), \quad \alpha \in \mathbb{N}.\]

(iii) The Euler-Borel-methods:

\[p_n = 1/n! : (P, p) = (B), \quad (N, p^{*\alpha}, p) = (E_{1/n}), \quad \alpha \in \mathbb{N}.\]

(We use the notation \(1\) for the sequence \((1, 1, \ldots)\).

We now generalize the above results to our general setting, provided some regularity assumptions are satisfied.

2. **Main results.** In [10], Proposition 1, R. Kiesel showed that for \(\alpha \leq \beta, \alpha, \beta \in \mathbb{N}\) the following inclusions hold true:

\[s_n \to s(N, p^{*\alpha}, p) \Rightarrow s_n \to s(N, p^{*\beta}, p) \Rightarrow s_n \to s(P, p),\]

provided that for all \(\gamma \in \mathbb{N}\) the methods \((N, p^{*\gamma}, p)\) are regular (for the second inclusion only the regularity of the \((P, p)\)-method is needed.) This is especially the case, if one of the following conditions is satisfied.

\[(A) \quad p_n \sim n^\alpha L(n), \quad \sigma \geq 0, \quad n^\sigma L(n) \text{ is nondecreasing and } L(.) \text{ is slowly varying, see } [3] \S 1.2 \text{ for the definition; } \\
(2.1) \quad p_n \sim \exp\{-g(n)\}, \text{ where } g \in C_2[0, \infty), \text{ with } g''(x) \downarrow 0, \quad x^2g''(x) \uparrow \infty \quad (x \to \infty).\]

Using the sequence of “maximal weights” \((\Delta_n)\) defined by

\[(2.2) \quad \Delta_n = \inf_{0 < t < R} p(t)r^{-n},\]
we have in the above cases the following relationship

\[(2.3) \quad \Delta_n = \sqrt{2\pi} \phi(n)p_n(n \to \infty),\]

where \(\phi(.)\) is a suitable, positive function.

For \((x \to \infty)\) we have in case (A) that \(\sqrt{2\pi} \phi(x) \sim \Gamma(\sigma + 1) (\frac{\alpha+1}{e})^{-\sigma} x\) and in case (B) that \(\phi(x) \sim (g''(x))^{-\frac{1}{2}}\).

Following [2, 3 §2.11] we call a function \(\psi: (0, \infty) \to (0, \infty)\) self-neglecting if \(\psi\) satisfies \(\psi(x) = o(x)\) \((x \to \infty)\), and if \(\psi(x + t\psi(x))/\psi(x) \to 1\) \((x \to \infty)\) locally uniformly in \(t \in \mathbb{R}\).

Observe that \(g''(x)^{-\frac{1}{2}}\) is self-neglecting because of (2.1) and since for \(e.g. t \geq 0\)

\[
\left(\frac{g''(x + t g''(x))^{-\frac{1}{2}}}{g''(x)}\right)^{-\frac{1}{2}} \geq 1, \quad \text{and}
\]

\[
\left(\frac{g''(x + t g''(x))^{-\frac{1}{2}}}{g''(x)}\right)^{-\frac{1}{2}} = \left(\frac{g''(x + t g''(x))^{-\frac{1}{2}}}{g''(x)}\right)^{-\frac{1}{2}} \left(1 + \frac{t}{x^2 g''(x)}\right) \leq 1 + \frac{t}{x^2 g''(x)} \to 1 (x \to \infty), \text{locally uniformly in } t.
\]

Because of this locally uniform convergence \(\phi(.)\) is self-neglecting, too.

We can now state our main theorem

**THEOREM 1.** Let \(\alpha, \beta, \gamma, \delta \in \mathbb{N}\) with \(\alpha < \delta \leq \beta\) and assume that \((p_n)\) satisfies (2.1). Then

(i) \(s_n \to s(P, \rho^*)\) and \(s_n = O(1) (N, p^*, \rho^*)\) imply \(s_n \to s(N, p^*, \rho^*)\).

(ii) \(s_n \to s(N, P^*, \rho^*)\) and \(s_n = O(1) (N, p^*, \rho^*)\) imply \(s_n \to s(N, p^*, \rho^*)\).

**REMARK 2.** In case \(p_n \equiv 1, \gamma = 1\) resp. \(p_n = 1/n\!, \gamma = 1\) Theorem 1 is Theorem TC1 resp. TC2 in the discrete index case.

In our paradigms Abel- and Borel-method we have the following relations of the methods (see [5]):

(i) Abel-case: \((\alpha_{\alpha-1}) = (P, (\alpha + \sigma - 1)) = (P, 1^\alpha)\), \(\alpha > 0\), then for \(\mu > \lambda > -1\):

\(s_n \to s(A_{\mu}) \Rightarrow s_n \to s(A_{\lambda}).\)

(ii) Borel-case: Since \(p^* = \alpha^* / n\!, we have

\[(B) = (P, 1/n!) \approx (P, ((\alpha^*)/n!)) = (P, (1/n!)\alpha^*).\]

(Where we use \(\approx\) to note that two methods are equivalent.)

So the question arises what the relation of \((P, p^*)\) and \((P, p^*)\) resp. \((N, p, p^*)\) and \((N, p, p^*)\) in the general case is. Unfortunately we can only present answers to the question under additional assumptions.
PROPOSITION 1. Suppose \( \alpha, \beta \in \mathbb{N} \) and the sequence \( (p_n) \) satisfies (1.1) with \( R = 1 \) or \( R = \infty \) and \( p_n^{*\alpha} > 0 \). If we have furthermore that \( \mu_n = (p_n^{*\beta})/(p_n^{*\alpha}) \) is a totally monotone sequence, i.e.

\[
\mu_n = \int_0^R r^\beta \, d\chi(t) < \infty
\]

for all \( n = 0, 1, \ldots \) with some (bounded) nondecreasing function \( \chi \), then we have

\[
s_n \to s(P, p^{*\alpha}) \quad \text{implies} \quad s_n \to s(P, p^{*\beta}).
\]

This result can also be obtained using a theorem of Borwein in [5], but we are able to present a somewhat easier proof. An answer to the question of inclusion in case of the \( (N, p, p^{*\alpha}) \)-means, was already given by Das [8], but again only under restricting additional assumptions.

PROPOSITION 2. Let \( \alpha, \beta \in \mathbb{N} \) and \( (p_n) \) a sequence of strictly positive reals. If

\[
\frac{p_{n+1}}{p_n} \uparrow 1 \quad (n \to \infty)
\]

and if additionally either

\[
\frac{p_n^{*\beta}}{p_n^{*\alpha}} \geq \frac{p_{n+1}^{*\beta}}{p_{n+1}^{*\alpha}} \quad \text{and} \quad (N, p, p^{*\beta}) \text{ is regular},
\]

or

\[
\frac{p_n^{*\beta}}{p_n^{*\alpha}} \leq \frac{p_{n+1}^{*\beta}}{p_{n+1}^{*\alpha}} \quad \text{and} \quad \frac{p_n^{*\alpha} p_{n+1}^{*\alpha}}{p_n^{*\beta} p_{n+1}^{*\beta}} = O(1) \quad \text{and} \quad (N, p, p^{*\alpha}) \text{ is regular},
\]

then \( (N, p, p^{*\alpha}) \) convergence implies \( (N, p, p^{*\beta}) \)-convergence.

3. Auxiliary results. First we discuss the asymptotic properties of the \( (N, p, q) \)-means.

LEMMA 1. Assume that \( (p_n) \) satisfies (2.1).

(i) In case (A), i.e. \( p_n = n^\sigma L(n) \), we have

\[
p_n^{*\alpha} \sim \begin{cases} n^{2\sigma+1} L^2(n) B(\sigma+1, \sigma+1), & \text{if } \sigma > -1, \\ L^*(n) n^{-1}, & \text{if } \sigma = -1, \end{cases}
\]

with \( B(\ldots) \) denoting the beta-integral and \( L^*(\ldots) \) some slowly varying function.

(ii) In case (B), we have for any \( \alpha \in \mathbb{N} \)

\[
p_n^{*\alpha} \sim \sqrt{(2\pi)^{\alpha-1}/\alpha} \phi(n/\alpha)^{\alpha-1} \exp\{-\alpha g(n/\alpha)\} \quad (n \to \infty),
\]
\( \phi(.) \) as in (2.3).

**Proof.** (i) is a slight generalization of Theorem 42 in [9] and Theorem 2.3.1 in Chapter 5 of [20]. (ii) For \( \alpha = 2 \) the result is contained in Proposition 3 of [10]. We use induction on \( \alpha \) for the general case. By Definition we have

\[
p_n^{(\alpha+1)} = \sum_{\nu=0}^{n} p_\nu^{\alpha} p_{n-\nu}.
\]

We define a function

\[
\varepsilon(x) = x \left( x^2 g''(x) \right)^{-1/4}.
\]

Then we can show that the essential part of the sum occurs for \( \nu \in M(n) \) with

\[
M(n) := \left\{ \nu : \left| \nu - \frac{\alpha n}{\alpha + 1} \right| \leq \varepsilon \left( \frac{\alpha n}{\alpha + 1} \right) \right\}.
\]

(Use techniques similar to those in the proof of Lemma 2 in [6], see also related calculations in [12, 13].)

By the induction hypotheses we find

\[
p_n^{(\alpha+1)} \sim \sum_{\nu \in M(n)} \sqrt{(2\pi)^{\alpha-1} / \alpha \phi(\nu / \alpha)^{\alpha-1}} \exp\{-\alpha g(\nu / \alpha)\} \exp\{-g(n - \nu)\}.
\]

We now use the asymptotics for \( p_n \) and the Taylor-expansion (\( \theta, \varphi \in (0, 1) \)):

\[
p_n^{(\alpha+1)} \sim \sum_{\nu \in M(n)} \frac{1}{\alpha} \left( \frac{2\pi}{g''(\nu / \alpha)} \right)^{\alpha-1} \exp\left\{-\alpha \left( g\left( \frac{n}{\alpha + 1} \right) + g'\left( \frac{n}{\alpha + 1} \right) \left( \frac{\nu}{\alpha} - \frac{n}{\alpha + 1} \right) \right) \right\}
+ \frac{1}{2} g''\left( \frac{n}{\alpha + 1} + \frac{\nu}{\alpha} - \frac{n}{\alpha + 1} \right) \left( \frac{\nu}{\alpha} - \frac{n}{\alpha + 1} \right)^2 \right)
- g\left( \frac{n}{\alpha + 1} \right) - g'\left( \frac{n}{\alpha + 1} \right) \left( n - \nu - \frac{n}{\alpha + 1} \right) \left( n - \nu - \frac{n}{\alpha + 1} \right)^2 \right\}.
\]

Now we use the basic inequality (13) in [6], namely

\[
\left| \frac{g''(t)}{g''(x)} - 1 \right| \leq 4 \left| \frac{t - x}{x} \right| \quad \text{for all sufficiently large } t, x, \quad \text{if } |t - x| \leq x/4,
\]

which is satisfied in our range \( M(n) \), and the fact that \( \varepsilon(n)/n \to 0 \) as \( n \to \infty \) to obtain

\[
p_n^{(\alpha+1)} \sim \sqrt{\left( \frac{2\pi}{g''(\nu / \alpha)} \right)^{\alpha-1}} \times \exp\left\{-\left( \alpha + 1 \right) g\left( \frac{n}{\alpha + 1} \right) \right\}
+ \sum_{\nu \in M(n)} \exp\left\{-\left( \alpha + 1 \right) g\left( \frac{n}{\alpha + 1} \right) \right\} \left( \frac{\nu}{\alpha} - \frac{n\alpha}{\alpha + 1} \right)^2 \right\} \left( 1 + o(1) \right)
\]

\[
\sim \sqrt{\left( \frac{2\pi}{g''(\nu / \alpha)} \right)^{\alpha}} \phi\left( \frac{n}{\alpha + 1} \right)^{\alpha} \exp\left\{-\left( \alpha + 1 \right) g\left( \frac{n}{\alpha + 1} \right) \right\}.
\]

For the last step use the approximation of the sum with the integral of a Gaussian density with variance \( \alpha / \left( \alpha + 1 \right) g''(n / (\alpha + 1)) \).

\[\square\]
COROLLARY. If \((p_n)\) satisfies (2.1(B)) and \(\alpha, \beta \in \mathbb{N}\), then we have for the entry \(a_{n,k}\) of the \((N, p^{\alpha}, p^{\beta})\)-matrix the asymptotic relation

\[
a_{n,k} \sim \left(\frac{\alpha + \beta}{2\pi\alpha\beta}\right)^{-1} \exp\left\{-\frac{\alpha + \beta}{2\alpha\beta} \left(\frac{k - \frac{n\beta}{\alpha + \beta}}{\phi\left(\frac{n}{\alpha + \beta}\right)}\right)^2\right\}
\]

if \(|k - \frac{n\beta}{\alpha + \beta}| \leq \varepsilon(n)\) with \(\varepsilon(.)\) as in (3.2) and furthermore

\[
\sum_{|k - \frac{n\beta}{\alpha + \beta}| \geq \varepsilon(n)} a_{n,k} \to 0 \quad (n \to \infty).
\]

PROOF. If \(|k - \frac{n\beta}{\alpha + \beta}| \leq \varepsilon(n)\) then \((\theta, \xi \in (0, 1))\)

\[
p^{\ast -k}P_k^\beta \sim \frac{\exp\{-\alpha g\left(\frac{n-k}{\alpha + \beta}\right) - \beta g\left(n\right)\}}{\exp\{-\alpha + \beta\}g\left(n\right)} \sqrt{\frac{(2\pi)^{\alpha-1}(2\pi)^{\beta-1}\beta^n\left(n\right)^{\alpha^\beta-1}(\alpha + \beta)}{(2\pi)^{\alpha^\beta-1}\beta^n\left(n\right)^{\alpha^\beta-1}(\alpha + \beta)}}
\]

\[
\sim \exp\left\{-\alpha \left(g\left(\frac{n-k}{\alpha + \beta}\right) + g'\left(\frac{n}{\alpha + \beta}\right)\left(\frac{n-k}{\alpha} - \frac{n}{\alpha + \beta}\right)\right)
+ \frac{1}{2} \frac{\beta^n\left(n\right)}{\alpha + \beta + \theta\left(\frac{n-k}{\alpha} - \frac{n}{\alpha + \beta}\right)\left(\frac{n-k}{\alpha} - \frac{n}{\alpha + \beta}\right)^2}
- \beta \left(g\left(\frac{n}{\alpha + \beta}\right) + g'\left(\frac{n}{\alpha + \beta}\right)\left(\frac{k}{\beta} - \frac{n}{\alpha + \beta}\right)\right)
+ \frac{1}{2} \frac{\beta^n\left(n\right)}{\alpha + \beta + \xi\left(\frac{k}{\beta} - \frac{n}{\alpha + \beta}\right)\left(\frac{k}{\beta} - \frac{n}{\alpha + \beta}\right)^2}\right\}
\times \exp\left\{(\alpha + \beta)\left(\frac{n}{\alpha + \beta}\right)\right\} \sqrt{\frac{1 + o(1)}{2\pi\alpha\beta\beta^n\left(n\right)}}
\]

Now \(|\frac{k}{\beta} - \frac{n}{\alpha + \beta}| \leq \varepsilon(n)\) and \(|\frac{n-k}{\alpha} - \frac{n}{\alpha + \beta}| \leq \varepsilon(n)\). Therefore we obtain the desired result by the same calculations as used in Lemma 1. For the second part observe that

\[
\sum_{k=0}^{n} p^{\ast -k}P_k^\beta = 1 \sim \left(1 + o(1)\right) \sum_{|k - \frac{n\beta}{\alpha + \beta}| \leq \varepsilon(n)} \exp\{\cdots\} \sqrt{\cdots}.
\]

We now give the asymptotics of the relevant power-series methods and show that for bounded sequences these methods are equivalent to certain generalized Valiron-type means, compare [6, 11].

LEMMA 2. Assume that \((p_n)\) satisfies (2.1(B)). Then we have as \(x \to \infty\)

\[
(i) \left\{P\left(\exp\left\{g'(\frac{x}{\mu})\right\}\right)\right\}^\mu \sim \left(2\pi\mu\phi\left(\frac{x}{\mu}\right)\right)^\mu \exp\left\{-\left(g\left(\frac{x}{\mu}\right) - \frac{x}{\mu}g'\left(\frac{x}{\mu}\right)\right)\right\}.
\]
(ii) For bounded sequences \((s_n)\) the following equivalence holds true
\[
s_n \to s(P, p^{*\mu}) \Leftrightarrow \int_{-\infty}^{\infty} \exp \left\{ -\frac{1}{2\mu} \left( \frac{x-t}{\phi(\frac{x}{\mu})} \right)^2 \right\} s(t) \frac{dt}{\sqrt{2\pi\mu\phi(\frac{x}{\mu})}} \to s,
\]
where \(s(t) = s_{[t]}\), for \(t \geq 0\) and \(s(t) = 0\) elsewhere.

**Proof.** (i) follows directly from [12], Lemma 5, resp. [13], Lemma 8, see also Lemma 2 in [6].

(ii) In this case the calculations are similar to the calculations used in [6], Lemma 2 and [11], Theorem 2, so we only outline the major steps. We have by using Lemma 1 and part (i) (For the notation see (1.2)).

\[
\sigma_{P^{\mu}}(e^x(\frac{1}{\mu})) = \frac{1 + o(1)}{\sqrt{2\pi\mu\phi(\frac{x}{\mu})}} \sum_{n=0}^{\infty} s_n \exp \left\{ -\mu g \left( \frac{n}{\mu} \right) + ng' \left( \frac{x}{\mu} \right) + \mu g \left( \frac{x}{\mu} \right) - xg' \left( \frac{x}{\mu} \right) \right\}
\]
\[
= \frac{1 + o(1)}{\sqrt{2\pi\mu\phi(\frac{x}{\mu})}} \sum_{n=0}^{\infty} s_n \exp \left\{ -\frac{\mu}{2} g'' \left( \frac{x}{\mu} \right) + \theta \left( \frac{n}{\mu} - \frac{x}{\mu} \right) \right\} \left( \frac{n}{\mu} - \frac{x}{\mu} \right)^2
\]
\[
= \frac{1 + o(1)}{\sqrt{2\pi\mu\phi(\frac{x}{\mu})}} \sum_{n=0}^{\infty} s_n \exp \left\{ -\frac{1}{2} g'' \left( \frac{x}{\mu} \right) \left( \frac{n}{\mu} - \frac{x}{\mu} \right)^2 \right\}
\]
\[
= \left( 1 + o(1) \right) \int_{-\infty}^{\infty} \exp \left\{ -\frac{1}{2\mu} \left( \frac{x-t}{\phi(\frac{x}{\mu})} \right)^2 \right\} s(t) \frac{dt}{\sqrt{2\pi\mu\phi(\frac{x}{\mu})}}.
\]

Next we show that the \((N, p^{*\alpha}, p^{*\beta})\)-means generalize some important properties of the Euler means.

First we consider the well known product-formula for the Euler-means
\[
E_{\alpha} \circ E_{\beta} = E_{\alpha+\beta}.
\]
This becomes

**Lemma 3.** Assume that \((p_n)\) and \((q_n)\) satisfy (1.1) (with possibly different radii of convergence) and let \(\alpha, \beta, \gamma \in \mathbb{N}, \alpha \leq \beta\).

(i) With \(r^{*(\alpha+\beta)} := p^{*\alpha} \ast q^{*\beta}\), we have
\[
(N, p^{*\beta}, q^{*\gamma}) = (N, p^{*\gamma}, r^{*(\alpha+\beta)}) \circ (N, p^{*\alpha}, q^{*\gamma})
\]
resp. in case \((p_n) = (q_n)\)
\[
(N, p^{*\beta}, q^{*\gamma}) = (N, p^{*\gamma}, r^{*(\alpha+\beta)}) \circ (N, p^{*\alpha}, q^{*\gamma}).
\]

(ii) If \((N, p^{*\gamma}, r^{*(\alpha+\beta)})\) is regular, then \(s_n \to s(N, p^{*\alpha}, q^{*\gamma})\) implies \(s_n \to s(N, p^{*\beta}, q^{*\gamma})\).

**Proof.** (ii) is a trivial consequence of (i).
To prove (i) observe that

\[(p^{*(\beta-\alpha)} \ast r^{*(\alpha+\gamma)})_n = (p^{*\beta} \ast q^{*\gamma})_n\]

and

\[
\sum_{k=0}^{n} p_{n-k}^{*(\beta-\alpha)} r_k^{*(\alpha+\gamma)} \frac{1}{r_k^{*(\alpha+\gamma)}} \sum_{\nu=0}^{k} p_{n-k-\nu}^{*\alpha} q_{\nu}^{*\gamma} s_{\nu} = \sum_{\nu=0}^{n-\nu} p_{n-\nu}^{*(\beta-\alpha)} r_{\nu}^{*(\alpha+\gamma)} \sum_{k=0}^{n-\nu} p_{n-\nu-k}^{*\alpha} p_{k}^{*\alpha} = \sum_{\nu=0}^{n} p_{n-\nu}^{*\beta} q_{\nu}^{*\gamma} s_{\nu}.
\]

Now \(s_n \to s\) \((N,p^{*(\beta-\alpha)},r^{*(\alpha+\gamma)}) \circ \phi\) \((N,p^{*\alpha},q^{*\gamma})\) means that

\[
\frac{1}{(p^{*(\beta-\alpha)} \ast r^{*(\alpha+\gamma)})_n} \sum_{k=0}^{n} p_{n-k}^{*(\beta-\alpha)} r_k^{*(\alpha+\gamma)} \frac{1}{r_k^{*(\alpha+\gamma)}} \sum_{\nu=0}^{k} p_{n-k-\nu}^{*\alpha} q_{\nu}^{*\gamma} s_{\nu} \to s \quad (n \to \infty),
\]

but by the above identities this is the same as

\[
\frac{1}{(p^{*\beta} \ast q^{*\gamma})_n} \sum_{\nu=0}^{n} p_{n-\nu}^{*\beta} q_{\nu}^{*\gamma} s_{\nu} \to s \quad (n \to \infty),
\]

which is \((N,p^{*\beta},q^{*\gamma})\) convergence.

A classical result of Knopp [9, Theorem 149] gives a connection between Cesàro convergence with speed and Euler convergence. We generalize this for general \((p_n)\) with an additional condition on the sequence \((s_n)\). (In [10, Theorem 2] this generalization is given with an additional condition on the \((p_n)\), but without conditions on the \((s_n)\).)

**Lemma 4.** Let \((p_n)\) be a sequence of weights satisfying (2.1(B)) and \(\phi(.)\) as in (2.3). Furthermore assume that \(s_n = O(1)\). Then

\[
\frac{1}{n+1} \sum_{k=0}^{n} (s_k + \epsilon_k) = s + o\left(\frac{\phi(n)}{n}\right), \quad (n \to \infty), \quad \text{with some nullsequence } (\epsilon_n).
\]

implies \(s_n \to s(N,p^{*\alpha},p^{*\beta})\) for every \(\alpha, \beta \in \mathbb{N}\).

**Proof.** Since \(s_n = O(1)\) we can use the asymptotic weights computed in the Corollary to Lemma 1 in the \((N,p^{*\alpha},p^{*\beta})\) method. By inclusion we have only to show the implication for the \((N,p,p^{*\beta})\) method. Because of regularity and linearity we can suppose \(s = 0\) and omit the convergent sequence \((\epsilon_k)\). Thus the hypothesis becomes

\[
\sum_{k=0}^{n} s_k = o\left(\frac{\phi(n)}{n}\right) \quad (n \to \infty).
\]

For given \(\epsilon > 0\) we can find a \(N \in \mathbb{N}\) such that for \(n \geq l \geq m \geq N\)

\[
\left|\sum_{k=m}^{l} s_k\right| \leq \epsilon \phi(l) \leq \epsilon \phi(n),
\]

using also the monotonicity of \(\phi(.)\). By the Corollary to Lemma 1 and since \(s_n = O(1)\) we have for the \((N,p,p^{*\beta})\)-transform \(t_n\)

\[
t_n = \left[\frac{\beta+1}{2\pi\beta}\frac{n}{\beta+1}\right]^{-1} \sum_{k=n^{-\beta}}^{n^{-\beta+1}} \exp\left(-\frac{\beta+1}{2\beta} \left(\frac{k-n^{-\beta+1}}{\phi(n)}}{\phi(n)}\right)^2\right) s_k + o(1),
\]

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with a function \( \varepsilon(.) \) as in (3.2). So the weights are piecewise monotone and the maximal weight is for \( k = \frac{n}{\beta+1} \). We therefore split the sum in two parts, namely

\[
 t_n = \sum_{\frac{n}{\beta+1} + \varepsilon(n)}^{\frac{n}{\beta+1}} \cdots + \sum_{\frac{n}{\beta+1} + \varepsilon(n)}^{\frac{n}{\beta+1} + o(1)}.
\]

Using Abels partial summation and the monotonicity of the weights we find that each of the two sums is bounded by \( \varepsilon \frac{\phi(n)}{\phi(n)/(\beta+1)} \). Since \( \phi(n/\gamma) = O(\phi(n)) \) for any fixed \( \gamma > 0 \), we obtain the desired result.

Cesàro-convergence with speed is also connected to the methods of moving-averages by the following

**PROPOSITION 3.** The following statements are equivalent for a self-neglecting function \( \phi(.) \)

(i) \( \frac{1}{m+1} \sum_{k=m+1}^{m+n} s_k = s + o(\frac{\phi(n)}{n}) \) \((n \to \infty)\) for some \( \varepsilon_n \to 0 \).

(ii) \( \frac{1}{\phi(n)} \sum_{n \leq k < n+\phi(n)} s_k \to s, \forall u > 0, (n \to \infty) \).

For the proof see [2], for notation and properties of self-neglecting functions consult [3, §2.11].

In the Euler-Borel case we have the identity \((B) \circ (E_p) \approx (B)\). A similar identity can be obtained in the general case. For a related calculation compare [7].

**LEMMA 5.** Assume that \((p_n)\) and \((q_n)\) satisfy (1.1) with the same radius of convergence \( R \) and let \( \alpha, \beta \in \mathbb{N} \) then

\[
 s_n \to s(P, q^* \beta) \iff s_n \to s(P, r^{*(\alpha+\beta)}) \circ (N, p^* \alpha, q^* \beta).
\]

**PROOF.** \( s_n \to s(P, q^* \beta) \) means that \( \frac{\sum_{u=0}^{\infty} s_{n+u} q_n^\alpha}{(q(x))^{\beta}} \to s, (x \to R) \), and \( s_n \to s(P, r^{*(\alpha+\beta)}) \circ (N, p^* \alpha, q^* \beta) \) means that

\[
 \frac{\sum_{u=0}^{\infty} \left( \sum_{k=0}^{n} p_n \alpha \cdot s_k q_k^\beta \right) x^n}{(p(x))^\alpha (q(x))^\beta} \to s \quad (x \to R).
\]

But

\[
 \frac{\sum_{u=0}^{\infty} s_n q_n^\beta x^n}{(q(x))^\beta} = \frac{\sum_{u=0}^{\infty} \left( \sum_{k=0}^{n} p_n \alpha \cdot s_k q_k^\beta \right) x^n}{(p(x))^\alpha (q(x))^\beta},
\]

and this proves the proposition.

Using Borwein’s Theorem, i.e. Proposition 1, we obtain

**COROLLARY.** If the assumptions of Lemma 5 hold true and if \( \frac{r_n}{n} q^\beta \) is a totally monotone sequence, then

\[
 s_n \to s(P, q^* \beta) \Rightarrow s_n \to s(P, q^* \beta) \circ (N, p^* \alpha, q^* \beta).
\]

Generalizing Theorem 1 in [10] slightly we obtain the following Tauberian theorem:
THEOREM 2. Assume that \((p_n)\) satisfies (1.1) and (2.1(B)). Then we have under the Tauberian condition \(s_n = O(1)\) that for any \(\gamma \in \mathbb{N}\)

\[ s_n \to s(P, p^{*\gamma}) \implies s_n \to s(N, p^{*\alpha}, p^{*\beta}) \]

for all \(\alpha, \beta \in \mathbb{N}\).

REMARK 3. (i) Under (2.1) \((N, p^{*\alpha}, p^{*\beta})\) is regular for all \(\alpha, \beta \in \mathbb{N}\).

(ii) \(s_n \to s(N, p^{*\alpha}, p^{*\beta})\) implies always \(s_n \to s(P, p^{*\gamma})\), since

\[
\sigma_{p^\sigma}(t) = \frac{\sum_{n=0}^{\infty} s_n p^\beta x^n}{(p(x))^\beta} = \frac{\sum_{n=0}^{\infty} p_n^{*(\alpha+\beta)}}{p_n^{*(\alpha+\beta)}} \left(\sum_{k=0}^{n} p_{n-k}^{*\alpha} p_k^{*\beta} s_k \right) x^n,
\]

and since \((P, p^{*\alpha+\beta})\) is regular, the Abelian conclusion follows.

PROOF. By Lemma 3(ii), it is sufficient to consider \(\alpha = 1\). Define \(s(u) = s_{[u]}\) if \(u \geq 0\) and \(s(u) = 0\) if \(u < 0\) and \(K(x) = 1/\sqrt{2\pi} \exp\{-x^2/2\}\).

Since \(s_n = O(1)\) we have by Lemma 2(ii), that \(s_n \to s(P, p^{*\gamma})\) implies

\[
(3.4) \lim_{x \to \infty} \int_{-\infty}^{\infty} K\left(\frac{x - t}{\sqrt{\gamma} \phi(x/\gamma)}\right) s(t) \frac{dt}{\sqrt{\gamma} \phi(x/\gamma)} = s.
\]

The conditions of Theorem 1 of [15], i.e. \(K(x) \in L^1(-\infty, \infty)\), the Fourier-transform of \(K\) is nonvanishing for any real argument and \(\phi(.)\) is self-neglecting, are trivially satisfied.

It follows now from that theorem that if we choose \(\varepsilon > 0\) and define

\[ H(x) = \begin{cases} \frac{1}{\varepsilon}, & \text{if } x \in (-\varepsilon, 0), \\ 0, & \text{if } x \notin (-\varepsilon, 0), \end{cases} \]

that

\[
\lim_{x \to \infty} \int_{-\infty}^{\infty} H\left(\frac{x - t}{\sqrt{\gamma} \phi(x/\gamma)}\right) s(t) \frac{dt}{\sqrt{\gamma} \phi(x/\gamma)} = \lim_{x \to \infty} \frac{1}{\varepsilon \sqrt{\gamma} \phi(x/\gamma)} \sum_{x \leq k < x + \varepsilon} s_k = s.
\]

Because \(\phi(.)\) is self-neglecting and \(\phi(x/\gamma) = O\left(\phi(x)\right)\), for any fixed \(\gamma > 0\), we obtain by Proposition 3, that

\[
\frac{1}{n+1} \sum_{k=0}^{n} (s_k + \varepsilon_k) = s + o\left(\frac{\phi(n)}{n}\right),
\]

which in turn by Lemma 4 implies that \(s_n \to s(N, p, p^{*\beta})\).
4. Proofs.

PROOF OF THEOREM 1. Part (i) by Lemma 5:

\[ s_n \to s(P, p^{\gamma}) \iff s_n \to s(P, p^{(\alpha+\gamma)}) \circ (N, p^{\alpha}, p^{\gamma}). \]

In case (A): We apply Karamata's Tauberian theorem (observe Lemma 1) (see [2, Theorem 1.7.6, 18]) and obtain

\[ s_n \to s(N, 1, p^{\alpha+\gamma}) \circ (N, p^{\alpha}, p^{\gamma}). \]

Since \( s_n = O(1)(N, p^{\alpha}, p^{\gamma}) \) we can use the asymptotic weights and assume w.l.o.g that \( p_n^{(\beta-\alpha)} \) is nondecreasing and by Theorem 3 in Das [8] we get

\[ s_n \to s(N, p^{(\beta-\alpha)}, p^{(\alpha+\gamma)}) \circ (N, p^{\alpha}, p^{\gamma}), \]

which by Lemma 3(i) implies our result.

In case (B): Since \( s_n = O(1)(N, p^{\alpha}, p^{\gamma}) \) we can use Theorem 2 to obtain directly

\[ s_n \to s(N, p^{(\beta-\alpha)}, p^{(\alpha+\gamma)}) \circ (N, p^{\alpha}, p^{\gamma}). \]

The last step is as above.

Part (ii) is directly implied by part (i) and by the Abelian inclusion. •

REMARK 4. Boos/Tietz [4] gave an alternative proof of Theorem 1 in the Borel-case. The basic steps are as follows (\( \alpha = \gamma = 1, \beta = 2 \))

(i) \( s_n \to s(P, p) \Rightarrow s_n \to s(P, p^3)(N, p^{x^2}, p) \)

(ii) \( (N, p^{x^2}, p) = (N, p, p^2)(N, p, p) \). Hence if \( (*) \) \( (N, p, p^2) \to (N, p, p^2) \), one can use the \( O \)-Tauberian theorems in [12, 13] to conclude

(iii) \( s_n \to s(N, p^{x^2}, p) \).

The statement \( (*) \) in (ii) is true for some special cases, like \( p_n = 1/n! \), but has not been obtained in general so far.

PROOF OF PROPOSITION 1. Observe that e.g. in case \( R = \infty \)

\[ \sigma_{p^\alpha}(x) = \frac{\sum_{n=0}^{\infty} s_n^\phi p_n^\alpha p^\alpha x^n}{(p(x))^\beta} = \int_0^\infty \frac{p(\alpha)}{p(x)^\beta} \sigma_{p^\alpha}(x) d\chi(t) = L(\sigma_{p^\alpha}(\cdot), x). \]

The interchange of integral and sum is allowed because of the absolute convergence for \( x > 0 \). We now follow the arguments in an unpublished paper by A. Jakimovski (oral communication, see also [16] for details.)

\( L(f, x) \) is a positive linear operator on a linear space of real functions in \( C[0, \infty) \) with the properties:

(i) There exists \( e(t) > 0, e(t) \to 1, t \to \infty \) such that \( L(e(\cdot), x) \to 1, x \to \infty \), namely \( e(t) = \sigma_{p^\alpha}(t) \) with the sequence \( (s_n) \) chosen to be \( (1, 1, \ldots) \).
(ii) There exists some $e_0(t) > 0$ such that $L(e_0(\cdot), x) \to 0$, $x \to \infty$, namely $e_0(t) = \sigma(p^\alpha(t)) = p_0^\alpha / p(t)^\alpha$, with the sequence $(s_n)$ chosen to be $(1, 0, 0, \ldots)$.

From (i) and the assumptions we find

$$|f(t) - se(t)| < \epsilon / 2 \leq \epsilon e(t), \quad t \geq t_0(\epsilon),$$

and by (ii)

$$|f(t) - se(t)| \leq M \leq M e_0(t), \quad t \in [0, t_0(\epsilon)],$$

with suitable $M, m$. Hence for $t \geq 0$:

$$|f(t) - se(t)| \leq \epsilon e(t) + \frac{M}{m} e_0(t).$$

Since $L$ is linear and positive we obtain that $L(f(\cdot), x) \to s$ if $f(x) \to s$, which yields the desired result.

REFERENCES