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THE SURFACE PAIR CORRELATION FUNCTION FOR STATIONARY BOOLEAN MODELS

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Abstract

The random surface measure S_Ξ of a stationary Boolean model Ξ with grains from the convex ring is considered. A sufficient condition and a necessary condition for the existence of the density of the second-order moment measure of S_Ξ are given and a representation of this density is derived. As applications, the surface pair correlation functions of a Boolean model with spheres and a Boolean model with randomly oriented right circular cylinders in $\mathbb{R}^3$ are determined.

Keywords: Random set; Boolean model; surface measure; second-order moment measure; product density; pair correlation function

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1. Introduction

In connection with the investigation of *random closed sets*, several related random measures are helpful tools. Given a random closed set $\Xi \subset \mathbb{R}^d$, $d \geq 2$, defined on some probability space $[\Omega, \mathcal{F}, \mathrm{P}]$, a well-known example is the random volume measure V_Ξ, where $V_\Xi(B)$ is the volume content of Ξ in the Borel set $B \in \mathcal{B}(\mathbb{R}^d)$. More formally, it is defined by $V_\Xi(\cdot) = \mathcal{H}^d(\cdot \cap \Xi)$, where $\mathcal{H}^k$, $k \geq 0$, is the k-dimensional Hausdorff measure. Assuming henceforth that Ξ has only realizations in the extended convex ring $\mathcal{S}^d$ (Schneider and Weil (2000)), another example, which is of interest here, is the *random surface measure* S_Ξ, where $S_\Xi(B)$ is the surface content of Ξ inside $B \in \mathcal{B}(\mathbb{R}^d)$. A formal definition of S_Ξ can be obtained in the following way.

Let $\mathcal{K}^d$ denote the system of all nonempty compact and convex sets in $\mathbb{R}^d$. Then, for any $K \in \mathcal{K}^d$, the $(d-1)$th curvature measure $C_{d-1}(K, B)$ gives the surface content of K inside $B \in \mathcal{B}(\mathbb{R}^d)$; see Schneider (1993). The measure C_{d-1} can be extended to $\mathcal{S}^d$ additively and nonnegatively; see, e.g. Hug and Last (2000). Therefore, for the random $\mathcal{S}^d$-set Ξ, the random surface measure S_Ξ can be formally defined by

$$S_\Xi(B) = C_{d-1}(\Xi, B), \qquad B \in \mathcal{B}(\mathbb{R}^d),$$

in which case S_Ξ is also the random curvature measure of order $d-1$.

If Ξ is stationary then so is S_Ξ. From a first-order point of view, S_Ξ is then characterized by its intensity, $S_V^{(d)}$, which is also called the *specific surface* or *surface density* of Ξ. For many applications this is not enough: it is desirable also to have knowledge about the second-order behaviour of S_Ξ. For the random volume measure V_Ξ, a corresponding second-order characteristic is the covariance, $\mathrm{P}(x \in \Xi,\ y \in \Xi)$, of Ξ, which has been extensively discussed in the literature (see, e.g. Böhm and Schmidt (2003), and Stoyan *et al.* (1995) for an overview).

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It is well known and easy to prove that the covariance is a density of the second-order moment measure of V_Ξ with respect to $\mathcal{H}^{2d}$.

The second-order characteristic for S_Ξ is more complicated. To be more precise, let $M_S^{(2)}$, defined by

$$M_S^{(2)}(A, B) = \mathrm{E}[S_\Xi(A) S_\Xi(B)], \qquad A, B \in \mathcal{B}(\mathbb{R}^d),$$

be the second-order moment measure of S_Ξ. It is not directly apparent whether or not $M_S^{(2)}$ has a density, $\varrho_S^{(2)}$, the *second-order surface product density*, with respect to $\mathcal{H}^{2d}$, and, if so, how formulae for $\varrho_S^{(2)}$ can be derived for certain models.

The product density $\varrho_S^{(2)}$ has already been frequently used in applied science. It plays a role, e.g. in the physics of porous media (in which context it is usually denoted by F_{ss} and called the *surface–surface correlation function*), where one is interested in bounds for permeability constants (Torquato (2002)). Further applications in physics are given in a recent paper by Arns *et al.* (2005). Therefore, it seems worthwhile, if not necessary, to investigate $\varrho_S^{(2)}$ more theoretically.

Although a general theory has hitherto not been developed, $\varrho_S^{(2)}$ has been exactly determined for some special cases using different approaches. In particular, the paper of Mecke (2001) should be mentioned. There a formula for the second-order surface product density was derived for the stationary Boolean model with smooth grains, using arguments from differential geometry. Using this formula, $\varrho_S^{(2)}$ was given for the respective Boolean models with discs in $\mathbb{R}^2$ and with segments in $\mathbb{R}^2$. Nevertheless, conditions for the existence of $\varrho_S^{(2)}$ were not given. Furthermore, $\varrho_S^{(2)}$ has been determined for the Boolean model with spheres in $\mathbb{R}^3$ (Doi (1976)) as well as approximately for the hard sphere Gibbs model in $\mathbb{R}^3$ (Torquato (1986)). In the case of stationary and isotropic Boolean models where all grains are $(d-1)$-dimensional, i.e. infinitely thin, one can also think of determining $\varrho_S^{(2)}$ as the derivative of the K-function of S_Ξ (see Section 7.2.2 of Stoyan *et al.* (1995)) as in the point process case. Furthermore, there are similarities to the investigation of product densities for random fibre processes; see, e.g. Stoyan *et al.* (1980) for the planar case.

This paper considers the special case of a stationary Boolean model Ξ with grains from the convex ring $\mathcal{R}^d$. This means that Ξ can be expressed in the form

$$\Xi = \bigcup_{n=1}^{\infty} (\xi_n + Z_n),$$

where the (ξ_n, Z_n), $n = 1, 2, \dots$, are given by a stationary, independently marked Poisson point process $\Psi = \sum_{n=1}^{\infty} \delta_{(\xi_n, Z_n)}$ in $\mathbb{R}^d$ with marks (grains) Z_n, $n = 1, 2, \dots$, in $\mathcal{R}^d$, and where it is also assumed that $\Psi(\{(x, K) \colon (x + K) \cap B \neq \varnothing\}) < \infty$ almost surely (a.s.) for all compact sets $B \subset \mathbb{R}^d$. Here the grains Z_n are all independently and identically distributed according to a distribution Q on $\mathcal{R}^d$, which is also called the distribution of the *typical grain* Z.

In this case, denoting by λ the intensity, by $\overline{V}$ the mean volume, and by $\overline{S}$ the mean surface area of the typical grain Z, the specific surface is given by

$$S_V^{(d)} = \lambda \overline{S} \exp(-\lambda \overline{V});$$

see, e.g. Stoyan *et al.* (1995, p. 76).

The present paper presents both a sufficient condition and a necessary condition for the existence of $\varrho_S^{(2)}$ and rigorously proves a representation of $\varrho_S^{(2)}$ using an integral-geometric approach which is relatively easy to use. Nevertheless, in order to apply the general formula

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for $\varrho_S^{(2)}$ to concrete grain distributions, some potentially difficult geometrical calculations have to be carried out. This is demonstrated in two examples in $\mathbb{R}^3$, first for spheres with random radii and then for the more complicated case of randomly oriented right circular cylinders.

In a manner similar to the point process case, it makes sense to consider a normalized version of $\varrho_S^{(2)}$, namely the *surface pair correlation function*, g_S, of S_Ξ:

$$g_S = \frac{\varrho_S^{(2)}}{(S_V^{(d)})^2}.$$

Owing to the standardization, several models can be compared using g_S. Furthermore, for a stationary random set Ξ, we have $g_S(x, y) \to 1$ as $\|x - y\| \to \infty$, and values of $g_S(x, y)$ close to unity can be interpreted by saying that point pairs on the boundary of Ξ with distance vector $x - y$ are uncorrelated.

2. Second-order surface product density

For a set $A \subset \mathbb{R}^d$ and $t \geq 0$, denote by A_t the outer parallel set $A \oplus tB^d$ of A, where B^d is the Euclidean unit ball in $\mathbb{R}^d$ and '$\oplus$' denotes Minkowski addition. Furthermore, $S^{d-1} = \partial B^d$ is the boundary of the unit ball. In analogy to the random surface measure S_Ξ, henceforth $S_K(\cdot) = C_{d-1}(K, \cdot)$ will denote the surface measure for a fixed $K \in \mathcal{R}^d$. In preparation for statements concerning the existence of a density of $M_S^{(2)}$, we first present an auxiliary result.

Lemma 1. *Let $K \in \mathcal{R}^d$. Then*

$$\frac{\partial}{\partial t}\bigg|_{t=0+} \int \mathbf{1}_{K_t}(x) f(x)\,\mathrm{d}x = \int f(z) S_K(\mathrm{d}z)$$

for all continuous functions f on $\mathbb{R}^d$.

Proof. The assertion is an immediate consequence of the local Steiner formula

$$\int \mathbf{1}_{K_t \setminus K}(x) f(x)\,\mathrm{d}x = \frac{1}{d}\sum_{i=0}^{d-1}\binom{d}{i}\int_0^t \int f(z+sb)s^{d-i-1}\mathbf{1}_{\{\delta(K,z,b)>s\}}\Theta_i^+(K, \mathrm{d}(z,b))\,\mathrm{d}s;$$

see Theorem 3.3 of Hug and Last (2000). Here, for a pair (z, b) in the normal bundle, $N(K)$, of K, $\delta(K, z, b)$ is the reach from $z \in \partial K$ in direction z to the exoskeleton of K (see Hug and Last (2000)). Furthermore, $\Theta_i^+(K, \cdot)$ is the nonnegative extension of the ith generalized curvature measure of K defined on $\mathcal{B}(\mathbb{R}^d \times S^{d-1})$. In particular, $S_K(\cdot) = \Theta_{d-1}^+(K, \cdot \times S^{d-1})$; see Theorem 3.9 of Hug and Last (2000). Therefore, for all $t \geq 0$ we have

$$\frac{\partial}{\partial t}\int \mathbf{1}_{K_t}(x) f(x)\,\mathrm{d}x = \frac{1}{d}\sum_{i=0}^{d-1}\binom{d}{i}t^{d-i-1}\int f(z+tb)\mathbf{1}_{\{\delta(K,z,b)>t\}}\Theta_i^+(K, \mathrm{d}(z,b)).$$

Since $K_{\bar{t}}$, $\bar{t} > 0$, is compact and f is continuous, f is also bounded. Furthermore, for a fixed $(z, b) \in N(K)$, we have $\delta(K, z, b) > 0$ and, hence, for all sufficiently small $t < \bar{t}$, we also have $\delta(K, z, b) > t$, which implies that

$$\lim_{t \to 0+} \mathbf{1}_{\{\delta(K,z,b)>t\}} = \mathbf{1}_{\{\delta(K,z,b)>0\}}.$$

The asserted equality now follows by the dominated convergence theorem and the continuity of f.

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We now need some further notation. Denote by $v(x, y) = \mathrm{P}(x \in \Xi^{\mathrm{c}},\ y \in \Xi^{\mathrm{c}})$ the covariance of Ξ^{c}, where $A^{\mathrm{c}} = \mathbb{R}^d \setminus A$ is the set-theoretic complement of any set $A \subseteq \mathbb{R}^d$. Because stationarity of Ξ implies stationarity of S_Ξ, the second-order surface product density $\varrho_S^{(2)}(x, y)$ depends only on the distance vector $y - x$. Therefore, it is completely described by the *reduced* product density $\breve{\varrho}_S^{(2)}(y - x) = \varrho_S^{(2)}(x, y)$ (see Daley and Vere-Jones (2003) for the point process case). If, in addition, Ξ is isotropic, then so is S_Ξ, meaning that $\varrho_S^{(2)}(x, y)$ depends only on the length, $\|y - x\|$, of $y - x$ and is completely described by the *radial part* of the reduced product density, $\tilde{\varrho}_S^{(2)}(\|y - x\|) = \varrho_S^{(2)}(x, y)$. Nevertheless, since for these densities the domain of the argument is unambiguous, henceforth $\varrho_S^{(2)}(q)$ is written for $\breve{\varrho}_S^{(2)}(q)$, $q \in \mathbb{R}^d$, and $\varrho_S^{(2)}(r)$ is written for $\tilde{\varrho}_S^{(2)}(r)$, $r \leq 0$. Similar notation will be used for v and g_S.

Theorem 1. *Let Ξ be a stationary Boolean model with intensity λ, $0 < \lambda < \infty$, and typical grain $Z \in \mathcal{R}^d$ a.s. satisfying $\mathrm{E}[\mathcal{H}^d(Z \oplus tB^d)] < \infty$ for some $t > 0$ and $\mathrm{E}[S_Z^2(\mathbb{R}^d)] < \infty$. Moreover, assume that $(\partial/\partial t)|_{t=0+} \mathrm{E}[S_Z(Z_t - q)]$ exists for almost all $q \in \mathbb{R}^d$. Then the second-order moment measure, $M_S^{(2)}$, of S_Ξ is absolutely continuous with respect to $\mathcal{H}^{2d}$, and for the reduced second-order surface product density we have*

$$\varrho_S^{(2)}(q) = v(q)\left(\lambda^2\,\mathrm{E}[S_Z(Z^{\mathrm{c}} - q)]\,\mathrm{E}[S_Z(Z^{\mathrm{c}} + q)] + \lambda \frac{\partial}{\partial t}\bigg|_{t=0+} \mathrm{E}[S_Z(Z_t - q)]\right) \quad \textit{for almost all } q \in \mathbb{R}^d. \qquad (1)$$

Proof. Substantial simplification of the proof is possible by applying Lemma 4.2 of Heinrich and Molchanov (1999). First, with the definition $H_K(\Gamma, \cdot) = S_K(\cdot)$, $K \in \mathcal{R}^d$ (the meaning of Γ is unimportant here), we must show that the random measure $\eta(\Gamma, \cdot)$, given by

$$\eta(\Gamma, B) = \sum_{n=1}^{\infty} H_{\xi_n + Z_n}\left(\Gamma, B \setminus \bigcup_{m=1,\, m\neq n}^{\infty} (\xi_m + Z_m)\right), \qquad B \in \mathcal{B}(\mathbb{R}^d),$$

is a.s. equal to S_Ξ. Note that each $Z_n \in \mathcal{R}^d$ can be expressed as $Z_n = \bigcup_{i=1}^{k_n} Z_{ni}$, where k_n is a finite number and $Z_{ni} \in \mathcal{K}^d$. By repeatedly using Corollary 3.5 and Theorem 3.9 of Hug and Last (2000), we find that, a.s.,

$$\begin{aligned}
\eta(\Gamma, \cdot) &= \sum_{n=1}^{\infty} S_{\xi_n + \bigcup_{i=1}^{k_n} Z_{ni}}\left(\cdot \setminus \bigcup_{m=1,\, m\neq n}^{\infty} (\xi_m + Z_m)\right) \\
&= \sum_{n=1}^{\infty}\sum_{i=1}^{k_n} S_{\xi_n + Z_{ni}}\left(\left[\cdot \setminus \bigcup_{m=1,\, m\neq n}^{\infty} (\xi_m + Z_m)\right] \setminus \bigcup_{j=1,\, j\neq i}^{k_n} (\xi_n + Z_{nj})\right) \\
&= \sum_{n=1}^{\infty}\sum_{i=1}^{k_n} S_{\xi_n + Z_{ni}}\left(\cdot \setminus \left[\bigcup_{m=1,\, m\neq n}^{\infty} \bigcup_{j=1}^{k_m} (\xi_m + Z_{mj}) \cup \bigcup_{j=1,\, j\neq i}^{k_n} (\xi_n + Z_{nj})\right]\right) \\
&= S_{\bigcup_{n=1}^{\infty}(\xi_n + Z_n)}(\cdot).
\end{aligned}$$

Note that $S_K(\cdot) = S_K(\cdot \cap K)$ and $S_{K+x}(\cdot + x) = S_K(\cdot)$, $x \in \mathbb{R}^d$, and that, by Lemma 3.2.1 of Schneider and Weil (2000), the marked point process $\Psi = \{(\xi_n, Z_n)\}$ is simple. Then, in view of Remark 4.2 of Heinrich and Molchanov (1999), all assumptions necessary for the application of Lemma 4.2 are satisfied. Letting $D \subseteq \mathbb{R}^{2d}$ be a Borel set, we obtain the following

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representation of $M_S^{(2)}$:

$$M_S^{(2)}(D) = \lambda^2 \iiiint\!\!\iint \mathbf{1}_D(x_1+z_1, x_2+z_2)v(x_1+z_1, x_2+z_2) \times (1-\mathbf{1}_{-K_1+z_2}(x_1-x_2))(1-\mathbf{1}_{-K_2+z_1}(x_2-x_1)) \times S_{K_1}(\mathrm{d}z_1)S_{K_2}(\mathrm{d}z_2)Q(\mathrm{d}K_1)Q(\mathrm{d}K_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 + \lambda \iiiint \mathbf{1}_D(x+z_1, x+z_2)v(x+z_1, x+z_2) \times S_K(\mathrm{d}z_1)S_K(\mathrm{d}z_2)Q(\mathrm{d}K)\,\mathrm{d}x. \tag{2}$$

Here, owing to the Slivnyak–Mecke formula (Mecke (1967)), v can be written instead of a Palm quantity. Let $\mu^{(2)}(D)$ be the first summand in (2) and let $\mu^{(1)}(D)$ be the second. By Fubini's theorem it follows, with the change of variables $(x_1, x_2) \mapsto (x_1 - z_1, x_2 - z_2)$, that $\mu^{(2)}(D)$ can be rewritten as

$$\mu^{(2)}(D) = \lambda^2 \iiiint\!\!\iint \mathbf{1}_D(x_1, x_2)v(x_1, x_2)\mathbf{1}_{(K_2+x_2-z_2)^{\mathrm{c}}}(x_1)\mathbf{1}_{(K_1+x_1-z_1)^{\mathrm{c}}}(x_2) \times S_{K_1}(\mathrm{d}z_1)S_{K_2}(\mathrm{d}z_2)Q(\mathrm{d}K_1)Q(\mathrm{d}K_2)\,\mathrm{d}x_1\,\mathrm{d}x_2;$$

hence,

$$\mu^{(2)}(D) = \lambda^2 \int \mathbf{1}_D(x_1, x_2)v(x_1, x_2) \int S_{K_1}((K_1+x_1-x_2)^{\mathrm{c}})Q(\mathrm{d}K_1) \times \int S_{K_2}((K_2+x_2-x_1)^{\mathrm{c}})Q(\mathrm{d}K_2)\,\mathrm{d}(x_1, x_2). \tag{3}$$

In analyzing $\mu^{(1)}$, let $f\colon \mathbb{R}^{2d} \to [0, \infty)$ be continuous. Then the changes of variable $z_1 \mapsto z_1 - x$ and $x \mapsto x - z_2$ and Lemma 1 yield

$$\begin{aligned}
&\int f(z_1, z_2)\mu^{(1)}(\mathrm{d}(z_1, z_2)) \\
&= \lambda \iiiint f(z_1, x)v(z_1, x)S_{K+x-z_2}(\mathrm{d}z_1)S_K(\mathrm{d}z_2)Q(\mathrm{d}K)\,\mathrm{d}x \\
&= \lambda \iiint \frac{\partial}{\partial t}\bigg|_{t=0+} \int \mathbf{1}_{K_t+x-z_2}(y)f(y, x)v(y, x)\mathcal{H}^d(\mathrm{d}y)S_K(\mathrm{d}z_2)Q(\mathrm{d}K)\,\mathrm{d}x \\
&= \lambda \int \frac{\partial}{\partial t}\bigg|_{t=0+} \iiint \mathbf{1}_{K_t+x-y}(z_2)f(y, x)v(y, x)\,\mathrm{d}y\,S_K(\mathrm{d}z_2)Q(\mathrm{d}K)\,\mathrm{d}x \\
&= \lambda \int \frac{\partial}{\partial t}\bigg|_{t=0+} \iint f(y, x)v(y, x)S_K(K_t+x-y)Q(\mathrm{d}K)\,\mathrm{d}y\,\mathrm{d}x \\
&= \lambda \int f(y, x)v(y, x)\frac{\partial}{\partial t}\bigg|_{t=0+} \int S_K(K_t+x-y)Q(\mathrm{d}K)\,\mathrm{d}(y, x),
\end{aligned}$$

where the dominated convergence theorem and Fubini's theorem have been repeatedly applied. This is justified by the assumed existence of the right-hand derivative of $\mathrm{E}[S_Z(Z_t+x-y)]$ at

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$t = 0$. Together with (3), this yields the asserted absolute continuity of $M_S^{(2)}$ with respect to $\mathcal{H}^{2d}$, and the corresponding reduced density (1).

Remark 1. In the case in which the typical grain Z is a.s. from $\mathcal{K}^d$, representation (2) can also be deduced from Theorem 4.2 of Molchanov (1995) if the boundary of the typical grain is smooth enough in the sense that there is only one tangent point in each direction.

The following corollary gives a simpler formula for $\varrho_S^{(2)}$ when Ξ is a stationary and *isotropic* Boolean model. Recall that, for the stationary Boolean model, isotropy is equivalent to the fact that the typical grain is isotropic.

Corollary 1. *Let the assumptions of Theorem 1 be satisfied and assume that the typical grain Z is isotropic. Then, for any vector $q \in \mathbb{R}^d$ of length $r > 0$, we have*

$$\varrho_S^{(2)}(r) = v(r)\left[\lambda^2\,\mathrm{E}[S_Z(Z^c - q)]^2 + \lambda\frac{\partial}{\partial t}\bigg|_{t=0+}\mathrm{E}[S_Z(Z_t - q)]\right] \tag{4}$$

for the radial part of the reduced second-order surface product density.

Remark 2. Because Ξ is a stationary Boolean model, $v(q)$ can be specified more explicitly. It is

$$v(q) = \exp(-2\lambda\overline{V})\exp(\lambda\gamma_Z(q)), \qquad q \in \mathbb{R}^d,$$

where $\gamma_Z(q) = \mathrm{E}[\mathcal{H}^d(Z \cap [Z - q])]$ is the set covariance function. If Ξ is also isotropic then so is Z, and $\gamma_Z(q)$ can be replaced by the isotropized set covariance function $\overline{\gamma}_Z(r)$, $r = \|q\|$; see Stoyan *et al.* (1995, p. 68).

Remark 3. It should be stressed that the existence of the right-hand derivative of $\mathrm{E}[S_Z(Z_t - q)]$ is a condition only on the typical grain Z or the corresponding grain distribution Q, and therefore has no bearing on the associated stationary (unmarked) Poisson point process.

For a more detailed discussion of the term $(\partial/\partial t)|_{t=0+}\,\mathrm{E}[S_Z(Z_t - q)]$ in the case in which Z is a.s. in $\mathcal{K}^d$, the reader is referred to Ballani (2006), where conditions ensuring its existence were given. Assuming, for example, that the typical grain is C^2 and *strictly* convex, it was shown that

$$\frac{\partial}{\partial t}\bigg|_{t=0+}\mathrm{E}[S_Z(Z_t - q)] = \mathrm{E}\left[\int_{\partial Z \cap \partial[Z-q]} |\sin\angle(n(Z, y), n(Z - q, y))|^{-1}\mathcal{H}^{d-2}(\mathrm{d}y)\right]$$

for almost all $q \in \mathbb{R}^d$, where $n(Z, y)$ is the outer normal unit vector at $y \in \partial Z$. This representation is useful if the surface of Z can be easily parametrized (see also Mecke (2001)). Otherwise the direct evaluation of $(\partial/\partial t)|_{t=0+}\,\mathrm{E}[S_Z(Z_t - q)]$ seems to be preferable.

Moreover, in the stationary and isotropic case with a convex typical grain Z *with inner points*, it was proved for dimensions $d = 2$ and $d = 3$ that, under milder conditions, namely when each grain has a *piecewise* C^1 $(d = 2)$ or *piecewise* C^2 $(d = 3)$ boundary,

$$\begin{aligned}\frac{\partial}{\partial t}\bigg|_{t=0+}&\mathrm{E}[S_Z(Z_t - q)] \\ &= \frac{1}{db_d r^{d-1}}\,\mathrm{E}\left[\int_{\partial Z \cap \partial[B^d(z,r)]} |\sin\angle(n(B^d(z,r), y), n(K, y))|^{-1}\mathcal{H}^{d-2}(\mathrm{d}y)S_Z(\mathrm{d}z)\right]\end{aligned}$$

for almost all $q \in \mathbb{R}^d$ with $r = \|q\|$, where $B^d(z, r) = z + rB^d$. Nevertheless, it might be conjectured that, apart from some integrability conditions, isotropy is sufficient and any smoothness conditions are redundant.

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The next theorem gives a necessary condition for the existence of the second-order surface product density $\varrho_S^{(2)}$. Denote by $S_{d-1}(K, \cdot) = \Theta_{d-1}(K, \partial K \times \cdot)$ the *area measure of order* $d-1$ (see Schneider (1993)) of $K \in \mathcal{R}^d$, which is a measure on $\mathcal{B}(S^{d-1})$.

Theorem 2. *Let Ξ be a stationary Boolean model with intensity λ, $0 < \lambda < \infty$, and typical grain $Z \in \mathcal{R}^d$ a.s. satisfying $\mathrm{E}[\mathcal{H}^d(Z \oplus tB^d)] < \infty$ for some $t > 0$ and $\mathrm{E}[S_Z^2(\mathbb{R}^d)] < \infty$. Assume that the second-order moment measure, $M_S^{(2)}$, of S_Ξ is absolutely continuous with respect to $\mathcal{H}^{2d}$. Then the measure $\mathrm{E}[S_{d-1}(Z, \cdot)]$ is diffuse.*

Proof. Assume, to the contrary, that $\mathrm{E}[S_{d-1}(Z, \cdot)]$ has an atom. Without loss of generality, assume that this atom is at $e_d = (0, \dots, 0, 1)^\top \in S^{d-1}$. This means that the boundary of the typical grain a.s. contains a flat piece with normal vector e_d. Let

$$D = \{(a_1 + a_{d+1}, \dots, a_{d-1} + a_{2d-1}, a_d, a_{d+1}, \dots, a_{2d-1}, a_d) \colon (a_1, \dots, a_{2d-1}) \in \mathbb{R}^{2d-1}\}$$

be a hyperplane in $\mathbb{R}^{2d}$. Hence, $\mathcal{H}^{2d}(D) = 0$. Furthermore, consider the measure $\mu^{(1)}$ defined in the proof of Theorem 1. With the change of variable $z_1 \mapsto z_1 + x - z_2$ and the mapping $H \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ given by $(x, y) \mapsto (x - y, y)$, we obtain

$$\begin{aligned}
\mu^{(1)}(D) &= \lambda \iiiint \mathbf{1}_{\{(z_1,x)\in D\}} v(z_1, x) S_{K+x-z_2}(\mathrm{d}z_1) S_K(\mathrm{d}z_2) Q(\mathrm{d}K)\, \mathrm{d}x \\
&= \lambda \iiiint \mathbf{1}_{\{(z_1-z_2+x,x)\in D\}} v(z_1 - z_2 + x, x) S_K(\mathrm{d}z_1) S_K(\mathrm{d}z_2) Q(\mathrm{d}K)\, \mathrm{d}x \\
&= \lambda \iiiint \mathbf{1}_{\{(z_1-z_2,x)\in \mathbb{R}^{d-1}\times\{0\}\times\mathbb{R}^d\}} v(z_1 - z_2) S_K(\mathrm{d}z_1) S_K(\mathrm{d}z_2) Q(\mathrm{d}K)\, \mathrm{d}x.
\end{aligned}$$

Since $\overline{V} = \mathrm{E}[\mathcal{H}^d(Z)] < \infty$, by Remark 2 we have $v(z_1 - z_2) > 0$. Furthermore, for all z_1 and z_2 in the flat pieces orthogonal to e_d, we have $z_1 - z_2 \in \mathbb{R}^{d-1} \times \{0\}$. This implies that $\mu^{(1)}(D) > 0$ and, hence, $\mu^{(1)}$ and $M_S^{(2)}$ are not absolutely continuous with respect to $\mathcal{H}^{2d}$, which contradicts the assumption.

3. Applications

Corollary 1 easily yields the formula for the second-order surface product density of the stationary Boolean model in $\mathbb{R}^3$ with spheres of constant radius $R > 0$ as stated by Doi (1976). In view of (4), for a sphere K the quantities $S_K(K^{\mathrm{c}} - q)$, $(\partial/\partial t)|_{t=0+} S_K(K_t - q)$, and, by Remark 2, $\overline{\gamma}_K(r)$ have to be determined.

The surface area of a sphere K_1 with radius R_1 inside a sphere K_2 with radius $R_2 \geq R_1$ and whose centre is a distance r from that of K_1 is, for $0 < r < R_1 + R_2$,

$$S_{K_1}(K_2) = 2\pi R_1^2 - \pi R_1 \frac{R_1^2 - R_2^2 + r^2}{r}.$$

Therefore,

$$S_K(K^{\mathrm{c}} - q) = 4\pi R^2 - (2\pi R^2 - \pi Rr)\mathbf{1}_{(0,2R)}(r)$$

and, for $0 < r < 2R$,

$$\frac{\partial}{\partial t}\bigg|_{t=0+} S_K(K_t - q) = \frac{\partial}{\partial t}\bigg|_{t=0+} \left(2\pi R^2 - \pi R \frac{R^2 - (R+t)^2 + r^2}{r}\right) = \frac{2\pi R^2}{r}.$$

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Finally,

$$\overline{\gamma}_K(r) = \frac{4\pi}{3}\left(1 - \frac{3r}{4R} + \frac{r^3}{16R^3}\right)\mathbf{1}_{(0,2R)}(r)$$

(see, e.g. Stoyan *et al.* (1995, p. 70)), which leads to

$$\varrho_S^{(2)}(r) = \left[\lambda^2(4\pi R^2 - (2\pi R^2 - \pi Rr)\mathbf{1}_{(0,2R)}(r))^2 + \lambda\frac{2\pi R^2}{r}\mathbf{1}_{(0,2R)}(r)\right]v(r) \tag{5}$$

for the radial part of the reduced second-order surface product density, where

$$v(r) = \exp\left(-\lambda\left(\frac{8\pi}{3}R^3 - \frac{4\pi}{3}\left(1 - \frac{3r}{4R} + \frac{r^3}{16R^3}\right)\mathbf{1}_{(0,2R)}(r)\right)\right).$$

When using spheres of *random radii* ξ with finite third moment, the surface product density $\varrho_S^{(2)}$ is obtained in a similar manner. It is

$$\varrho_S^{(2)}(r) = \left((4\pi\lambda)^2\,\mathrm{E}\left[\xi^2 - \mathbf{1}_{(r/2,\infty)}(\xi)\left(\frac{\xi^2}{2} - \frac{r\xi}{4}\right)\right]^2 + 4\pi\lambda\,\mathrm{E}\left[\mathbf{1}_{(r/2,\infty)}(\xi)\frac{\xi^2}{2r}\right]\right)v(r)$$

for $r > 0$, where

$$v(r) = \exp\left(-\frac{8\pi\lambda}{3}\,\mathrm{E}\left[\xi^3 - \mathbf{1}_{(r/2,\infty)}(\xi)\frac{4\pi}{3}\xi^3\left(1 - \frac{3r}{4\xi} + \frac{r^3}{16\xi^3}\right)\right]\right),$$

which coincides with the formula stated by Torquato (2002, p. 165).

As another application, we use Corollary 1 in the case of a stationary and isotropic Boolean model with randomly oriented, identical right circular cylinders in $\mathbb{R}^3$. Because such cylinders have both planar and curved faces they might serve well for studying the effects of both face types on the surface product density. With this grain type the derivation of $\varrho_S^{(2)}$ is much more complicated than in the case of spherical grains. Let the typical grain Z be a right circular cylinder in $\mathbb{R}^3$ with height h, base radius R, and base normal vector uniformly distributed on S^2. Thus, only the orientation of each cylinder, not its size, is random. Here only the case in which $h \le 2R$ (*short* cylinders) is considered; however, the calculations for $h > 2R$ (*long* cylinders) are very similar.

By (4), $\overline{\gamma}_Z(r)$, $\mathrm{E}[S_Z(Z^c - q)]$, and $(\partial/\partial t)|_{t=0+}\,\mathrm{E}[S_Z(Z_t - q)]$ again have to be considered. Let

$$A(r, R_1, R_2) = R_1^2 \arccos\left(\frac{r^2 + R_1^2 - R_2^2}{2rR_1}\right) + R_2^2 \arccos\left(\frac{r^2 + R_2^2 - R_1^2}{2rR_2}\right) - \frac{1}{2}\sqrt{4r^2R_2^2 - (r^2 + R_2^2 - R_1^2)^2}$$

be the intersection area of two discs with radii R_1 and R_2, $R_1 \le R_2$, and centre-to-centre distance r satisfying $R_2 - R_1 < r < R_1 + R_2$; see, e.g. Stoyan and Stoyan (1994, p. 365).

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Writing

$$q_V(\alpha, r, R, h) = \frac{2}{\pi}(h - r\cos\alpha)A(r\sin\alpha, R, R)\sin\alpha,$$

where $(h - r\cos\alpha)A(r\sin\alpha, R, R)$ is the intersection volume of the cylinders Z and $Z - q(\alpha)$, the isotropized set covariance function is

$$\overline{\gamma}_Z(r) = \begin{cases} \displaystyle\int_0^{\pi/2} q_V(\alpha, r, R, h)\,\mathrm{d}\alpha, & 0 < r \le h, \\ \displaystyle\int_{\arccos h/r}^{\pi/2} q_V(\alpha, r, R, h)\,\mathrm{d}\alpha, & h < r \le 2R, \\ \displaystyle\int_{\arccos h/r}^{\arcsin 2R/r} q_V(\alpha, r, R, h)\,\mathrm{d}\alpha, & 2R < r \le \sqrt{h^2 + 4R^2}, \\ 0, & r > \sqrt{h^2 + 4R^2}; \end{cases}$$

see, e.g. Gille (1987). In a similar manner we obtain

$$\mathrm{E}[S_Z(Z - q)] = \begin{cases} \displaystyle\int_0^{\pi/2} q_S(\alpha, r, R, h)\,\mathrm{d}\alpha, & 0 < r \le h, \\ \displaystyle\int_{\arccos h/r}^{\pi/2} q_S(\alpha, r, R, h)\,\mathrm{d}\alpha, & h < r \le 2R, \\ \displaystyle\int_{\arccos h/r}^{\arcsin 2R/r} q_S(\alpha, r, R, h)\,\mathrm{d}\alpha, & 2R < r \le \sqrt{h^2 + 4R^2}, \\ 0, & r > \sqrt{h^2 + 4R^2}, \end{cases}$$

where

$$q_S(\alpha, r, R, h) = \frac{2}{\pi}((h - r\cos\alpha)l(r\sin\alpha, R, R) + A(r\sin\alpha, R, R))\sin\alpha$$

and

$$l(r, R_1, R_2) = 2R_1 \arccos\left(\frac{r^2 + R_1^2 - R_2^2}{2rR_1}\right)$$

is the length of that part of the boundary of the smaller disc which lies inside the larger disc.

The derivation of $(\partial/\partial t)|_{t=0+}\,\mathrm{E}[S_Z(Z_t - q)]$ is far more elaborate because each of the cases $0 < r \le h$, $h < r \le 2R$, and $2R < r \le \sqrt{h^2 + 4R^2}$ has to be investigated separately. The necessary calculations are given in Appendix A. With the abbreviations

$$I_V(r, R, h, \alpha_1, \alpha_2) = 2\pi R^2 h - \int_{\alpha_1}^{\alpha_2} q_V(\alpha, r, R, h)\,\mathrm{d}\alpha,$$

$$I_S(r, R, h, \alpha_1, \alpha_2) = 2\pi R(R + h) - \int_{\alpha_1}^{\alpha_2} q_S(\alpha, r, R, h)\,\mathrm{d}\alpha,$$

$$I_T(r, R, h, \alpha_1, \alpha_2) = \int_{\alpha_1}^{\alpha_2} \frac{8R\sin\alpha}{\pi}\arccos\left(\frac{r\sin\alpha}{2R}\right) + \frac{8R^2 h}{\pi r\sqrt{4R^2 - r^2\sin^2\alpha}}\,\mathrm{d}\alpha,$$

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the second-order product density of the Boolean model with randomly oriented cylinders is then given by

$$\varrho_S^{(2)}(r) = \exp\left(-\lambda I_V\left(r, R, h, 0, \frac{\pi}{2}\right)\right) \times \left[\lambda^2 I_S\left(r, R, h, 0, \frac{\pi}{2}\right)^2 + \lambda\left(I_T\left(r, R, h, 0, \frac{\pi}{2}\right) + \frac{12R^2}{\pi r}\arccos\left(\frac{r}{2R}\right) - \frac{4R^2}{r} - \frac{\sqrt{4R^2 - r^2}}{\pi}\right)\right] \quad \text{for } 0 < r \leq h,$$

by

$$\varrho_S^{(2)}(r) = \exp\left(-\lambda I_V\left(r, R, h, \arccos\frac{h}{r}, \frac{\pi}{2}\right)\right) \times \left[\lambda^2 I_S\left(r, R, h, \arccos\frac{h}{r}, \frac{\pi}{2}\right)^2 + \lambda\left(I_T\left(r, R, h, \arccos\frac{h}{r}, \frac{\pi}{2}\right) + \frac{12R^2}{\pi r}\arccos\left(\frac{r}{2R}\right) - \frac{\sqrt{4R^2 - r^2}}{\pi} - \frac{4R^2}{\pi r}\arccos\left(\frac{\sqrt{r^2 - h^2}}{2R}\right) - \frac{\sqrt{r^2 - h^2}}{\pi r}\sqrt{4R^2 - (r^2 - h^2)}\right)\right] \quad \text{for } h < r \leq 2R,$$

by

$$\varrho_S^{(2)}(r) = \exp\left(-\lambda I_V\left(r, R, h, \arccos\frac{h}{r}, \arcsin\frac{2R}{r}\right)\right) \times \left[\lambda^2 I_S\left(r, R, h, \arccos\frac{h}{r}, \arcsin\frac{2R}{r}\right)^2 + \lambda\left(I_T\left(r, R, h, \arccos\frac{h}{r}, \arcsin\frac{2R}{r}\right) - \frac{4R^2}{\pi r}\arccos\left(\frac{\sqrt{r^2 - h^2}}{2R}\right) - \frac{\sqrt{r^2 - h^2}}{\pi r}\sqrt{4R^2 - (r^2 - h^2)}\right)\right] \quad \text{for } 2R < r \leq \sqrt{h^2 + 4R^2},$$

and by

$$\varrho_S^{(2)}(r) = \lambda^2 4\pi^2 R^2 (R + h)^2 \exp(-\lambda 2\pi R^2 h) \quad \text{for } r > \sqrt{h^2 + 4R^2}.$$

The integrals in I_V, I_S, and I_T cannot be expressed in closed form. Nevertheless, $\varrho_S^{(2)}(r)$ or $g_S(r)$ can be obtained from some simple numerical computations.

Figure 1 presents typical examples of the surface pair correlation function $g_S(r)$ for different values of h. It shows that there are discontinuities, namely a finite jump at $r = h$ and a pole at $r = 2R$. When the height of the cylinder equals the base diameter then the two discontinuities coincide. For infinitely thin cylinders ($h = 0$), $g_S(r)$ is continuous for $r > 0$. The two discontinuities are consequences of the special geometry of the typical grain. They will vanish if both the height and the base radius are continuously distributed.

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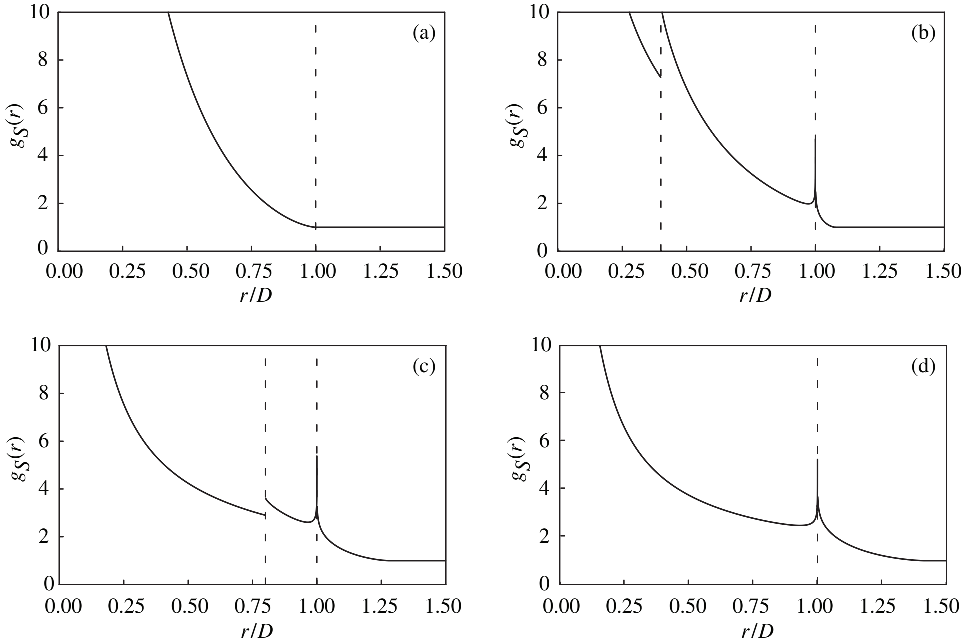

FIGURE 1: The pair correlation function $g_S(r) = \varrho_S^{(2)}(r)/(S_V^{(d)})^2$ for the random surface measure of the stationary Boolean model with uniformly oriented right circular cylinders of height h and base diameter D for the intensity $\lambda = 0.05/D^3$. In (a) $h = 0$ (disc), in (b) $h = 0.4D$, in (c) $h = 0.8D$, and in (d) $h = D$.

However, the pole observable at $r = 0$ does not depend on a particular grain distribution. It is rather an inherent part of any surface pair correlation function and results from the fact that any point on a $(d-1)$-dimensional surface has infinitely many neighbouring points on the surface for any arbitrarily small distance $r > 0$.

Remark 4. Examples suggest that the discontinuities in $\varrho_S^{(2)}(r)$ and $g_S(r)$ smooth out as the dimension increases. For the stationary Boolean model with identical spheres in $\mathbb{R}^3$, the radial part of the reduced surface product density $\varrho_S^{(2)}(r)$ has a finite jump at $r = 2R$, as can be seen from (5). Like all the discontinuities here, it results from the term $(\partial/\partial t)|_{t=0+} S_K(K_t - q)$. Considering a stationary Boolean model with identical spheres in $\mathbb{R}^d$, it was shown in Ballani (2006) that

$$\left.\frac{\partial}{\partial t}\right|_{t=0+} S_K(K_t - q) = \frac{(d-1)b_{d-1}}{2^{d-3}} \frac{(\sqrt{4R^2 - r^2})^{d-3} R^2}{r}.$$

Therefore, for $d = 2$ there is even a pole at $r = 2R$, whereas for $d > 3$ there is no discontinuity.

Appendix A. Further details of the calculation for the cylinder case

Let Z be a right circular cylinder in $\mathbb{R}^3$ with height h, base radius R, and base normal vector uniformly distributed on S^2. We will show how the right-hand derivative of $\mathrm{E}[S_Z(Z_t - q)]$ at $t = 0$ can be evaluated, which completes the discussion of Section 3. Because only short cylinders are considered in this paper (see Section 3), it is henceforth assumed that $h \leq 2R$.

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Let $q = (r, 0, 0)^\top$. Furthermore, let K be a right circular cylinder with height h and base radius R which has its centre at the origin and whose axis coincides with the axis $(0, 0, z)$, $z \in \mathbb{R}$. Then Z can be represented as θK where θ is a random rotation around the origin. Since $S_Z(Z_t - q) = S_{\theta K}(\theta K_t - q) = S_K(K_t - \theta^{-1} q)$ (see Schneider (1993)), the randomness of the cylinder can be transferred to the distance vector. Because of cylinder symmetry it suffices then to consider distance vectors $q(\alpha) = (r \sin\alpha, 0, r\cos\alpha)^\top$ with $\alpha \in [0, \pi/2)$, where α is the polar angle in polar coordinates. The corresponding expectation is then taken with respect to the probability measure $(2/\pi)\sin\alpha\,\mathrm{d}\alpha$.

A.1. The case $0 < r \le h$

Let $0 \le t < (\sqrt{2}/2)r$. Then $\arcsin t/r < \arccos t/r$ and

$$S_K(K_t - q(\alpha)) = \begin{cases} \pi R^2 + 2\pi R(h - r\cos\alpha) + T_{11}(t,\alpha), & 0 < \alpha < \arcsin t/r, \\ A(r\sin\alpha, R, R+t) + l(r\sin\alpha, R, R+t) & \\ \quad \times (h - r\cos\alpha) + T_{12}(t,\alpha), & \arcsin t/r < \alpha < \arccos t/r, \\ A(r\sin\alpha, R, R+t) & \\ \quad + A(r\sin\alpha, R, R + \sqrt{t^2 - r^2\cos^2\alpha}) & \\ \quad + l(r\sin\alpha, R, R+t)(h - r\cos\alpha) + T_{13}(t,\alpha), & \arccos t/r < \alpha < \pi/2, \end{cases}$$

where

$$T_{12}(t,\alpha) = \int_0^t l(r\sin\alpha, R, R + \sqrt{t^2 - s^2})\,\mathrm{d}s$$

and T_{11} and T_{13} are unimportant contributions to $S_K(K_t - q)$ which will vanish after differentiating at $t = 0$. We have $T_{11}(t,\alpha) \le 2\pi Rt$ and $T_{13}(t,\alpha) \le 2\pi Rt$; see Figure 2. This implies that

$$\frac{\partial}{\partial t}\bigg|_{t=0+} \int_0^{\arcsin t/r} T_{11}(t,\alpha)\sin\alpha\,\mathrm{d}\alpha = \frac{\partial}{\partial t}\bigg|_{t=0+} \int_{\arccos t/r}^{\pi/2} T_{13}(t,\alpha)\sin\alpha\,\mathrm{d}\alpha = 0. \tag{6}$$

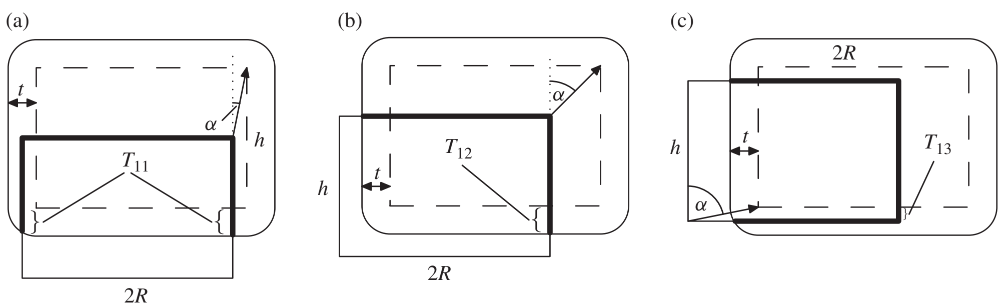

FIGURE 2: Section along the plane through the rotational axes of the cylinder and its translated outer parallel set for the case $0 < r \le h$. The cylinder surface inside the parallel set is indicated by the thick segments. In (a) $0 < \alpha < \arcsin t/r$, in (b) $\arcsin t/r < \alpha < \arccos t/r$, and in (c) $\arccos t/r < \alpha < \pi/2$.

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Since

$$\frac{\partial}{\partial t}\bigg|_{t=0+}\int_{\alpha_1(t)}^{\alpha_2(t)} A(r\sin\alpha,\,R,\,R+t)\sin\alpha\,\mathrm{d}\alpha = \int_{\alpha_1(0)}^{\alpha_2(0)} 2R\arccos\left(\frac{r\sin\alpha}{2R}\right)\sin\alpha\,\mathrm{d}\alpha$$
$$+ A(r\sin\alpha_2(0),\,R,\,R)\sin\alpha_2(0)\frac{\partial}{\partial t}\bigg|_{t=0+}\alpha_2(t)$$
$$- A(r\sin\alpha_1(0),\,R,\,R)\sin\alpha_1(0)\frac{\partial}{\partial t}\bigg|_{t=0+}\alpha_1(t),$$

we have

$$\frac{\partial}{\partial t}\bigg|_{t=0+}\int_{\arcsin t/r}^{\pi/2} A(r\sin\alpha,\,R,\,R+t)\sin\alpha\,\mathrm{d}\alpha = \int_0^{\pi/2} 2R\arccos\left(\frac{r\sin\alpha}{2R}\right)\sin\alpha\,\mathrm{d}\alpha. \quad (7)$$

Similarly, from

$$\frac{\partial}{\partial t}\bigg|_{t=0+}\int_{\alpha_1(t)}^{\alpha_2(t)} l(r\sin\alpha,\,R,\,R+t)(h-r\cos\alpha)\sin\alpha\,\mathrm{d}\alpha$$
$$= \int_{\alpha_1(0)}^{\alpha_2(0)} \frac{4R^2h}{r\sqrt{4R^2-r^2\sin^2\alpha}}\,\mathrm{d}\alpha + \frac{4R^2}{r}\left[\arccos\left(\frac{r\sin\alpha_2(0)}{2R}\right) - \arccos\left(\frac{r\sin\alpha_1(0)}{2R}\right)\right]$$
$$+ l(r\sin\alpha_2(0),\,R,\,R)(h-r\cos\alpha_2)\sin\alpha_2(0)\frac{\partial}{\partial t}\bigg|_{t=0+}\alpha_2(t)$$
$$- l(r\sin\alpha_1(0),\,R,\,R)(h-r\cos\alpha_1)\sin\alpha_1(0)\frac{\partial}{\partial t}\bigg|_{t=0+}\alpha_1(t),$$

we obtain

$$\frac{\partial}{\partial t}\bigg|_{t=0+}\int_{\arcsin t/r}^{\pi/2} l(r\sin\alpha,\,R,\,R+t)(h-r\cos\alpha)\sin\alpha\,\mathrm{d}\alpha$$
$$= \int_0^{\pi/2} \frac{4R^2h}{r\sqrt{4R^2-r^2\sin^2\alpha}}\,\mathrm{d}\alpha - \frac{4R^2}{r}\arcsin\left(\frac{r}{2R}\right). \quad (8)$$

Furthermore,

$$\frac{\partial}{\partial t}\bigg|_{t=0+}\int_{\alpha_1(t)}^{\alpha_2(t)}\int_0^t l(r\sin\alpha,\,R,\,R+\sqrt{t^2-s^2})\,\mathrm{d}s\sin\alpha\,\mathrm{d}\alpha$$
$$= \int_{\alpha_1(0)}^{\alpha_2(0)} 2R\arccos\left(\frac{r\sin\alpha}{2R}\right)\sin\alpha\,\mathrm{d}\alpha.$$

Hence,

$$\frac{\partial}{\partial t}\bigg|_{t=0+}\int_{\arcsin t/r}^{\arccos t/r} T_{12}(t,\alpha)\sin\alpha\,\mathrm{d}\alpha = \int_0^{\pi/2} 2R\arccos\left(\frac{r\sin\alpha}{2R}\right)\sin\alpha\,\mathrm{d}\alpha. \quad (9)$$

Taking into account the relations

$$\frac{\partial}{\partial t}\bigg|_{t=0+}\int_{\arccos t/r}^{\pi/2} A(r\sin\alpha,\,R,\,R+\sqrt{t^2-r^2\cos^2\alpha})\sin\alpha\,\mathrm{d}\alpha = \frac{1}{r}A(r,\,R,\,R)$$

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and

$$\frac{\partial}{\partial t}\bigg|_{t=0+}\int_0^{\arcsin t/r}(\pi R^2+2\pi R(h-r\cos\alpha))\sin\alpha\,\mathrm{d}\alpha=0,$$

from (6), (7), (8), and (9) we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\bigg|_{t=0+}\mathrm{E}[S_Z(Z_t-q)]&=\frac{8R}{\pi}\int_0^{\pi/2}\arccos\Big(\frac{r\sin\alpha}{2R}\Big)\sin\alpha\,\mathrm{d}\alpha+\frac{12R^2}{\pi r}\arccos\Big(\frac{r}{2R}\Big)\\&\quad+\frac{8R^2h}{\pi r}\int_0^{\pi/2}\frac{1}{\sqrt{4R^2-r^2\sin^2\alpha}}\,\mathrm{d}\alpha-\frac{4R^2}{r}-\frac{1}{\pi}\sqrt{4R^2-r^2}.\end{aligned}$$

We can determine $(\partial/\partial t)|_{t=0+}\,\mathrm{E}[S_Z(Z_t-q)]$ in a similar fashion in the next two cases.

A.2. The case $h<r\le 2R$

Let $t<\min\{\sqrt{r^2-h^2},h\}$. Then $\arccos h/r<\arccos t/r$, and for $t<\sqrt{r^2-h^2}$ we have

$$S_K(K_t-q(\alpha))$$

$$=\begin{cases}A(r\sin\alpha,R,R+\sqrt{t^2-(r\cos\alpha-h)^2})\\ \quad+T_{21}(t,\alpha), & \arccos(h+t)/r<\alpha<\arccos h/r,\\ A(r\sin\alpha,R,R+t)+l(r\sin\alpha,R,R+t)\\ \quad\times(h-r\cos\alpha)+T_{22}(t,\alpha), & \arccos h/r<\alpha<\arccos t/r,\\ A(r\sin\alpha,R,R+t)\\ \quad+A(r\sin\alpha,R,R+\sqrt{t^2-r^2\cos^2\alpha})\\ \quad+l(r\sin\alpha,R,R+t)(h-r\cos\alpha)\\ \quad+T_{23}(t,\alpha), & \arccos t/r<\alpha<\pi/2,\\ 0, & \text{otherwise,}\end{cases}$$

where $T_{21}(t,\alpha)\le 2\pi Rt$, $T_{23}(t,\alpha)\le 2\pi Rt$, and $T_{22}(t,\alpha)=T_{12}(t,\alpha)$.

A.3. The case $2R<r\le\sqrt{h^2+4R^2}$

Let $t<\min\{\sqrt{r^2-4R^2},2R\}$. Then $\arccos h/r\le\arcsin 2R/r$ and

$$S_K(K_t-q(\alpha))$$

$$=\begin{cases}A(r\sin\alpha,R,R+\sqrt{t^2-(r\cos\alpha-h)^2})\\ \quad+T_{31}(t,\alpha), & \arccos(h+t)/r<\alpha<\arccos h/r,\\ A(r\sin\alpha,R,R+t)+l(r\sin\alpha,R,R+t)\\ \quad\times(h-r\cos\alpha)+T_{32}(t,\alpha), & \arccos h/r<\alpha<\arcsin 2R/r,\\ A(r\sin\alpha,R,R+t)+l(r\sin\alpha,R,R+t)\\ \quad\times(h-r\cos\alpha)+T_{33}(t,\alpha), & \arcsin 2R/r<\alpha<\arcsin(2R+t)/r,\\ 0, & \text{otherwise,}\end{cases}$$

where $T_{31}(t,\alpha)\le 2\pi Rt$, $T_{33}(t,\alpha)\le 2\pi Rt$, and $T_{32}(t,\alpha)=T_{12}(t,\alpha)$.

A.4. The case $r>\sqrt{h^2+4R^2}$

Let $t<r-\sqrt{h^2+4R^2}$. Then $K\cap[K_t-q]=\varnothing$ always and, hence,

$$\frac{\partial}{\partial t}\bigg|_{t=0+}\mathrm{E}[S_Z(Z_t-q)]=0.$$

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