Forum

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Orthodrome-Loxodrome Correlation by the Middle Latitude Rule

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In this note the Middle Latitude Rule is derived. Namely, to reach Great Circle vertex in two steps such that the total distance is a minimum, an initial rhumb line course equal to the orthodromic course at Middle latitude is to be used. The shortest distance is achieved if the rhumb line course is altered towards the vertex at the orthodrome-loxodrome intersection point.

KEY WORDS

1. Middle Latitude. 2. Loxodrome. 3. Orthodrome.

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1. INTRODUCTION.

The loxodrome is a transcendental curve spiralling between the north pole and the south pole. The orthodrome (Great Circle) is the intersection of the sphere with a plane containing the centre of the sphere. An intrinsic property of these two curves can be found by Napier's Rules of Circular Parts that have been widely used in navigation. The following problem will demonstrate their advantage. Finding the infinitesimal distance dD_O that the vessel must cover in Great Circle sailing for the orthodromic course to change by a small value dD_O illustrates the problem. Referring to Figure 1, from a spherical right angled triangle TVP with any two parts given any third can be found as follows:

$$\cot C_O = \sin D_O \ \tan \varphi_V. \tag{1}$$

Having differentiated the expression (1) with respect to the geographic latitude (φ), as $C_O = C_O(\varphi)$ and $D_O = D_O(\varphi)$ obtains:

$$dC_O = -dD_O \cos D_O \ \tan \varphi_V \ \sin^2 C_O. \tag{2}$$

From the same triangle:

$$\cos D_O = \frac{\sin \varphi}{\sin \varphi_V}, \tag{3}$$

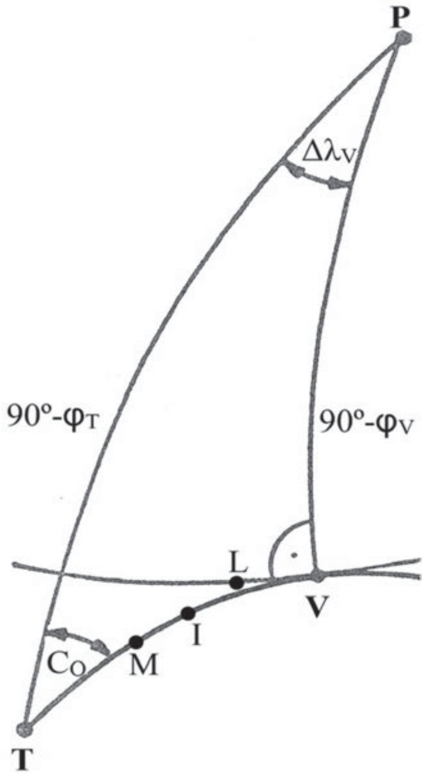

Figure 1. Middle Latitude Rule on a Sphere.

and

$$\sin C_O = \frac{\cos \varphi_V}{\cos \varphi}. \tag{4}$$

Thus, inserting Equations (3) and (4) in Equation (2) it becomes:

$$dC_O = -dD_O \cos \varphi_V \tan \varphi \sec \varphi. \tag{5}$$

When φ equals φ_V, $dC_O = -dD_O \tan \varphi_V$. For $\varphi = 0$, $dC_O = 0$ showing no course change at the intersection with the equator (node point). Finally, transposing Equation (5) gives an answer to the above problem i.e. for the unit course alteration the unit distance is defined:

$$dD_O = -dC_O \sec \varphi_V \cot \varphi \cos \varphi. \tag{6}$$

Since the orthodrome is a curved line whose true direction changes continually (except for a meridian or the equator), a number of points along the Great Circle are selected, connected by loxodromes and followed by rhumb line courses.

2. ANALYSIS.

2.1. *Middle latitude rule for the sphere.* Equation (4) is of a particular interest as it gives an orthodromic course at any latitude ($0 \leqslant \varphi \leqslant \varphi_V$) once vertex latitude is

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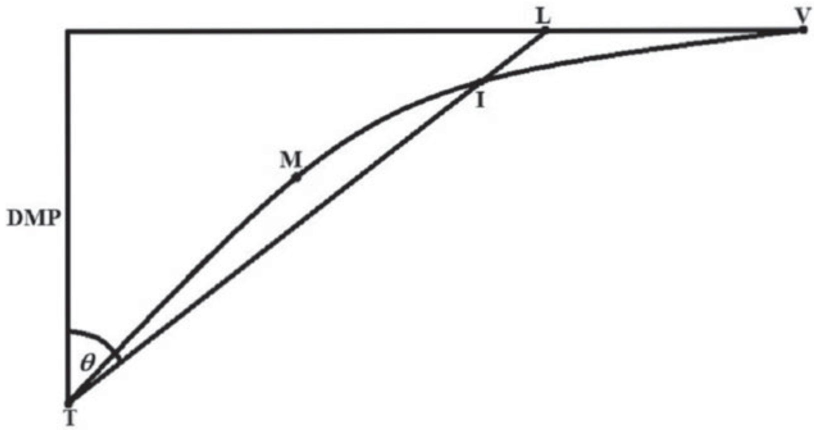

Figure 2. Middle Latitude Rule on a Mercator chart.

known. Introducing *Middle* latitude *(M)* at which the arc length of the parallel is equal to the *departure* in proceeding between two points, shown with the relation:

$$\varphi_{Mid} = \arccos\frac{DLat}{DMP}, \tag{7}$$

Equation (4) gives orthodromic course at *Middle* latitude:

$$C_{O_{Mid}} = \arcsin\frac{\cos\varphi_V}{\cos\varphi_{Mid}}, \tag{8}$$

where:

DLat-Difference of Latitude, and
DMP-Difference of Meridional Parts for the Sphere.

Taking $C_{O_{Mid}}$ as an initial rhumb line course (θ_T) from the point of departure (T) leads to the intersection (I) of a loxodrome with an orthodrome (Great Circle) as shown in Figure 2. As determination of the intersection of the orthodrome and the loxodrome cannot be formulated in a closed form, an iterative solution is to be derived. Transcendental equation:

$$f(\lambda_I) = \sinh[\cot\theta_T \cdot (\lambda_I \sim \lambda_0)] - \tan|\varphi_V| \cdot \cos(\lambda_V \sim \lambda_I), \tag{9}$$

(λ_0 – equatorial intersection longitude of the loxodrome) and its derivatives with respect to λ_I:

$$f'(\lambda_I) = \cosh[\cot\theta_T \cdot (\lambda_I \sim \lambda_0)] \cdot \cot\theta_T - \tan|\varphi_V| \cdot \sin(\lambda_V \sim \lambda_I), \tag{10}$$

$$f''(\lambda_I) = \sinh[\cot\theta_T \cdot (\lambda_I \sim \lambda_0)] \cdot \cot^2\theta_T + \tan|\varphi_V| \cdot \cos(\lambda_V \sim \lambda_I), \tag{11}$$

form a base for an iterative solution of intersection using the *Newton-Raphson* method. If $\varphi_T = 0 \Rightarrow \lambda_T = \lambda_0$. In order to use the above formulae for all quadrants, the difference of longitude sign ($\sim$) represents the shorter arc of the equator between the two meridians. Vertex latitude (φ_V) is taken as an absolute value while the rhumb line course is $\left(0 < \theta_T < \frac{\pi}{2}\right)$. Thus, determining the zero of $f(\lambda_I)$ crossing is obtained. As an initial approximation for the longitude of intersection (λ_I) geographic longitude

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Table 1. Middle Latitude Rule (Sphere).

φ_T/φ_V	λ_0/λ_V	φ_I/λ_I	θ_T/θ_I	D_{TLV}/D_{TIV}	D_{GC}
5°/25°	– 14°17·6′/79°11·2′	22°25·3′/51°25·1′	70·7°/84·2°	4702·7′/4695·1′	4685·9′
15°/35°	– 33°24·8′/67°30·0′	32°11·2′/41°30·7′	65·6°/82·6°	3813·8′/3803·4′	3790·6′
25°/45°	– 46°02·0′/62°12·3′	42°05·5′/36°48·0′	60·7°/81·0°	3226·4′/3213·7′	3197·8′
35°/55°	– 54°22·5′/60°38·4′	52°03·9′/34°34·8′	55·5°/79·3°	2767·0′/2752·1′	2733·4′
45°/65°	– 58°28·0′/62°12·3′	62°05·6′/33°54·3′	49·2°/77·0°	2361·7′/2344·6′	2323·2′

(λ_1) which satisfies the condition $f(\lambda_1)\cdot f''(\lambda_1)>0$ is to be used.

$$\lambda_2 = \lambda_1 - \left[{}^{f(\lambda_1)}\!/_{f'(\lambda_1)}\right]. \tag{12}$$

If $f(\lambda_2)$ using Equation (12) is larger than the allowed error ($E=10^{-6}$), λ_2 is taken as the approximation and the iteration process is repeated until the error falls below E. The iterations converge in a few steps. With current computer capabilities, the procedure can be completed within a fraction of a second.

In the examples presented in Table 1, the Greenwich meridian is taken as a longitude of departure ($\lambda_T=0$), while θ_I represents a rhumb line course connecting intersection point (I) with vertex (V). Distance (D_{TLV}) is based on rhumb line (θ_T) sailing to vertex latitude then due east (or west) along parallel to vertex. The shorter distance (D_{TIV}) is obtained if course is altered to rhumb line (θ_I) at I, then proceeding towards the vertex. Great Circle Distance (D_{GC}) serves as a reference value.

For practical navigation the latitudes of points on the Great Circle track for equal *Difference of Longitude* (*DLo*) intervals from Vertex (usually 5°) are transferred from a Gnomonic to a Mercator chart and connected by rhumb lines. One of these legs cuts the said loxodrome near to the calculated value of I (Figure 2). The smaller is the interval *DLo* from the Vertex, the more accurate intersection point (I) on the Mercator chart is obtained.

2.2. *Middle latitude rule for the spheroid.* By introducing *Difference of Latitude Parts (DLP)* and *Difference of Meridional Parts (DMP)* for the oblate Earth, with a slight modification the method may be used on the rotational ellipsoid (spheroid). Specifically, Equation (7) becomes:

$$\varphi_{Mid} = \arccos\frac{DLP}{DMP}, \tag{13}$$

and consequently modified Equation (8) gives the *geodesic* course at *Middle* latitude:

$$C_{G_{Mid}} = \arcsin\frac{\cos\varphi_V}{\cos\varphi_{Mid}(1-e^2\sin^2\varphi_V)^{1/2}}. \tag{14}$$

As the *geodesic* (line) on the spheroid is defined by differential equations, finding its vertex longitude and intersection with a loxodrome will require a different mathematical model and a more laborious calculation, which falls beyond the scope of this article.

3. CONCLUSION.

The mathematical tools i.e. Napier's Rules of Circular Parts and *Newton-Raphson* method have been well known since the 17th century and are still very relevant. The study gives a few findings worth remembering:

- To reach Great Circle vertex in two steps with minimum distance, the initial rhumb line course (θ_T) must be equal to the orthodromic course at the *Middle* latitude ($C_{O_{Mid}}$).
- The method finds a turn point (I) from where a rhumb line course (θ_I) is followed towards vertex which gives a further reduction in the overall distance as inferred in the paper (Han-Fei et al., 1991).
- The higher the *Middle* latitude the more distance will be saved on the sphere.

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