Can. J. Math., Vol. XXV, No. 4, 1973, pp. 687–692

GRAPHS WITH 6-WAYS

JOHN L. LEONARD

In a finite graph with no loops nor multiple edges, two points a and b are said to be connected by an *r-way*, or more explicitly, by a line r-way $a - b$ if there are r paths, no two of which have lines in common (although they may share common points), which join a to b. In this note we demonstrate that any graph with n points and $3n - 2$ or more lines must contain a pair of points joined by a 6-way, and that $3n - 2$ is the minimum number of lines which guarantees the presence of a 6-way in a graph of n points. In the language of [**3**], this minimum number of lines needed to guarantee a 6-way is denoted $l_6(n)$. For the background of this problem, the reader is referred to [**3**].

We denote a graph with p points and q lines by $[p, q]$ or $G[p, q]$. If G may have more than q lines, we denote it as $G[p, \geqq q]$. A J-graph is an $[n, 3n - 3]$ which contains no 6-way. For any point a in a graph G, $N(a)$ denotes the subgraph of G induced by the points adjacent to a. Adding a and its incident edges to $N(a)$ gives $N[a]$. A graph $\langle n, r \rangle$ is any graph which results from removing r lines from the complete graph K_n. A path is external to a graph G if it contains no lines in common with G. A point of degree s is called an s-point. We may contract a subgraph or point set P by replacing P with a single point p, which is joined to all points which had neighbors in P. Indiscriminate contradiction may introduce multiple edges in a graph.

We will frequently demonstrate the existence of a 6-way in a graph which was supposedly without one. Of use for this purpose is the following lemma.

LEMMA. *If six line-disjoint paths join a point f to the points of K_5, then there is a 6-way from f to a single point of K_5.*

Proof. There are ten possible ways in which the six paths may be partitioned among the five points of K_5. Each partitioning may be seen to yield a 6-way between f and that point of K_5 upon which the greatest number of the six paths is incident.

In particular, the lemma shows that the operation of contracting a subgraph K_5 to a point cannot introduce a 6-way into a graph which did not previously contain a 6-way.

Figure 1 establishes the existence of J-graphs $[n, 3n - 3]$ for any n. We call graphs of the types illustrated *bi-wheels*. By increasing the number of inner-ring neighbors which each outer-ring point adjoins one can construct a bi-wheel with n points, $[r(n - 1)/2]$ lines, and no r-way, for any r, establishing a lower bound for $l_r(n)$.

Received November 22, 1971 and in revised form, June 19, 1973.

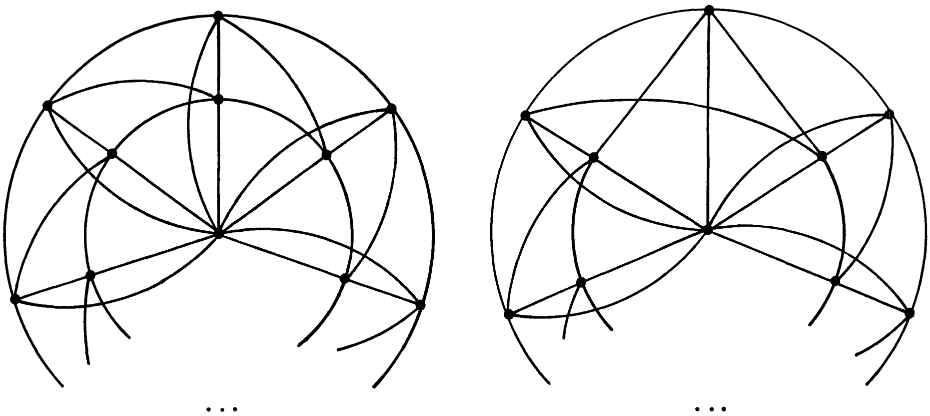

FIGURE 1. Bi-wheel blocks $J[n, 3n-3]$, for n odd and even

That any $[n, 3n-2]$ must contain a 6-way is demonstrated in the following theorem.

THEOREM. (1) *Any* $G[n, 3n-2]$ *contains a* 6-*way.*

(2) *No* $J[n, 3n-3]$ *contains a point of degree less than five.*

(3) *Addition of an external path to a* $J[n, 3n-3]$ *creates a* 6-*way. Furthermore, if the endpoints of the external path lie in the same block of* J, *then there is a* 6-*way between these endpoints.*

Proof. For $n \leqq 5$ all three statements are vacuous, as is (1) for $n = 6$. Since $J[6, 15] = K_6$, statements (2) and (3) hold for $n = 6$. For $n = 7$, $G[7, 19]$ must contain at least two 6-points, which by joining all other points of G make a 6-way, proving (1). If $J[7, 18]$ contains a point of degree less than five, then the degrees of the other six points must sum to at least 32, showing J has two 6-points, and hence a 6-way, contrary to the definition of a J-graph. We demonstrate (3) by noting that Dirac's extension of Turán's Theorem [**1**, Theorem 2] states that $J[7, 18]$ contains as a subgraph a $\langle 6, 2\rangle$, which has either three or two 5-points. Since J contains five lines in addition to the $\langle 6, 2\rangle$, the former case would give J at least two 6-points, which we have seen is impossible. Thus the latter must hold, J having one 6-point and six 5-points, so any two points of J are joined by a 5-way, and any external path makes a 6-way.

We proceed by induction, assuming the theorem valid for all $7 \leqq n \leqq k-1$, and showing that it also holds for $n = k$.

(1) Assume there is a $G[k, 3k-2]$ which has no 6-way. Since $6k/2 > 3k-2$, G must contain a point a of degree less than six. If deg $a \leqq 3$, removing a leaves a $[k-1, \geqq 3(k-1)-2]$, which by (1) of the induction hypothesis contains a 6-way. If deg $a = 4$, removing a leaves a $[k-1, 3(k-1)-3]$. The removed point a and its incident edges form an external path for this graph, and hence by (3) reveal a 6-way in G. If deg $a = 5$ and $N(a) \neq K_5$,

then removing a and adding a line bc missing in $N(a)$ yields $R[k-1, 3(k-1)-3]$ and introduces no 6-way, since the path bac in G plays the role of bc in R. Replacing the line bc with bac gives a graph homeomorphic to R. The other lines incident to a in G provide an external path to this graph, and hence, by (3), a 6-way in G. If $N(a) = K_5$, then $N[a] = K_6$. If $N[a]$ is a block of G, contracting it to a point will give a $[k-5, 3(k-5)-2]$, which for $k < 12$ is nonexistent, and for $k \geqq 12$ contains a 6-way by (1), which must have been present in G. If $N[a]$ is properly contained in a block, then the remainder of the block provides an external path for K_6, and hence a 6-way by (3).

(2) If $J[k, 3k-3]$ consists of two or more blocks, write

$$J = B[k_1, \leqq 3k_1 - 3] \cup C[k_2, \geqq 3k_2 - 3],$$

where B is a block and $k_1 + k_2 - 1 = k$. To avoid a 6-way in J, the equality must hold in both B and C, and since $k_1 < k$, $k_2 < k$, no point of B or C has degree less than five by the induction hypothesis. If $J[k, 3k-3]$ is a block, assume it contains a point a with deg $a < 5$. If deg $a \leqq 3$, removing a leaves $[k-1, \geqq 3(k-1)-3]$, which either contains a 6-way by (1), or gains a 6-way when a is replaced, by (3). If deg $a = 4$ and $N(a) \neq K_4$, then removing a and adding a line bc missing in $N(a)$ introduces no 6-way and yields $R[k-1, 3(k-1)-3]$. Substituting bac for bc gives a graph homeomorphic to R. The other two lines incident to a in J make a path external to R and hence a 6-way. But the graph which we have reconstructed and shown to contain a 6-way is J, a contradiction.

If $N(a) = K_4$, we also reach a contradiction, albeit with more effort. We shall proceed by contracting $N[a] = K_5$ to a point. Denote the set of points in $J - N[a]$ which have more than one neighbor in $N(a)$ by M. If $M \neq \emptyset$ this contraction will introduce multiple lines, and we must then remove enough of these lines to eliminate the multiplicity. If we contract $N[a]$ and have to remove fewer than two lines, we obtain a $[k-4, \geqq 3(k-4)-2]$, which is either nonexistent or contains a 6-way by (1). Since contracting K_5 cannot introduce a 6-way into a graph, this 6-way must have existed in J, a contradiction.

If it is necessary to remove two lines upon contracting, we obtain $R[k-4, 3(k-4)-3]$, and by (3), addition of any external path to R will make a 6-way. Choose any point f in M, and call the point to which $N[a]$ was contracted b. Since the graph induced by $N[a] \cup M$ lies in a block, addition of a path external to R, joining b to f, will form a 6-way $b - f$. The removed line from f to $N[a] = K_5$ is just such an external path, so J contained six paths from f to $N[a]$, and thus by the Lemma a 6-way, *contra hypothesis*.

If the removal of three lines is necessary to eliminate duplicate lines, we first observe that $N(a)$ cannot contain two points d and e both of which have two or more neighbors in M. For assume the contrary. Let us call two points in $N(a)$ *associated* if they share a common neighbor in M. Since each point of M

has at least two neighbors in $N(a)$, each point of M associates two points of $N(a)$. If d and e share two common neighbors in M, there is a 6-way $d - e$. If they share only one common neighbor, then either d and e are both associated with a common point in $N(a)$, making a 6-way $d - e$, or they are associated with distinct points b and c of $N(a)$, and the line bc in $N[a]$ makes a 6-way $d - e$. If d and e share no common neighbors in M, they must each be associated with both other points b and c of $N(a)$, and this makes a 6-way $b - c$. Thus at most one point of $N(a)$ can have more than one neighbor in M, so M contains at most three points. Removal of four lines is therefore never necessary, and only three configurations of J require removal of three lines:

(i) One point e in $N(a)$ joining the three points f, g, and h of M, with the other points b, c, and d of $N(a)$ joining f, g, and h by lines, say bf, cg, dh. We distinguish two possible subcases. If none of the lines fg, fh, or gh are present in J, then we may contract $N[a]$ to a point j and add the line gh, which plays the role of path $gcdh$ in J, yielding the graph $R[k - 4, 3(k - 4) - 3]$, which contains no 6-way. Then addition of an external path $j - f$ to R yields a 6-way, by the induction hypothesis. The removed line from f to $N[a] = K_5$ is just such an external path, so J contained six paths from f to $N[a]$, and thus by the Lemma a 6-way, *contra hypothesis*.

Secondly, suppose at least one of the lines fg, fh, gh, say fh, is contained in J. Then if a sixth line leave b, since J is a block, this line must be in a path which returns to the point set $\{c, d, e, f, g, h\}$, and this path is easily seen to make a 6-way between b and either e or the point of return. Thus deg $b = 5$ and similarly deg $d = 5$. Now remove a, b, and d from J, yielding $F[k - 3, 3k - 14]$, and add the lines hc (for hdc in J) and fc (for fbc), yielding $E[k - 3, 3(k - 3) - 3]$. Addition of a path $e - c$, external to E, will thus yield a 6-way. Add to F the points b, d and lines hd, dc, fb, bc, creating D, a subgraph of J which is homeomorphic to E. Now add to D the point a and lines ea and ac, which creates the external path $e - c$ and hence a 6-way. But this final graph is a subgraph of J, so J contains a 6-way, contrary to its definition, and completing case (i).

(ii) Two points of $N(a)$, say b and c, join a point f of M, and the points c, d, and e of $N(a)$ join a second point g of M. We first show that e, d, and b all have degree five. Suppose a sixth line left e. Since J is a block, this line must be in a path which returns to the set $\{b, c, d, f, g\}$, and this path makes a 6-way from e to either the point of return or to c. The same argument shows that deg $d = 5$. Suppose deg $b > 5$. Then remove a, d, and e from J, leaving $R[k - 3, 3(k - 3) - 5]$. If R contains only a 2-way $b - c$, then Menger's Theorem [**2**, p. 49] implies that removing two lines from R disconnects R into E and F, with b in one component and c in the other. But

$$E \cup F = [k - 3, 3(k - 3) - 7] = E[k_1, \geqq 3k_1 - 3] \cup F[k_2, \leqq 3k_2 - 4].$$

Since both b and c have degrees in R exceeding two, both k_1 and k_2 exceed one. Then if E has more than $3k_1 - 3$ lines, it contains a 6-way by (1). If E con-

https://doi.org/10.4153/CJM-1973-069-x Published online by Cambridge University Press

tains just $3k_1 - 3$ lines, the remainder of J provides paths external to E (such as bac), and thus creates a 6-way, by (3). (For future reference, we call this Mengerian decomposition technique 'Argument M'.) If R contains a 3-way $b - c$, then bdc, bec, and bac make a 6-way in J. Thus deg $b = 5$.

Next note that the line fg is in J, else we can contract $N[a]$, add the line fg (for $fbdg$ in J), producing $[k - 4, 3(k - 4) - 3]$. The remainder of $N[a]$ provides an external path $c - g$, and hence a 6-way in J. Furthermore, deg $g \neq 4$, else we could remove g, add line fd (for fgd) and produce $[k - 1, 3(\mathrm{k} - 1) - 3]$ which contains the 4-point a, contrary to (2). Similarly, deg $g \neq 5$, for suppose g joins a fifth point h. Removing g, adding he (for hge) and fd (for fgd) again makes a $[k - 1, 3(k - 1) - 3]$ containing the 4-point a. We therefore conclude that deg $g \geqq 6$.

We now shift our attention to the degree of c. It must exceed six, else we could remove $N[a]$, leaving $[k - 5, 3(k - 5) - 3]$, for which $N[a]$ provides an external path, and hence a 6-way in J. If deg $c = 7$, denote the seventh neighbor of c by j. The lines gj and fj must be in J, else we could remove $N[a]$ and add the missing line, say gj (for gcj), making a $[k - 5, 3(k - 5) - 3]$ with no 6-way, for which the remainder of $N[a]$ makes an external path $gdbf$, and hence a 6-way in J. But the lines gj and fj make a 6-way $g - c$ as follows: three paths through $N[a]$; one path gjc; another gfc; and finally a path through the sixth neighbor of g, which path must return to $\{b, c, d, e, f, j\}$ either at c, or at f, whence it may reach c via b, or at j, and hence to c via jfb.

Lastly, suppose deg $c > 7$. Remove a, b, d, and e, leaving $R[k - 4, 3(k - 4) - 4]$. If only a 3-way joins g to c in R, then we can again apply Argument M. Removing three lines from R will disconnect R into $E[k_1, \geqq 3k_1 - 3]$ and $F[k_2, \leqq 3k_2 - 4]$, with c and g in distinct components. Since both c and g have degrees in R which exceed three, removing three lines leaves them with degrees of at least one, so both k_1 and k_2 exceed one. If E contains more than $3k_1 - 3$ lines, it contains a 6-way by (1). If E contains exactly $3k_1 - 3$ lines, the remainder of J provides a path external to E, such as cdg, which by (3) makes a 6-way in J. Thus R contains a 4-way $g - c$, and the paths gdc and gec in J make a 6-way. In all, case (ii) is impossible.

(iii) Finally, all four points b, c, d, and e of $N(a)$ may join the only point g of M. The points of $N(a)$ are thus all connected by 5-ways, so a sixth line leading from any one of them can return to $\{b, c, d, e, g\}$ only at g without making a 6-way. Furthermore, if two points of $N(a)$ have such a sixth line, they share a sixth path through g, hence a 6-way. Thus only one point of $N(a)$, call it b, has degree exceeding five. But deg $b > 6$, lest removing $N[a]$ leaves $R[k - 5, 3(k - 5) - 3]$, for which $N[a]$ makes an external path and a 6-way in J. Similarly, deg $g > 5$, lest removing a, c, d, e, and g leave a $[k - 5, 3(k - 5) - 3]$ for which g and its incident edges make a 6-way. Now remove a, c, d, e, leaving $R[k - 4, 3(k - 4) - 4]$. If b and g are joined by only a 2-way in R, removing two lines and using Argument M reveals a 6-way in the original J, completing case (iii) and part (2).

(3) Suppose we have a $J[k, 3k - 3]$ which is a block, and an external path $a - b$ fails to create a 6-way $a - b$. Then there is at most a 4-way joining a to b in J, and removal of four lines will disconnect J into nontrivial components.

Then Argument M establishes the existence of a 6-way in J, which is impossible. Therefore, the external path $a - b$ must create a 6-way $a - b$. If J is not a block, then an external path from a in block A to b in block B will provide a path, external to A, between a and the cutpoint c of A which provides a connection toward B, and thus a 6-way $a - c$.

Acknowledgement. The author wishes to thank the referee for pointing out various obscurities, and for suggesting the possibility that various special cases might be comprehended by a lemma.

References

1. G. A. Dirac, Extensions of the theorems of Turán and Zarankiewicz, *Theory of graphs and its applications, Proceedings of the symposium held in Smolenice, Czechoslovakia, June, 1963* (M. Fiedler, ed., Academic Press, New York, 1964), 127–132.

2. F. Harary, *Graph theory* (Addison-Wesley, Reading, Mass., 1969).

3. John L. Leonard, *On graphs with at most four line-disjoint paths connecting any two vertices*, J. Combinational Theory Ser. B. *13* (1972), 242–250.

University of Arizona,
Tucson, Arizona