AN INTEGRAL FOR CESÀRO SUMMABLE SERIES

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1. Introduction. The P^{k+2} - integral of James [2] is strong enough to integrate a trigonometric series of the form

$$(1.1) \qquad \frac{1}{2} a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \frac{1}{2} a_o + \sum_{n=1}^{\infty} a_n (x)$$

which is summable (C, k) in $[0, 2\pi]$, provided an extra condition holds involving the conjugate series

$$(1.2) \qquad \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) \equiv \sum_{n=1}^{\infty} b_n (x) .$$

Considering series with coefficients $o(n)$, Taylor [5] constructed an integral (the AP-integral) which successfully integrates series of the form (1.1) which are Abel summable provided an extra condition holds involving the Abel means of the conjugate series (1.2). In particular, James' result is ([3], Theorem 6.2) :

THEOREM A. Suppose that the series (1.1) is summable (C, k) to a finite function $f(x)$ for all $x \epsilon [0, 2\pi] - E$, where E is at most countable, and let $f(x) = 0$, $x \epsilon E$. If $A_n^{k-1}(x) = o(n^k)$ for $x \epsilon E$ and $B_n^{k-1}(x) = o(n^k)$ for $x \epsilon [0, 2\pi]$, then $f(x)$, $f(x) \cos px$, $f(x) \sin px$, $p = 1, 2, \ldots,$ are each P^{k+2} - integrable and the coefficients of (1.1) are given in modified Fourier form

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using the P^{k+2} - *integral*, while Taylor's result ([5], Theorem 11) may be stated as follows:

THEOREM B. *Suppose that*

$$f(r, x) = a_o/2 + \sum_{n=1}^{\infty} a_n(x) r^n$$

and $a_n = o(n)$, $b_n = o(n)$. *Let* $\limsup_{r\to 1-} f(r, x)$, $\liminf_{r\to 1-} f(r, x)$ *be finite except at points of an enumerable set* E ; *and let* $f(x) = \lim_{r\to 1-} f(r, x)$ *exist and be finite* p.p. *At points of* E *let*

$$\lim_{r\to 1-} (1-r)\ f(r, x) = 0 .$$

Suppose further that

$$\lim_{r\to 1-} [(1-r) \sum_{n=1}^{\infty} b_n(x) r^n] = 0 ,$$

for all x . *Then* f(x) , f(x) cos px , f(x) sin px , p = 1, 2, . . . *are* (AP)-*integrable and the given series is the* (AP)-*Fourier series of* f(x) .

In his definition of major and minor functions, James [2] used the generalized symmetric derivative $D^n F(x)$ and the concept of n-convexity. The proof of Theorem A involves the construction of major and minor functions from the sum function F(x) of the series obtained by integrating (1.1) formally k + 2 times; and the fact that no derivative $D^{k+2-2r} F(x)$, $1 \leq r \leq (k+1)/2$, has an ordinary discontinuity is used. To obtain his integral, Taylor [5] used known properties of Abel summable trigonometric series with coefficients o(n) and constructed "upper-" and "lower-approximating pairs" from the sum function G(x) of the series obtained by integrating (1.1) formally twice. His theory is stated in terms of ordinary convexity, continuity and approximate continuity of G(x) .

It is the purpose of this paper to use the method of Taylor and properties of (C, k) summable series with coefficients o(n) to construct an integral which will successfully integrate such series. This integral is less general than Taylor's but

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the conditions that must be imposed in Theorem 3.1 are very similar to the conditions that James imposed and reveal to some extent the connection between Theorems A and B .

2. Definition and Notation. All functions considered will be real-valued or extended real-valued. As in the previous section, the notation of Hardy ([1], Section 5.4) will be adopted. For example, if

$$A_n^o(x) = A_n(x) = a_o/2 + \sum_{r=1}^{n} (a_r \cos rx + b_r \sin rx)$$

(2.1) $$\equiv a_o/2 + \sum_{r=1}^{n} a_r(x) ,$$

$$B_n^o(x) = B_n(x) = \sum_{r=1}^{n} (b_r \cos rx - a_r \sin rx) \equiv \sum_{r=1}^{n} b_r(x)$$

then the Cesàro means of order k , $k = 1, 2, \ldots,$ of series (1.1) are defined by

$$\sigma_n^k(x) = \frac{A_n^k(x)}{E_n^k} ,$$

where $A_n^k(x) = \sum_{r=0}^{n} A_r^{k-1}(x)$ and $E_n^k = (n+k)!/n!k!$.

Series (1.1) is said to be summable (C, k) to A at x if $A_n^k(x) / E_n^k \to A$ as $n \to \infty$. Similar statements hold for series (1.2) where

$$B_n^k(x) = \sum_{r=1}^{n} B_r^{k-1}(x) .$$

DEFINITION 2.1. Let $F(x)$ be a Lebesgue integrable function of period 2π defined on $[0, 2\pi]$ with Fourier series

(2.2) $$a_o/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) .$$

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Denote the Cesàro means of order k for series (2.2) at the point x_0 by $\sigma^k(F, n, x_0)$. Let

$$H^k\underline{D}\,F(x_0) = \liminf_{n\to\infty} \left\{\frac{\partial^2}{\partial x^2}[\sigma^k(F, n, x)]\right\}_{x = x_0}$$

$$H^k\bar{D}\,F(x_0) = \limsup_{n\to\infty} \left\{\frac{\partial^2}{\partial x^2}[\sigma^k(F, n, x)]\right\}_{x = x_0}.$$

If $H^k\underline{D}\,F(x_0) = H^k\bar{D}\,F(x_0)$, the common value will be called the kth Cesàro derivative of $F(x)$ at the point $x = x_0$ and will be denoted by $H^kD\,F(x_0)$. If

$$F(x) = \frac{1}{2}Cx^2 + \phi(x)$$

where C is a finite constant and $\phi(x)$ is integrable and 2π - periodic, then $H^kD\,F(x)$, $H^k\underline{D}\,F(x)$, $H^k\bar{D}\,F(x)$ are defined as $C + H^kD\,\phi(x)$, and so on.

It is clear that if the series

$$(2.3) \qquad -\sum_{n=1}^{\infty} n^2 (a_n \cos nx_0 + b_n \sin nx_0)$$

is summable (C, k) to $f(x_0)$, then $H^kD\,F(x_0)$ exists and equals $f(x_0)$. More generally, it is known ([8], p.80 and 353) that if series (2.2) is summable (C, k) to $F(x)$ then

$$(2.4) \qquad \begin{aligned} \underline{D}^2 F(x) &\le \limsup_{r\to 1-} f(r, x) \le \limsup_{n\to\infty} \sigma_n^k(x), \\ \bar{D}^2 F(x) &\ge \liminf_{r\to 1-} f(r, x) \ge \liminf_{n\to\infty} \sigma_n^k(x), \end{aligned}$$

where $\underline{D}^2 F(x)$, $\bar{D}^2 F(x)$ denote the lower and upper Riemann derivatives of order two, and the Abel and Cesàro means are for series (2.3).

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DEFINITION 2.2. *A function* $F(x)$ *which is Lebesgue integrable and* 2π - *periodic will be said to be* H^k - *continuous at the point* x_o *if*

$$\lim_{n\to\infty} \sigma^k(F, n, x_o) = F(x_o) .$$

3. *The* H^k - *integral.* The procedure in this section will be to give a general definition of an integral and then to show that this definition is equivalent to a second definition. It will then be shown that the integral is finite valued.

Let $g(x)$ be defined in $[0, 2\pi]$ and, by 2π - periodicity, for all real x .

DEFINITION 3.1. *The real number* M *and the real-valued function* $F(x)$ *will be called an* H^k - *upper approximating pair* $[M, F(x)]$ *for* $g(x)$ *if*

(i) $\Phi(x) = F(x) - Mx^2/4\pi$ *is* 2π - *periodic;*

(ii) $\Phi(x)$ *is Lebesgue integrable and* H^k - *continuous for all* x ;

(iii) $\Phi(x)$ *is approximately continuous and has the property* R^* (see e.g. [5]);

(iv) $F(-2\pi) = F(2\pi) = 0$;

(v) $\left.\begin{array}{l} H^k\underline{D}\ F(x) \geq g(x) \\ H^k\underline{D}\ F(x) > -\infty \end{array}\right\}$ *except possibly on a countable set* E ;

(vi) $\displaystyle\lim_{n\to\infty} \frac{1}{n} \frac{\partial^2}{\partial x^2} \sigma^k(\Phi, n, x) = 0, \ x \in E$.

An H^k - lower approximating pair $[m, f(x)]$ is defined similarly.

DEFINITION 3.2. *The function* $g(x)$ *will be said to be* H^k - *integrable over* $[0, 2\pi]$ *if*

$$\inf M = \sup m = I ,$$

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where the bounds are taken over the class of all approximating pairs. The notation will be

$$H^k - \int_0^{2\pi} g(x)\,dx = I .$$

LEMMA 3.1. *Given* $\epsilon > 0$ *and* $x_0 \in [0, 2\pi]$, *there exists an* H^k - *upper approximating pair* $[M, F(x)]$ *for the function* $J(x) \equiv 0$ *such that*

(i) $F(x)$ *is continuous for all* x ;

(ii) $H^k \underline{D} F(x) \geq 0$, *for all* x ;

(iii) $H^k \underline{D} F(x_0) = +\infty$,

$$\frac{1}{n} \frac{\partial^2}{\partial x^2} \{\sigma^k (G, n, x)\}_{x=x_0} > L > 0 ,$$

for all sufficiently large n , *where*

$$G(x) = F(x) - Mx^2/4\pi ;$$

(iv) $0 < M < \epsilon$, $|F(x)| < \epsilon$, $-2\pi \leq x \leq 2\pi$.

Proof. Let $C \equiv 2[\sum_{n=1}^{\infty} (1/n^2) + \pi^2 + 1]$. Consider the series

$$\epsilon/C \left[\frac{1}{2} + \sum_{n=1}^{\infty} \cos n(x-x_0)\right] \tag{3.1}$$

and

$$-\epsilon/C \left[\sum_{n=1}^{\infty} \frac{\cos n(x-x_0)}{n^2}\right] + \lambda , \tag{3.2}$$

and denote the sum of (3.2) by $G(x)$, where λ is chosen so that $G(-2\pi) = G(2\pi) = (-\pi^2 \epsilon)/C$. Then $G(x)$ is continuous for all x , $H^k \underline{D}\ G(x) \geq -\frac{\epsilon}{2C}$ for all x , and $H^k \underline{D}\ G(x_0) = +\infty$.

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Considering the series $\Sigma\alpha_n$ where $\alpha_n = 1$, $n = 0, 1, 2, \ldots$, it is clear that

$$A_n^k = \sum_{r=0}^{n} \binom{r+k}{k} > \frac{1}{k!} \sum_{r=1}^{n} r^k ,$$

and the expression on the extreme right hand side reduces to a polynomial in n of degree $k+1$, whose leading coefficient is $1/(k+1)!$. This shows that

$$\frac{1}{n} \frac{\partial^2}{\partial x^2} \{\sigma^k(G, n, x)\}_{x=x_0} > \frac{\epsilon}{C} \left[\frac{\frac{n^{k+1}}{(k+1)!} + C_k n^k + \ldots + C_o}{\frac{n}{k!} \quad (n+k)^k} \right] \longrightarrow \frac{\epsilon}{C(k+1)} > 0 .$$

Let the function F be defined by

$$F(x) = G(x) + Mx^2 / 4\pi ,$$

where $M = \epsilon\pi/C$. Then the pair $[M, F(x)]$ satisfy the conditions of the lemma.

LEMMA 3.2. Suppose $[M, F(x)]$ is an upper approximating pair for $f(x)$ on $[0, 2\pi]$ and let $\epsilon > 0$. Then there exists an upper approximating pair $[M_2, F_2(x)]$ such that

$$0 < M_2 - M < \epsilon , \quad |F_2(x) - F(x)| < \epsilon , \quad -2\pi \leq x \leq 2\pi ,$$

and

$$H^k \underline{D}\, F_2(x) > -\infty, \quad H^k \underline{D}\, F_2(x) \geq f(x) ,$$

for all x.

Proof. Let E be the set of points where either

$$H^k \underline{D}\, F(x) = -\infty, \text{ or } H^k \underline{D}\, F(x) < f(x) .$$

Then E is countable and for $x \in E$

$$\lim_{n\to\infty} \frac{1}{n} \frac{\partial^2}{\partial x^2} \{\sigma^k (\Phi, n, x)\} = 0 .$$

Let the points of E in $[0, 2\pi]$ be enumerated

$$x_1, x_2, \ldots, x_n, \ldots$$

and let $\{\epsilon_i\}$ be a sequence of positive numbers such that

$$\sum_{i=1}^{\infty} \epsilon_i < \epsilon .$$

Let $[M_i, F_i(x)]$ be the upper approximating pairs defined by Lemma 3.1, with ϵ, x_o replaced by ϵ_i, x_i. Write

(3.3) $$G(x) = \sum_{i=1}^{\infty} F_i(x) , \quad N = \sum_{i=1}^{\infty} M_i .$$

Then $G(x)$, as the sum of a uniformly convergent series of continuous functions, is continuous and the function Θ defined by

$$\Theta(x) = G(x) - (N/4\pi)\, x^2$$

is periodic and Lebesgue integrable. Moreover, since

$$\frac{\partial^2}{\partial x^2} \{\sigma^k (G, n, x)\} = \sum_{i=1}^{\infty} \frac{\partial^2}{\partial x^2} \sigma^k (F_i, n, x)$$

and since the Cesàro means of the series of the form $\frac{1}{2} + \Sigma \cos n(x_i - x)$ are non-negative, it follows that

$$\frac{1}{n} \frac{\partial^2}{\partial x^2} \{\sigma^k (\Theta, n, x)\} > L > 0$$

for $x \in E$ and sufficiently large $n = n(x)$, and $H^k \underline{D}\ G(x) \geq 0$ for all x. Now let

$$F_2(x) = F(x) + G(x) , \quad M_2 = M + N .$$

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Then

$$H^k\underline{D}\ F_2(x) \geq H^k\underline{D}\ F(x) + H^k\underline{D}\ G(x)$$

$$\geq f(x)\ ,\ x \notin E\ .$$

But at points of E

$$\frac{1}{n}\ \frac{\partial^2}{\partial x^2}\ \sigma^k(\Phi_2, n, x) =$$

$$\frac{1}{n}\ \frac{\partial^2}{\partial x^2}\ \sigma^k(\Phi, n, x) + \frac{1}{n}\ \frac{\partial^2}{\partial x^2}\ \sigma^k(\Theta, n, x) > L_1 > 0$$

for all sufficiently large n . This implies that

$$H^k\underline{D}\ F_2(x) = +\infty,\ x \in E\ ,$$

and the proof is complete.

LEMMA 3.3. <u>If</u> $[M, F(x)]$ <u>and</u> $[m, f(x)]$ <u>are upper and lower</u> H^k - <u>approximating pairs for a function</u> $g(x)$, <u>then</u> $M \geq m$ <u>and</u> $[F(x) - f(x)]$ <u>is convex for</u> $-2\pi \leq x \leq 2\pi$.

<u>Proof.</u> In view of Lemma 3.2, it may be assumed that the exceptional set E in Definition 3.1 is empty. Then for all x

$$H^k\underline{D}\ [F(x) - f(x)] \geq H^k\underline{D}\ F(x) - H^k\bar{D}\ f(x) \geq 0$$

and

$$H^k\underline{D}\ [\Phi(x) - \phi(x)] = -\frac{1}{2}\pi\ \{M-m\}\ ,$$

which, in the light of (2.4) implies that

$$\bar{D}^2\ [\Phi(x) - \phi(x)] \geq -\frac{1}{2}\pi\ \{M-m\}$$

and

$$\bar{D}^2\ [F(x) - f(x)] = \frac{1}{2}\pi(M-m) + \bar{D}^2\ [\Phi(x) - \phi(x)] \geq 0\ .$$

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Moreover, $\bar{D}^2 [F(x) - f(x)] > -\infty$ for all x. Now since $F(x) - f(x)$ is approximately continuous and has the property R^* it follows ([5], Theorem 1] that $F(x) - f(x)$ is convex. But then $F(0) \leq f(0)$ which implies that $M \geq m$.

The preceding sequence of lemmas shows that the H^k-integral is always finite valued.

THEOREM 3.1. _Suppose that series_ (1.1) _is summable_ (C, k) _to a function_ $f(x)$, _except possibly on a countable set_ E, _and suppose_ $a_n = o(n)$, $b_n = o(n)$. _At points of_ E _let_ $A_n^{k-1}(x) = o(n^k)$ _and let_ $B_n^{k-1}(x) = o(n^k)$ _for all_ x. _Then_

$$a_n = \frac{1}{\pi} H^k - \int_0^{2\pi} f(x) \cos n x \, dx, \quad n \geq 0,$$

$$b_n = \frac{1}{\pi} H^k - \int_0^{2\pi} f(x) \sin n x \, dx, \quad n \geq 1.$$

Proof. The series $-\sum_1^\infty \frac{A_n(x)}{n^2}$ is a Fourier series of a function $G(x) \in L^2(0, 2\pi)$ since the coefficients are $o(\frac{1}{n})$. By Lemma 21 of [6], this series is Abel-summable for all x to a function with the property R. Since the coefficients are $o(\frac{1}{n})$, the series is convergent everywhere to $G(x)$. By Lemma 9 of [7], $G(x)$ is approximately continuous. Let

$$F(x) = \frac{1}{4} a_o x^2 + G(x) + T,$$

where T is a finite constant chosen so that $F(-2\pi) = 0 = F(2\pi)$. Then the pair $[\pi a_o, \Phi(x)]$ forms both an upper and a lower H^k-approximating pair for $f(x)$ on $[0, 2\pi]$. (The condition $A_n^{k-1}(x_o) = o(n^k)$ implies $\frac{A_n^k(x_o)}{E_n^k} = o(n)$). We then have

$$\pi a_o = H^k - \int_0^{2\pi} f(x) \, dx.$$

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Let the given series be multiplied by $\cos px$ to give the trigonometric series

$$(3.4) \qquad \alpha_o + \sum_{n=1}^{\infty} \alpha_n(x)$$

with constant term $\alpha_o = \frac{1}{2} a_p$. Let the functions associated with series (3.4) and its conjugate (i.e., the functions corresponding to $A_n^k(x)$ and $B_n^k(x)$ for series (1.1) and (1.2)) be denoted by $U_n^k(x)$ and $V_n^k(x)$ respectively. By Theorem 2.1 of [3] the product series is summable (C, k) at points of $[0, 2\pi] - E$ to $f(x) \cos px$ and for $x \in [0, 2\pi]$, $V_n^{k-1}(x) = o(n^k)$. It follows from equation 2.11 of [3] that at points of E

$$(3.5) \qquad 2(U_n^{k-1} - A_n^{k-1}) = \sum_{r=n-p+2}^{n+p} c_{n+p-r} A_r^{k-3} + 0(n^{k-2}),$$

where

$$c_n = \begin{cases} n+1 & 0 \le n \le p-1 \\ 2p-n-1 & p \le n \le 2p-2 \end{cases}.$$

Since (3.5) implies that $U_n^{k-1}(x) = o(n^k)$ for $x \in E$, the first part of the proof may be used to prove the second part.

4. <u>The Relationship Between the H^k - integral and Other Integrals.</u> That the H^k - integral is less general than the AP-integral follows from Lemma 3.2 and inequalities 2.4. But it is not possible to show that the Perron integral is less general than the H^k - integral by the obvious method (Cf. [5], page 269), since Fatou's theorem ([8], pp. 99-100) is not known to be true for Cesàro means. The following theorem shows that under suitable additional assumptions a Lebesgue integrable function is H^k - integrable.

THEOREM 4.1. <u>Let</u> $f(x)$ <u>be finite-valued on</u> $[0,2\pi)$ <u>and defined by periodicity elsewhere. If</u> $f(x)$ <u>is Lebesgue</u>

integrable an $[0, 2\pi]$ and if there exists a continuous function $G(x)$ such that $D_2 G(x) = f(x)$ everywhere, then $f(x)$ is H_k-integrable. $k \geq 3$, and

$$L - \int_0^{2\pi} f(t)dt = H^k - \int_0^{2\pi} f(t)dt .$$

Proof. If

$$g(u) = \int_0^u f(t)dt ,$$

and

$$h(x) = \int_0^x g(u)du ,$$

then

$$D_2 h(x) = D_2 G(x) = f(x) \qquad ([4], \text{ p.37}) .$$

Let the Fourier series of $f(x)$ be

(4.1) $$\frac{1}{2} a_o + \Sigma (a_n \cos nx + b_n \sin nx) .$$

Then

(4.2) $$C_o + \frac{a_o x}{2} + \Sigma \left(\frac{a_n \sin nx - b_n \cos nx}{n} \right) = g(x) ,$$

(4.3) $$C_1 + C_o x + \frac{a_o x^2}{4} - \Sigma \frac{a_n \cos nx + b_n \sin nx}{n^2} = h(x) ,$$

where the series on the left hand side in both (4.2) and (4.3) are uniformly convergent. Then $D_2 h(x) = f(x)$ implies that series (4.1) is summable $(C, 3)$ to $f(x)$, i.e. $H_3 D\, h(x) = f(x)$ ([9], p.60). Now defining

$$F(x) \equiv h(x) - C_1 - C_o x ,$$

and

$$\Phi(x) \equiv F(x) - \frac{a_o x^2}{4} ,$$

it is clear that $[\pi a_o , F(x)]$ form both an upper and lower approximating pair and the theorem is proved.

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