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A CATEGORICAL QUANTUM TOROIDAL ACTION ON THE HILBERT SCHEMES

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Abstract We categorify the commutation of Nakajima's Heisenberg operators $P_{\pm 1}$ and their infinitely many counterparts in the quantum toroidal algebra $U_{q_1,q_2}(\ddot{gl}_1)$ acting on the Grothendieck groups of Hilbert schemes from [10, 24, 26, 32]. By combining our result with [26], one obtains a geometric categorical $U_{q_1,q_2}(\ddot{gl}_1)$ action on the derived category of Hilbert schemes. Our main technical tool is a detailed geometric study of certain nested Hilbert schemes of triples and quadruples, through the lens of the minimal model program, by showing that these nested Hilbert schemes are either canonical or semidivisorial log terminal singularities.

Key words and phrases: Derived category of coherent sheaves; Quantum group; Hilbert scheme

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1. Introduction

1.1. The description of our result

The quantum toroidal algebra $U_{q_1,q_2}(\ddot{gl}_1)$ (Definition 1.3) is an affinisation of the quantum Heisenberg algebra which has been realised in several contexts:

- the elliptic Hall algebra in [2, 30],
- the double shuffle algebra in [8, 9, 23],
- the trace of the deformed Khovanov Heisenberg category in [4](when $q_1 = q_2$).

Given a smooth quasiprojective surface S over $k = \mathbb{C}$, let

$$\mathcal{M} = \bigsqcup_{n=0}^{\infty} S^{[n]}$$

be the Hilbert schemes of points on S. Schiffmann-Vasserot [32], Feigin-Tsymbaliuk [10] and Neguţ [26] constructed the $U_{q_1,q_2}(\ddot{gl}_1)$ action on the Grothendieck group of $\mathcal{M}$. It generalises the action of

- the Heisenberg algebra (Nakajima [22] and Grojnowski [11])
- the W algebra (Li-Qin-Wang [21])

on the cohomology of Hilbert schemes.

The main result of this paper is a weak categorification of the above quantum toroidal algebra action. First let us state our results precisely. Given a nonnegative integer n, let $S^{[n,n+1]}$ be the nested Hilbert scheme

$$S^{[n,n+1]} := \{(\mathcal{I}_n,\mathcal{I}_{n+1},x) \in S^{[n]} \times S^{[n+1]} \times S | \mathcal{I}_{n+1} \subset \mathcal{I}_n, \mathcal{I}_n/\mathcal{I}_{n+1} = k_x\},$$

which is a closed subscheme of $S^{[n]} \times S^{[n+1]} \times S$. There is a tautological line bundle $\mathcal{L}$ on $S^{[n,n+1]}$, such that its fibre at each closed point is $\mathcal{I}_n/\mathcal{I}_{n+1}$. We abuse the notation to denote $\mathcal{I}$ as the universal ideal sheaf on $\mathcal{M}\times S$. We will write S_1,S_2 for two copies of S, in order to emphasise the factors of $S \times S$. Let $\Delta : \mathcal{M}\times S \to \mathcal{M}\times\mathcal{M}\times S_1\times S_2$ be the diagonal embedding and $\iota : \mathcal{M}\times\mathcal{M}\times S_1\times S_2 \to \mathcal{M}\times\mathcal{M}\times S_2\times S_1$ be the involution map which changes the order of two copies of S. We denote $D^b(X)$ to be the (bounded) derived categories of coherent sheaves on a given scheme X. We prove that

Theorem 1.1 (Theorem 5.1, see Section 1.8 for the notations). *Consider the Fourier-Mukai kernels $e_k, f_k : D^b(\mathcal{M}) \to D^b(\mathcal{M} \times S)$ induced from:*

$$e_k := \mathcal{L}^k \mathcal{O}_{S^{[n,n+1]}} \in D^b(S^{[n]} \times S^{[n+1]} \times S)$$
$$f_k := \mathcal{L}^{k-1} \mathcal{O}_{S^{[n,n+1]}}[1] \in D^b(S^{[n+1]} \times S^{[n]} \times S).$$

(1) *For every two integers m and r, there exists natural transformations*

$$\begin{cases} f_r e_{m-r} \to \iota_* e_{m-r} f_r & \text{if } m > 0 \\ \iota_* e_{m-r} f_r \to f_r e_{m-r} & \text{if } m < 0\,, \\ f_r e_{-r} = \iota_* e_r f_{-r} \oplus \mathcal{O}_\Delta[1] \end{cases} \tag{1.1}$$

where Δ is the diagonal of $\mathcal{M} \times \mathcal{M} \times S \times S$.

(2) *When $m \neq 0$, the cone of the natural transformations in (1.1) has a filtration with associated graded object*

$$\begin{cases} \bigoplus_{k=0}^{m-1} R\Delta_*(h^+_{m,k}) & \text{if } m > 0 \\ \bigoplus_{k=m+1}^{0} R\Delta_*(h^-_{m,k}) & \text{if } m < 0 \end{cases},$$

where $h^+_{m,k}, h^-_{m,k} \in D^b(\mathcal{M} \times S)$ are complexes of wedge and symmetric product of universal sheaves on $\mathcal{M} \times S$.

It is natural to expect that Theorem 1.1 should be compatible with the computation in [24], which we show in Theorem 3.2.

The nontriviality of the extension is a feature of the derived category statement, which is not visible at the level of Grothendieck groups. Proposition 6.3 provides a precise extension formula.

1.2. The weak categorification of an algebra action

Categorification can take place in a very general setting. Roughly speaking, it lifts a certain quantity to a chain complex whose Betti number is the quantity. Here we follow

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the notation of Savage [29] for the naive, weak and strong categorifications of algebra and representations.

Let A be a commutative ring and $B=(\{b_i\}_{i,\in I},\{c_j\}_{j\in J})$ be a unitial associative A-algebra, where $\{b_i\}_{i\in I}$ is the generating set and $\{c_j\}_{j\in J}$ is a set of relations which generate all the relations in B. Let M be a B-module. The action of each b_i defines an endomorphism b_i^M of M and each c_j defines a relation in the endomorphism ring of M which we denote as c_j^M. For a triangulated category $\mathcal{M}$, we denote $K_0(\mathcal{M})$ as the Grothendieck group of $\mathcal{M}$. For any $F\in\mathcal{M}$, we denote $[F]\in K_0(\mathcal{M})$ as the class generated by F.

Definition 1.2 (Weak categorification). A weak categorification of $(B,\{b_i\},\{c_i\},M)$ is a quadruple $(\mathcal{M},\phi,\{F_i\}_{i\in I},\{E_i\}_{j\in J})$, where

(1) $\mathcal{M}$ is a triangulated category with an isomorphism $\phi: K_0(\mathcal{M})\to M$;

(2) for each $i\in I$, $F_i:\mathcal{M}\to\mathcal{M}$ is an triangulated endofunctor of $\mathcal{M}$, such that $[F_i]=b_i^M$ under the isomorphism ϕ;

(3) for each $j\in J$, $E_j=\{E_j^1\xrightarrow{e_j^1}E_j^2\xrightarrow{e_j^2}E_j^3\}$, where E_j^1,E_j^2,E_j^3 are endofunctors of $\mathcal{M}$ which are generated by F_i, e_j^1 and e_j^2 are natural transformations, such that for each element $K\in\mathcal{M}$, $E_j^1(K)\xrightarrow{e_j^1(K)}E_j^2(K)\xrightarrow{e_j^3(K)}E_j^3(K)$ is an exact triangle in $\mathcal{M}$ and the relation $c_j^M=\{[E_j^1]-[E_j^2]+[E_j^3]=0\}$ under the isomorphism ϕ.

Now we recall an integral version of the quantum toroidal algebra $U_{q_1,q_2}(\ddot{gl}_1)$.

Definition 1.3 ([26]). Given two formal parameters q_1 and q_2, let $q=q_1q_2$. Let $\mathbb{K}=\mathbb{Z}[q_1+q_2,q,q^{-1}]$. The quantum toroidal algebra $U_{q_1,q_2}(\ddot{gl}_1)$ is the $\mathbb{K}$-algebra with generators:

$$\{E_k,F_k,H_l^{\pm}\}_{k\in\mathbb{Z},l\in\mathbb{N}}$$

modulo the following relations:

$$(z-wq_1)(z-wq_2)(z-\frac{w}{q})E(z)E(w)= \\ =(z-\frac{w}{q_1})(z-\frac{w}{q_2})(z-wq)E(w)E(z) \tag{1.2}$$

$$(z-wq_1)(z-wq_2)(z-\frac{w}{q})E(z)H^{\pm}(w)= \\ =(z-\frac{w}{q_1})(z-\frac{w}{q_2})(z-wq)H^{\pm}(w)E(z) \tag{1.3}$$

$$[[E_{k+1},E_{k-1}],E_k]=0 \quad \forall k\in\mathbb{Z} \tag{1.4}$$

together with the opposite relations for $F(z)$ instead of $E(z)$, as well as:

$$[E(z),F(w)]=\delta(\frac{z}{w})(1-q_1)(1-q_2)(\frac{H^+(z)-H^-(w)}{1-q}), \tag{1.5}$$

where

$$E(z)=\sum_{k\in\mathbb{Z}}\frac{E_k}{z^k},\quad F(z)=\sum_{k\in\mathbb{Z}}\frac{F_k}{z^k},\quad H^{\pm}(z)=\sum_{l\in\mathbb{N}\cup\{0\}}\frac{H_l^{\pm}}{z^{\pm l}}, \tag{1.6}$$

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where

$$\delta(z)=\sum_{n\in\mathbb{Z}} z^n \in \mathbb{Q}\{\{z\}\}.$$

We will set $H_0^+=q$ and $H_0^-=1$.

The weak categorification of relations (1.2), (1.3), (1.4) was obtained in the main theorem of [26]. Hence, we still need to categorify (1.5) to obtain a weak categorification of the quantum toroidal algebra action, which is the purpose of Theorem 1.1.

1.3. Outline of the proof

Let us first review the categorification of the commutation of e_k and e_l for different k and l in [26]. Consider the moduli spaces $\mathfrak{Z}_2$,$\mathfrak{Z}_2'$ which parameterise diagrams

$$\mathcal{I}_{n-1} \overset{x}{\hookrightarrow} \mathcal{I}_n \overset{y}{\hookrightarrow} \mathcal{I}_{n+1} \tag{1.7}$$

$$\mathcal{I}_{n-1} \overset{y}{\hookrightarrow} \mathcal{I}_n' \overset{x}{\hookrightarrow} \mathcal{I}_{n+1}, \tag{1.8}$$

respectively, of ideal sheaves, where each successive inclusion is colength 1 and supported at the point indicated on the diagrams. Then e_ke_l and e_le_k are the derived pushfoward of line bundles on $\mathfrak{Z}_2$ and $\mathfrak{Z}_2'$ to $S^{[n-1]}\times S^{[n+1]}\times S\times S$, respectively. In order to compare e_ke_l and e_le_k, [26] introduced the quadruple moduli space $\mathfrak{Y}$ which parameterises the diagram

$$\begin{array}{ccccc} & & \mathcal{I}_n & & \\ & {\scriptstyle x}\nearrow & & \searrow{\scriptstyle y} & \\ \mathcal{I}_{n+1} & & & & \mathcal{I}_{n-1} \\ & {\scriptstyle y}\searrow & & \nearrow{\scriptstyle x} & \\ & & \mathcal{I}_n' & & \end{array} \tag{1.9}$$

of ideal sheaves, where each successive inclusion is colength 1 and supported at the point indicated on the diagrams. $\mathfrak{Y}$ is smooth and induces resolutions of $\mathfrak{Z}_2$ and $\mathfrak{Z}_2'$. Proposition 2.30 of [26] proved that $\mathfrak{Z}_2$ and $\mathfrak{Z}_2'$ are rational singularities, based on the fact that any fibre of the resolution has dimension ≤ 1. Thus, e_ke_l and e_le_k could be compared through line bundles on $\mathfrak{Y}$.

Now in order to compare f_re_{m-r} and $e_{m-r}f_r$, we introduce the triple moduli spaces $\mathfrak{Z}_+$,$\mathfrak{Z}_-$ which parameterise diagrams

$$\begin{array}{ccc} & & \mathcal{I}_n \\ & {\scriptstyle x}\nearrow & \\ \mathcal{I}_{n+1} & & \\ & {\scriptstyle y}\searrow & \\ & & \mathcal{I}_n' \end{array} \tag{1.10}$$

$$\begin{array}{ccc} \mathcal{I}_n & & \\ & \overset{y}{\searrow} & \\ & & \mathcal{I}_{n-1} \\ & \underset{x}{\nearrow} & \\ \mathcal{I}'_n & & \end{array} \tag{1.11}$$

Then $f_r e_{m-r}$ and $e_{m-r} f_r$ are the derived pushforward of line bundles on $\mathfrak{Z}_+$,$\mathfrak{Z}_-$, respectively. The quadruple moduli space $\mathfrak{Y}$ still induces resolutions of $\mathfrak{Z}_-$, but $\mathfrak{Z}_+$ have two irreducible components. One irreducible component is $S^{[n,n+1]}$, denoted by W_1, and the other irreducible component is denoted by W_0 in Section 6. $\mathfrak{Y}$ induces a resolution of W_0.

In order to compare $f_r e_{m-r}$ and $e_{m-r} f_r$ through line bundles on $\mathfrak{Y}$, one must prove that $\mathfrak{Z}_-$ and W_0 are rational singularities. The approach in [26] did not work here, as the fibre could have pretty large dimensions. Instead, we study the singularity structure of $\mathfrak{Z}_+$ and $\mathfrak{Z}_-$ through the viewpoint of the minimal model program (MMP) [17, 18]. We prove that

Proposition 1.4 (Propositions 4.6 and 4.13). *The pair* $(\mathfrak{Z}_+,0)$ *is semi-dlt.* $\mathcal{Z}_-$ *and* W_0 *are canonical singularities.*

We prove Proposition 1.4 by explicitly computing the discrepancy (see Section 6 for the definitions of semi-dlt, canonical singularities and the discrepancy). Canonical singularities are always rational singularities by Theorem B.7.

Remark 1.5. One should notice the elliptic Hall algebra of [2] contains more relations than Definition 1.3, which we would investigate in the future. The main obstacle of generalising our result to the elliptic Hall algebra is that for the action of other operators, the corresponding nested moduli space is not Cohen-Macaulay, and, hence, the enhancement in derived algebraic geometry has to be considered. It is also the obstacle of generalising our result to the quantum toroidal algebra action on the Grothendieck group of higher rank stable sheaves [26], as $\mathfrak{Z}_+$ is no longer equidimensional in this situation.

1.4. Categorical Heisenberg actions

Khovanov [16] defined the Heisenberg category through graphical calculus. Cautis-Licata [5] constructed a categorical Heisenberg action on the derived category of Hilbert schemes of points of the minimal resolution of the type ADE singularities. Krug [20] constructed the weak categorical Heisenberg action on the derived category of Hilbert schemes of points on smooth surfaces. Our categorification is different from those above, as it is given in terms of explicit correspondences and independent of the derived McKay correspondence.

Although the higher Nakajima operators were categorified by the objects $e_{(0,\ldots,0)}$ of [26], the relations between them (as well as the morphisms between them in Khovanov's Heisenberg category) are still unclear to us.

1.5. Double categorified Hall algebra

The study of Cohomological Hall algebra was initiated by Kontsevich-Soibelman [19] and Schiffmann-Vasserot [31]. Kapranov-Vasserot [15] and the author [37] constructed the K-theoretic Hall algebra on surfaces, which was categorified by Porta-Sala [27]. It also categorified the positive half of $U_q(\ddot{gl}_1)$ when $S = \mathbb{A}^2$. The relation between the categorified Hall algebra of minimal resolution of type A singularities and quivers was studied by Diaconescu-Porta-Sala [6]. On the other hand, the Drinfeld double of the categorified Hall algebra is still mysterious. As an attempt to understand the action of the 'double Categorified Hall algebra', it is natural to expect that our approach could be generalised to categorifications in other settings, like those of Toda [36] and Rapcak-Soibelman-Yang-Zhao [28].

1.6. Other related work

Recently, Addington-Takahashi [35] studied certain sequences of moduli spaces of sheaves on $K3$ surfaces and showed that these sequences can be given the structure of a geometric categorical $\mathfrak{sl}_2$ action in the sense of [3]. It would be interesting to explore the interactions between their action and ours.

Another related work is Jiang-Leung's projectivisation formula [14]. Through this formula, they obtained a semiorthogonal decomposition of the derived category of the nested Hilbert schemes.

1.7. The organisation of the paper

The proof of the main theorem is in Section 5 and the extension formula is in Section 6. The other sections are organised as follows:

- **Section 2:** we review the action of $U_{q_1,q_2}(\ddot{gl}_1)$ on the Grothendieck group of Hilbert schemes [10, 24, 26, 32];
- **Section 3:** we define $h^+_{m,k} \in D^b(\mathcal{M} \times S)$ and prove the third part of Theorem 1.1;
- **Section 4:** we study the singularity structures of $\mathfrak{Z}_-$ and $\mathfrak{Z}_+$ through the singularity theory of the minimal model program.

1.8. Notations

In this paper, we will always work over $k = \mathbb{C}$.

1.8.1. Derived categories and the Grothendieck groups. For any scheme X, we denote $D_{qcoh}(X)$ as the derived category of quasicoherent sheaves on X. We denote $D^u(X)$ as the full subcategory of $D_{qcoh}(X)$ which consists of elements, such that all the cohomologies are coherent sheaves on X. We denote $D^b(X)$ as the full subcategory of $D^u(X)$, such that the cohomologies are bounded. We denote $K(X) := K_0(D^b(X))$.

1.8.2. Fourier-Mukai transforms associated to a surface. We will write S_1,S_2 for two copies of S, in order to emphasise the factors of $S \times S$ and write $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ for three copies of $\mathcal{M}$, in order to emphasise the factors of $\mathcal{M} \times \mathcal{M} \times \mathcal{M}$. We define elements P in $D^b(\mathcal{M}_1 \times \mathcal{M}_2 \times S)$ to be the Fourier-Mukai kernels associated to S. Given

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$P \in D^b(\mathcal{M}_1 \times \mathcal{M}_2 \times S_1)$ and $Q \in D^b(\mathcal{M}_2 \times \mathcal{M}_3 \times S_2)$, we define the composition $QP \in D^b(\mathcal{M}_1 \times \mathcal{M}_3 \times S_1 \times S_2)$ by

$$QP := \mathbf{R}\pi_{13_*}(\mathbf{L}\pi_{12}^* P \otimes \mathbf{L}\pi_{23}^* Q),$$

where π_{12},π_{23} and π_{13} are the projections from $\mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_3 \times S_1 \times S_2$ to $\mathcal{M}_1 \times \mathcal{M}_2 \times S_1$, $\mathcal{M}_2 \times \mathcal{M}_3 \times S_2$ and $\mathcal{M}_1 \times \mathcal{M}_3 \times S_1 \times S_2$, respectively.

1.8.3. Complexes. In this paper, we adapt the cohomological degree for complexes, that is the degree of a complex is always increasing. For any complex

$$\{\cdots \to C_{-1} \to C_0 \to 0\},$$

we will assume that C_0 has cohomological degree 0 unless explicitly pointing out the cohomological degree. Given a two term complex of locally free sheaves $U := \{W \xrightarrow{s} V\}$, we denote the symmetric product and the wedge product complexes:

$$S^k(U) := \{\wedge^k W \cdots \to \cdots \to W \otimes S^{k-1}V \to S^k(V)\}$$
$$\wedge^k(U) := \{S^k W \to \cdots \to \wedge^{k-1}(V) \otimes W \to \wedge^k(V)\}$$

and $S^k(U) = \wedge^k(U) = 0$ when $k < 0$. At the level of Grothendieck groups, we have

$$[\wedge^k(U)] := \sum_{i=0}^{k} (-1)^i [S^i W][\wedge^{k-i} V] \quad [S^k(U)] := \sum_{i=0}^{k} (-1)^i [\wedge^i W][S^{k-i} V].$$

We define $det(U) := \frac{det(V)}{det(W)}$ and $U^\vee$ as the two term complex

$$\{V^\vee \xrightarrow{\mathrm{u}^\vee} W^\vee\}.$$

Given complexes $\{C_i | i \in \mathbb{Z}\}$ with morphisms $d_i : C_i \to C_{i+1}$, such that $d_i \circ d_{i+1} = 0$, we will write

$$\{\cdots \to C_{i+1} \to C_i \to \cdots\}$$

for the total complex of the double complex $C_\bullet$. Given a complex $C_\bullet$ and an integer k, we denote the complex $C_\bullet[k]$, such that the degree n term is C_{n+k}.

2. The quantum toroidal algebra $U_{q_1,q_2}(\ddot{gl}_1)$ and the K-theory of Hilbert scheme of points on surfaces

In this section, we will review the action of $U_{q_1,q_2}(\ddot{gl}_1)$ on the K-theory on Hilbert scheme of points on surfaces from [10, 24, 26, 32]. The main theorem will be formulated in Theorem 2.6.

2.1. Hilbert and nested Hilbert schemes

Given an integer $n > 0$ and a smooth quasiprojective surface S over $k = \mathbb{C}$, let

$$S^{[n]} := \{\mathcal{I}_n \subset \mathcal{O} | \mathcal{O}/\mathcal{I}_n \text{ is dimension 0 and length } n\}$$

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be the Hilbert scheme of n points on S. There is a universal ideal sheaf on $S^{[n]}\times S$, which we still denote as $\mathcal{I}_n$. Let $\mathcal{Z}_n\subset S^{[n]}\times S$ be the closed scheme of $S^{[n]}\times S$ with the ideal sheaf $\mathcal{I}_n$. We define the Hilbert schemes of points on S as

$$\mathcal{M}:=\bigsqcup_{n=0}^{\infty} S^{[n]}.$$

Proposition 2.1 (Proposition 2.11 of [26]). *There exists a resolution of $\mathcal{I}_n$ by*

$$0\to W_n\xrightarrow{s} V_n\to\mathcal{I}_n\to 0, \tag{2.1}$$

where W_n and V_n are locally free coherent sheaves with the same determinant. Let w_n and v_n be the rank of W_n and V_n, respectively. Then $v_n-w_n=1$.

Definition 2.2. The nested Hilbert scheme $S^{[n,n+1]}$ is defined to be

$$S^{[n,n+1]}:=\{(\mathcal{I}_n,\mathcal{I}_{n+1},x)\in S^{[n]}\times S^{[n+1]}\times S|\mathcal{I}_{n+1}\subset\mathcal{I}_n,\mathcal{I}_n/\mathcal{I}_{n+1}=k_x\}$$

with natural projection maps

$$\begin{array}{ccccc} & & S^{[n,n+1]} & & \\ & \swarrow^{p_n^+} & \downarrow \pi_n & \searrow^{p_n^-} & \\ S^{[n]} & & S & & S^{[n+1]} \end{array} \tag{2.2}$$

$$\begin{array}{ccccc} & & (\mathcal{I}_n,\mathcal{I}_{n+1},x) & & \\ & \swarrow^{p_n^+} & \downarrow \pi_n & \searrow^{p_n^-} & \\ \mathcal{I}_n & & x & & \mathcal{I}_{n+1} \end{array} \tag{2.3}$$

and let

$$p_n:=(p_n^+,\pi_n):S^{[n,n+1]}\to S^{[n]}\times S.$$

We abuse the notation to denote $\mathcal{I}_n$ and $\mathcal{I}_{n+1}$ as the universal sheaf on $S^{[n,n+1]}\times S$. Then $\mathcal{I}_{n+1}\subset\mathcal{I}_n$ and the pushforward of $\mathcal{I}_n/\mathcal{I}_{n+1}$ to $S^{[n,n+1]}$ is a line bundle, which we denote as $\mathcal{L}$. The fibre of $\mathcal{L}$ at each closed point $(\mathcal{I}_{n+1}\subset\mathcal{I}_n)$ is $\mathcal{I}_n/\mathcal{I}_{n+1}$.

2.2. The quantum toroidal algebra action on the K-theory of Hilbert schemes

Definition 2.3 (Definitions 4.10 and 4.11 of [26]). Let $\Delta_S:S\to S\times S$ be the diagonal embedding. For any group homomorphisms $x,y:K(\mathcal{M})\to K(\mathcal{M}\times S)$, we define:

$$xy|_{\Delta_S}=\{K(\mathcal{M})\xrightarrow{y}K(\mathcal{M}\times S)\xrightarrow{x\times Id_S}K(\mathcal{M}\times S\times S)\xrightarrow{Id_{\mathcal{M}}\times\Delta_S^*}K(\mathcal{M}\times S)\}$$

$$\begin{aligned}[x,y]=&\{K(\mathcal{M})\xrightarrow{y}K(\mathcal{M}\times S_2)\xrightarrow{x\times Id_{S_2}}K(\mathcal{M}\times S_1\times S_2)\}\\ &-\{K(\mathcal{M})\xrightarrow{x}K(\mathcal{M}\times S_1)\xrightarrow{y\times Id_{S_1}}K(\mathcal{M}\times S_1\times S_2)\}.\end{aligned}$$

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We define

$$[x,y]_{red} = z$$

for a group homomorphism $z : K(\mathcal{M}) \to K(\mathcal{M} \times \mathcal{S})$ if

$$[x,y] = \Delta_{S*}(z).$$

The definition is unambiguous, since $\Delta_{S*} : K(S) \to K(S \times S)$ is injective, and so z is unique.

Definition 2.4. Let ω_S be the canonical line bundle of S. We define $h_0^+ := [\omega_S]$, $h_0^- := 1$, and when $m > 0$

$$h_m^+ := (1-\omega_S)\sum_{j=0}^{m-1}[\omega_S^{-j}]\sum_{i=0}^{j}(-1)^i[S^{m-i}\mathcal{I}_n][\wedge^i\mathcal{I}_n] \tag{2.4}$$

$$h_m^- := (1-\omega_S)\sum_{j=0}^{m-1}(-1)^j[\omega_S^j]\sum_{i=0}^{j}(-1)^i[\wedge^{m-i}\mathcal{I}_n^\vee][S^i\mathcal{I}_n^\vee] \tag{2.5}$$

as elements in $K(S^{[n]} \times S)$. Here, we abuse the notation to denote

$$\mathcal{I}_n := \{W_n \xrightarrow{s} V_n\}$$

in the short exact sequence (2.1).

Remark 2.5. Definition 2.4 is equivalent to the definition of $h_m^\pm$ in [24]. We will prove it in Appendix A.

Theorem 2.6 (Theorem 1.2 of [24]). *Let T^*S be the cotangent bundle of S and ω_S be the canonical bundle of S. The morphism:*

$$q_1 + q_2 \to [T^*S] \quad q = q_1q_2 \to [\omega_S]$$

induces a homomorphism:

$$\mathbb{K} \to K(S).$$

We regard $S^{[n,n+1]}$ as a closed subscheme of $S^{[n]}\times S^{[n+1]}\times S$ and $S^{[n]}\times S$ as a closed subscheme of $S^{[n]}\times S^{[n]}\times S$ through the diagonal embedding and consider the following element:

$$\begin{aligned}\tilde{e}_i &:= [\mathcal{L}^i\mathcal{O}_{S^{[n,n+1]}}] \in K(S^{[n]} \times S^{[n+1]} \times S),\\ \tilde{f}_i &:= -[\mathcal{L}^{i-1}\mathcal{O}_{S^{[n,n+1]}}] \in K(S^{[n+1]} \times S^{[n]} \times S),\\ \tilde{h}_i^\pm &:= [h_i^\pm\mathcal{O}_{S^{[n]}\times S}] \in K(S^{[n]} \times S^{[n]} \times S).\end{aligned}$$

All the elements $\tilde{e}_i$, $\tilde{f}_i$ and $h_i^\pm$ could be regarded as operators $K(\mathcal{M}) \to K(\mathcal{M} \times S)$ through the K-theoretic correspondences. Then there exists a unique $\mathbb{K}$-homomorphism

$$\Phi : U_{q_1,q_2}(\ddot{gl}_1) \to Hom(K_{\mathcal{M}}, K_{\mathcal{M}\times S}),$$

such that

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(1)

$$\Phi(E_i)=\tilde{e}_i,\quad \Phi(F_i)=\tilde{f}_i,\quad \Phi(H_i^{\pm})=\tilde{h}_i^{\pm}$$

(2) *For all* $x,y\in U_{q_1,q_2}(\ddot{gl}_1)$*, we have*

$$\Phi(xy)=\Phi(x)\Phi(y)|_{\Delta_S}$$
$$[\Phi(x),\Phi(y)]|_{red}=\Phi(\frac{[x,y]}{(1-q_1)(1-q_2)}).$$

The right-hand side is well defined due to the fact that all commutators in $U_{q_1,q_2}(\ddot{gl}_1)$ *are multiples of* $(1-q_1)(1-q_2)$ *(see Theorem 2.4 of [25]).*

3. Nested Hilbert schemes and $h^{\pm}_{m,k}$

In this section, we consider the nested Hilbert scheme $S^{[n-1,n,n+1]}$ by the Cartesian diagram:

$$\begin{array}{ccc} S^{[n-1,n,n+1]} & \xrightarrow{q_n} & S^{[n,n+1]} \\ \downarrow & & \downarrow{\scriptstyle (p_n^+,\pi_n)} \\ S^{[n-1,n]} & \xrightarrow{(p^-_{n-1},\pi_{n-1})} & S^{[n]}\times S \end{array} \tag{3.1}$$

which consists of

$$\{(\mathcal{I}_{n-1},\mathcal{I}_n,\mathcal{I}_{n+1},x)\in S^{[n-1]}\times S^{[n]}\times S^{[n+1]}\times S|\mathcal{I}_{n-1}/\mathcal{I}_n=k_x,\mathcal{I}_n/\mathcal{I}_{n+1}=k_x\}.$$

Like the definition of the line bundle $\mathcal{L}$ in Definition 2.2, we can also define two line bundles $\mathcal{L}_1,\mathcal{L}_2$ on $S^{[n-1,n,n+1]}$ whose fibres are $\mathcal{I}_n/\mathcal{I}_{n+1},\mathcal{I}_{n-1}/\mathcal{I}_n$, respectively. We denote

$$q_n: S^{[n-1,n,n+1]}\to S^{[n,n+1]}$$

as the projection morphism.

Example 3.1. Let $\Delta_S: S\to S\times S$ be the diagonal embedding and $\mathcal{I}_{\Delta_S}$ be the ideal sheaf of the diagonal. Then

$$S^{[1,2]}=Bl_{\Delta_S}(S\times S)=\mathbb{P}_{S\times S}(\mathcal{I}_{\Delta_S})\qquad S^{[2]}=Bl_{\Delta_S}(S\times S)/\mathbb{Z}_2,$$

where the $\mathbb{Z}_2$ action on $Bl_{\Delta_S}(S\times S)$ is induced by the $\mathbb{Z}_2$ action

$$i: S\times S\to S\times S\quad i(x,y)=(y,x).$$

By [33], the projection morphism

$$(p_2^-,\pi_2): S^{[1,2]}\to S^{[2]}\times S\quad (\mathcal{I}_1,\mathcal{I}_2,x)\to(\mathcal{I}_2,x)$$

is a closed embedding with image $\mathcal{Z}_2$. By (3.1), $S^{[1,2,3]}$ is the preimage of $\mathcal{Z}_2$.

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For two integers k,m, such that $m > k \geq 0$, we define $h^+_{m,k} \in D^b(S^{[n]} \times S)$ by

$$h^+_{m,k} := \begin{cases} \mathbf{R}(p_n \circ q_n)_*(\mathcal{L}_1^{m-1-k}\mathcal{L}_2^k)[1] & k > 0 \\ \mathbf{R}p_{n*}(\mathcal{L}^m)[2] & k = 0 \end{cases} \tag{3.2}$$

and if $m < k \leq 0$, we define $h^-_{m,k} \in D^b(S^{[n]} \times S)$ by

$$h^-_{m,k} := \begin{cases} \mathbf{R}(p_n \circ q_n)_*(\mathcal{L}_1^{m-1-k}\mathcal{L}_2^k)[1] & k < 0 \\ \mathbf{R}p_{n*}(\mathcal{L}^{-m-1})[1] & k = 0. \end{cases} \tag{3.3}$$

The purpose of this section is to prove that

Theorem 3.2. *At the level of Grothendieck groups,*

$$[h^+_{m,k}] = [\omega_S^{-k}] \sum_{i=0}^{k} (-1)^i [S^{m-i}\mathcal{I}_n][\wedge^i \mathcal{I}_n]$$

$$[h^-_{m,k}] = (-1)^k [\omega_S^k] \sum_{i=0}^{k} (-1)^i [\wedge^{m-i}\mathcal{I}_n^\vee][S^i \mathcal{I}_n^\vee]$$

and at the level of Grothendieck groups

$$\begin{cases} (1-[\omega_S]) \sum\limits_{k=0}^{m-1} [h^+_{m,k}] = h^+_m & m > 0 \\ (1-[\omega_S]) \sum\limits_{k=m+1}^{0} [h^-_{m,k}] = h^-_{-m} & m < 0. \end{cases} \tag{3.4}$$

3.1. Projectivisation and A categorical projection lemma

Definition 3.3. Let

$$U := \{W \xrightarrow{s} V\}$$

be a two term complex of locally free sheaves over a scheme X, such that W has rank w and V has rank v. Let $Z \subset \mathbb{P}_X(V)$ be the closed subscheme, such that $\mathcal{O}_Z$ is the cokernel of the composition of morphisms

$$\rho^* W \otimes \mathcal{O}_{\mathbb{P}_X(V)}(-1) \xrightarrow{\rho^*(s)} \rho^* V \otimes \mathcal{O}_{\mathbb{P}_X(V)}(-1) \xrightarrow{taut} \mathcal{O}_{\mathbb{P}_X(V)},$$

where $\rho : \mathbb{P}_X(V) \to X$ is the projection morphism. We define Z to be the projectivisation of U over X, denoted by

$$Z = \mathbb{P}_X(U)$$

if $\mathcal{O}_Z$ is resolved by the Koszul complex:

$$0 \to \wedge^w \rho^*(W) \otimes \mathcal{O}_{\mathbb{P}_X(V)}(-w) \to \cdots \to \rho^* W \otimes \mathcal{O}_{\mathbb{P}_X(V)}(-1) \to \mathcal{O}_{\mathbb{P}_X(V)} \to \mathcal{O}_Z \to 0.$$

When Z is a projectivisation of U over X, we have a categorical projection lemma for $\mathbf{R}\rho_*(\mathcal{O}_Z(k))$:

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Lemma 3.4 (Categorical projection lemma). *If Z is the projectivisation of U over X in Definition 3.3, then the tensor contraction*

$$det(U)^{-1}\wedge^{w-v-k}W^{\vee}=\wedge^{v}(V^{\vee})\wedge^{k+v}(W)\to\wedge^{k}(W) \tag{3.5}$$

induces a morphism of complexes

$$det(U)^{-1}\wedge^{w-v-k}(U^{\vee})[k]\to S^{k}(U) \tag{3.6}$$

and $\mathbf{R}\rho_*(\mathcal{O}_Z(k))$ is quasi-isomorphic to its cone.

Proof. $\mathcal{O}_Z(k)$ is quasi-isomorphic to the complex

$$\{\cdots\to\wedge^{j}\rho^*W\otimes\mathcal{O}_{\mathbb{P}_X(V)}(-j+k)\to\cdots\to\rho^*W\otimes\mathcal{O}_{\mathbb{P}_X(V)}(k-1)\to\mathcal{O}_{\mathbb{P}_X(V)}(k)\to 0\}.$$

Consider the following two complexes

$$F_0=\{\cdots\to\wedge^{k+v+1}\rho^*W\otimes\mathcal{O}_{\mathbb{P}_X(V)}(-v-1)\to\wedge^{k+v}\rho^*W\otimes\mathcal{O}_{\mathbb{P}_X(V)}(-v)\}[v+k-1]$$
$$F_1=\{\wedge^{k+v-1}\rho^*W\otimes\mathcal{O}_{\mathbb{P}_X(V)}(-v+1)\cdots\to\rho^*W\otimes\mathcal{O}_{\mathbb{P}_X(V)}(k-1)\to\mathcal{O}_{\mathbb{P}_X(V)}(k)\}.$$

Then the morphism $\wedge^{k+v}\rho^*W\otimes\mathcal{O}_{\mathbb{P}_X(V)}(-v)\to\wedge^{k+v-1}\rho^*W\otimes\mathcal{O}_{\mathbb{P}_X(V)}(-v+1)$ induces a morphism of $F_0\to F_1$ with the cone quasi-isomorphic to $\mathcal{O}_Z(k)$.

By Exercise III.8.4 of Hartshorne [12],

$$\mathbf{R}\rho_*(\mathcal{O}_{\mathbb{P}(V)}(j))=\begin{cases}S^j(V) & j\geq 0\\ 0 & -v<j<0\\ detV^{-1}\otimes S^{-j-v}V^{\vee}[1-v] & j\leq -v,\end{cases} \tag{3.7}$$

and thus,

$$\mathbf{R}\rho_*F_0\cong det(U)^{-1}\wedge^{w-v-k}(U^{\vee})[-k],\quad \mathbf{R}\rho_*F_1\cong S^k(U).$$

Hence, $\mathbf{R}\rho_*(\mathcal{O}_Z(k))$ is quasi-isomorphic to the cone

$$det(U)^{-1}\wedge^{w-v-k}(U^{\vee})[k]\to S^k(U).$$

□

3.2. Nested Hilbert schemes as projectivisation

Recall the short exact sequence (2.1):

$$0\to W_n\xrightarrow{s_n}V_n\to\mathcal{I}_n\to 0.$$

Nested Hilbert schemes can be realised as projectivisations, as in the following Propositions:

Proposition 3.5 (Proposition 2.2 and Lemma 3.1 of [7]). *The nested Hilbert scheme $S^{[n,n+1]}$ is the blow up of $\mathcal{Z}_n$ in $S^{[n]}\times S$ and*

$$S^{[n,n+1]}\cong\mathbb{P}_{S^{[n]}\times S}(\mathcal{I}_n)$$

is smooth of dimension $2n+2$. Moreover, $S^{[n,n+1]}$ is the projectivisation of

$$W_n\xrightarrow{s}V_n$$

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over $S^{[n]} \times S$. The tautological line bundle $\mathcal{L}$ is the restriction of $\mathcal{O}_{\mathbb{P}_{S^{[n]}\times S}(V_n)}(1)$ to $S^{[n,n+1]}$.

Corollary 3.6. *The line bundle $\mathcal{L}$ is the exceptional divisor of $S^{[n,n+1]}$ as the blow up of $\mathcal{Z}_n$, that is we have the short exact sequence:*

$$0 \to \mathcal{L} \to \mathcal{O}_{S^{[n,n+1]}} \to p_n^{-1}\mathcal{O}_{\mathcal{Z}_n} \to 0. \tag{3.8}$$

Proof. It is obvious from Proposition 7.13 of [12]. □

Let $\overline{V_n}$ be the kernel of the surjective morphism

$$p_n^* V_n \to \mathcal{O}_{S^{[n,n+1]}}(1) = \mathcal{L}.$$

Then $\overline{V_n}$ is also locally free. The morphism $p_n^*(W_n) \to p_n^*(V_n)$ factors through $\overline{V_n}$ and induces a morphsim

$$\overline{V_n}^\vee \otimes \omega_S \to p_n^*(W_n^\vee) \otimes \omega_S.$$

Proposition 3.7 (Proposition 4.15 of [25]). *The scheme $S^{[n-1,n,n+1]}$ is smooth of dimension $2n+1$. Moreover, it is the projectivisation of*

$$\overline{V_n}^\vee \otimes \omega_S \to p_n^*(W_n^\vee) \otimes \omega_S$$

over $S^{[n,n+1]}$. For the two tautological line bundles $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_1 = q_n^*(\mathcal{O}_{S^{[n,n+1]}}(1))$ and $\mathcal{L}_2$ is the restriction of $\mathcal{O}_{\mathbb{P}_{S^{[n,n+1]}}(p_n^*(W_n^\vee)\otimes\omega_S)}(-1)$ in $S^{[n,n+1]}$.

3.3. The derived pushforward of line bundles on $S^{[n-1,n,n+1]}$

In this subsection, we still abuse the notation to denote

$$\mathcal{I}_n := \{W_n \to V_n\}, \qquad \mathcal{I}_n^\vee := \{V_n^\vee \to W_n^\vee\}.$$

Lemma 3.8. *We have the following formula for the derived pushforward $\mathbf{R}p_{n*}\mathcal{L}^j$:*

$$\mathbf{R}p_{n*}(\mathcal{L}^j) = \begin{cases} S^j(\mathcal{I}_n) & j \geq 0 \\ \wedge^{-j-1}(\mathcal{I}_n^\vee)[1+j] & j < 0. \end{cases} \tag{3.9}$$

Proof. By Proposition 3.5, $S^{[n,n+1]}$ is the projectivisation of

$$W_n \xrightarrow{s_n} V_n.$$

Thus, (3.9) follows from Lemma 3.4. □

Lemma 3.9. *We have the following formula for $\mathbf{R}q_{n*}\mathcal{L}_2^k$:*

$$\mathbf{R}q_{n*}(\mathcal{L}_2^k) = \begin{cases} \{\mathcal{L} \to \mathcal{O}_{S^{[n,n+1]}}\} & k = 0 \\ \omega_S^{-k} \otimes \{\mathbf{L}p_n^*(\wedge^k \mathcal{I}_n)\mathcal{L} \to \cdots \to \mathbf{L}p_n^*(\mathcal{I}_n) \otimes \mathcal{L}^k \to \mathcal{L}^{k+1}\}[1-2k] & k > 0 \\ \omega_S^{-k} \otimes \{\cdots \to \mathcal{L}^{-1}\mathbf{L}p_n^*(S^{-k-1}\mathcal{I}_n^\vee)[1] \to \mathbf{L}p_n^*(S^{-k}\mathcal{I}_n^\vee)\} & k < 0, \end{cases} \tag{3.10}$$

where $\mathbf{L}p_n^* : D^b(S^{[n]} \times S) \to D^b(S^{[n,n+1]})$ is the derived pullback morphism.

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Proof. By Proposition 3.7, $S^{[n-1,n,n+1]}$ is the projectivisation of

$$\overline{V_n}^\vee \otimes \omega_S \to p_n^*(W_n^\vee) \otimes \omega_S$$

over $S^{[n,n+1]}$. We notice that $p_n^*(W_n^\vee) \otimes \omega_S$ and $\overline{V_n}^\vee \otimes \omega_S$ have the same rank and

$$\frac{det(p_n^*(W_n) \otimes \omega_S)}{det(\overline{V_n}) \otimes \omega_S} = \frac{det(p_n^*(W_n))}{det(\overline{V_n})} = \frac{det(p_n^*(V_n))}{det(p_n^*(V_n))\mathcal{L}^{-1}} = \mathcal{L}.$$

By Lemma 3.4, we have

- When $k = 0$,
$$\mathbf{R}q_{n*}(\mathcal{O}_{S^{[n-1,n,n+1]}}) = \{\mathcal{L} \to \mathcal{O}_{S^{[n,n+1]}}\}, \tag{3.11}$$
- When $k > 0$, we have
$$\mathbf{R}q_{n*}\mathcal{L}_2^k = \mathcal{L} \otimes \omega_S^{-k} \otimes \{S^k(p_n^* W_n) \to \overline{V_n} \otimes S^{k-1}(p_n^* W_n) \cdots \to \wedge^k(\overline{V_n})\}[1-k].$$
By the resolution of $\wedge^k(\overline{V_n})$:
$$0 \to \wedge^k(\overline{V_n}) \to \wedge^k(V_n) \to \wedge^{k-1}(V_n) \otimes \mathcal{L} \to \cdots \to \mathcal{L}^k \to 0,$$
we have
$$\mathbf{R}q_{n*}(\mathcal{L}_2^k) = \omega_S^{-k} \otimes \{\mathbf{L}p_n^*(\wedge^k \mathcal{I}_n)\mathcal{L} \to \cdots \to \mathbf{L}p_n^*(\mathcal{I}_n) \otimes \mathcal{L}^k \to \mathcal{L}^{k+1}\}[1-2k].$$
- When $k < 0$, we have
$$\mathbf{R}q_{n*}(\mathcal{L}_2^k) = \omega_S^{-k} \otimes \{\cdots \to \overline{V_n}^\vee \otimes S^{-k-1}(p_n^* W_n^\vee) \to S^{-k}(p_n^* W_n^\vee)\}$$
By the resolution of $\wedge^k(\overline{V_n}^\vee)$:
$$0 \to \mathcal{L}^k \to \cdots \to \wedge^{-k-1}(V_n^\vee)\mathcal{L}^{-1} \to \wedge^k(V_n^\vee) \to \wedge^k(\overline{V_n}^\vee) \to 0,$$
we have
$$\mathbf{R}q_{n*}(\mathcal{L}_2^k) = \omega_S^{-k} \otimes \{\cdots \to \mathcal{L}^{-1}\mathbf{L}p_n^*(S^{-k-1}\mathcal{I}_n^\vee)[1] \to \mathbf{L}p_n^*(S^{-k}\mathcal{I}_n^\vee)\} \tag{3.12}$$

□

By Lemmas 3.8 and 3.9, we have

Corollary 3.10. *We have the following formula for* $\mathbf{R}(p_n \circ q_n)_* \mathcal{L}_1^{m-1-k}\mathcal{L}_2^k$*:*

$$\mathbf{R}(p_n \circ q_n)_*(\mathcal{L}_1^{m-1-k}\mathcal{L}_2^k) = \tag{3.13}$$
$$\begin{cases} \omega_S^{-k} \otimes \{\wedge^{-m}(\mathcal{I}_n^\vee) \to \cdots \to \wedge^{-m+k}(\mathcal{I}_n^\vee) S^{-k}(\mathcal{I}_n^\vee)\}[m-k] & m \le k \le -1 \\ \omega_S^{-k} \otimes \{\wedge^k(\mathcal{I}_n) S^{m-k}(\mathcal{I}_n) \to \cdots \to \mathcal{I}_n \otimes S^{m-1}(\mathcal{I}_n) \to S^m(\mathcal{I}_n)\}[1-2k] & 1 \le k \le m. \end{cases}$$

Remark 3.11. When $m = k < 0$, same as the computation in the Grothendieck group, the complex

$$\{\wedge^{-k}(\mathcal{I}_n^\vee) \to \cdots \to S^{-k}(\mathcal{I}_n^\vee)\}$$

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is quasi-isomorphic to 0, and thus, we have

$$\mathbf{R}(p_n \circ q_n)_* \mathcal{L}_1^{-1} \mathcal{L}_2^k = 0. \tag{3.14}$$

Proof of Theorem 3.2. It follows from Definition 2.4, Lemma 3.8 and Corollary 3.10. □

4. The quadrulple/triple moduli spaces and the minimal model program

In this section, we introduce the triple moduli spaces $\mathfrak{Z}_+, \mathfrak{Z}_-$ and the quadruple moduli space $\mathfrak{Y}$ which parameterise diagrams:

$$\begin{array}{ccc} & & \mathcal{I}_n \\ & \overset{x}{\nearrow} & \\ \mathcal{I}_{n+1} & & \\ & \underset{y}{\searrow} & \\ & & \mathcal{I}'_n \end{array} \tag{4.1}$$

$$\begin{array}{ccc} \mathcal{I}_n & & \\ & \overset{y}{\searrow} & \\ & & \mathcal{I}_{n-1} \\ & \underset{x}{\nearrow} & \\ \mathcal{I}'_n & & \end{array} \tag{4.2}$$

$$\begin{array}{ccccc} & & \mathcal{I}_n & & \\ & \overset{x}{\nearrow} & & \overset{y}{\searrow} & \\ \mathcal{I}_{n+1} & & & & \mathcal{I}_{n-1} \\ & \underset{y}{\searrow} & & \underset{x}{\nearrow} & \\ & & \mathcal{I}'_n & & \end{array} \tag{4.3}$$

respectively, of ideal sheaves, where each successive inclusion is colength 1 and supported at the point indicated on the diagrams. We consider line bundles $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}'_1, \mathcal{L}'_2$ over triple/quadruple moduli spaces with fibre $\mathcal{I}_{n+1}/\mathcal{I}_n, \mathcal{I}_n/\mathcal{I}_{n-1}, \mathcal{I}_{n+1}/\mathcal{I}'_n, \mathcal{I}'_n/\mathcal{I}_{n-1}$, respectively.

We consider the Cartesian diagram:

$$\begin{array}{ccc} \mathfrak{Y} & \xrightarrow{\alpha_+} & \mathfrak{Z}_+ \\ \downarrow{\scriptstyle \alpha_-} & \overset{\theta}{\searrow} & \downarrow{\scriptstyle \beta_+} \\ \mathfrak{Z}_- & \xrightarrow{\beta_-} & S^{[n]} \times S^{[n]} \times S \times S \end{array} \qquad \begin{array}{ccc} (\mathcal{I}_{n-1}, \mathcal{I}_n, \mathcal{I}'_n, \mathcal{I}_{n+1}, x, y) & \xrightarrow{\alpha_+} & (\mathcal{I}_n, \mathcal{I}'_n, \mathcal{I}_{n+1}, x, y) \\ \downarrow{\scriptstyle \alpha_-} & \overset{\theta}{\searrow} & \downarrow{\scriptstyle \beta_+} \\ (\mathcal{I}_{n-1}, \mathcal{I}_n, \mathcal{I}'_n, x, y) & \xrightarrow{\beta_-} & (\mathcal{I}_n, \mathcal{I}'_n, x, y). \end{array} \tag{4.4}$$

(3.1) is the restriction of (4.4) to the diagonal $\Delta : S^{[n]} \times S \to S^{[n]} \times S^{[n]} \times S \times S$.

Example 4.1. When $n=1$, then $\mathcal{I}_0=\mathcal{O}_S$, and thus, $\mathfrak{Y}=S^{[1,2]}=Bl_{\Delta_S}(S\times S)$. The scheme $\mathfrak{Z}_-=S\times S$ and α_- is the projection morphism of the blow up. The scheme $\mathfrak{Z}_+$ is induced by the Cartesian diagram

$$\begin{array}{ccc} \mathfrak{Z}_+ & \longrightarrow & Bl_{\Delta_S}(S\times S) \\ \downarrow & & \downarrow \\ Bl_{\Delta_S}(S\times S) & \longrightarrow & Bl_{\Delta_S}(S\times S)/\mathbb{Z}_2, \end{array} \tag{4.5}$$

and has two irreducible components, such that each one is isomorphic to $Bl_\Delta(S\times S)$.

The main purpose of this section is to compute $\mathbf{R}\alpha_{+*}\mathcal{O}_{\mathfrak{Y}}$ and $\mathbf{R}\alpha_{-*}\mathcal{O}_{\mathfrak{Y}}$ explicitly.

Proposition 4.2. *We have the formula*

$$\mathbf{R}\alpha_{-*}\mathcal{O}_{\mathfrak{Y}}=\mathcal{O}_{\mathfrak{Z}_-},\quad \mathbf{R}\alpha_{+*}\mathcal{O}_{\mathfrak{Y}}=\mathcal{O}_{W_0}, \tag{4.6}$$

where W_0 will be defined in Section 4.3.

Proposition 4.2 follows from Proposition 4.7 and Corollary 4.14, which will be proved later in this section. This section would rely on the singularity theory of the minimal model program, which is summarised in Appendix B.

4.1. The geometry of $\mathfrak{Y}$

Theorem 4.3 (Propositions 2.28 and 5.28 of [26]). *The scheme $\mathfrak{Y}$ is smooth of dimension $2n+2$. The closed embedding:*

$$\Delta_{\mathfrak{Y}}:S^{[n-1,n,n+1]}\to\mathfrak{Y}\quad(\mathcal{I}_{n-1},\mathcal{I}_n,\mathcal{I}_{n+1},x)\to(\mathcal{I}_{n-1},\mathcal{I}_n,\mathcal{I}_n,\mathcal{I}_{n+1},x,x)$$

is a regular closed subscheme of codimension 1. If we abuse the notation to denote $S^{[n-1,n,n+1]}$ by $\Delta_{\mathfrak{Y}}$, then the morphism of coherent sheaves over $\mathfrak{Y}\times S$:

$$\mathcal{I}_n/\mathcal{I}_{n+1}\to\mathcal{I}_{n-1}/\mathcal{I}_{n+1}\to\mathcal{I}_{n-1}/\mathcal{I}'_n,\quad \mathcal{I}'_n/\mathcal{I}_{n+1}\to\mathcal{I}_{n-1}/\mathcal{I}_{n+1}\to\mathcal{I}_{n-1}/\mathcal{I}_n$$

induce short exact sequences

$$0\to\mathcal{L}_1\to\mathcal{L}'_2\to\mathcal{L}'_2\mathcal{O}_{\Delta_{\mathfrak{Y}}}\to 0$$
$$0\to\mathcal{L}'_1\to\mathcal{L}_2\to\mathcal{L}_2\mathcal{O}_{\Delta_{\mathfrak{Y}}}\to 0.$$

Moreover, $\mathcal{L}_1\mathcal{L}_2'^{-1}=\mathcal{L}'_1\mathcal{L}_2^{-1}=\mathcal{O}(-\Delta_{\mathfrak{Y}})$. The normal bundle $N_{\mathfrak{Y}/\Delta_{\mathfrak{Y}}}=\mathcal{L}_1^{-1}\mathcal{L}_2$.

Remark 4.4. From now on, we will abuse the notation to denote $S^{[n-1,n,n+1]}$ by $\Delta_{\mathfrak{Y}}$.

Lemma 4.5 (Claim 3.8 of [24]). *Let U be the complement of $\Delta_{\mathfrak{Y}}$ in $\mathfrak{Y}$. Then α_+ and α_- in (4.4) are isomorphisms when restricting to U.*

4.2. The geometry of $\mathfrak{Z}_-$

Proposition 4.6. *The scheme $\mathfrak{Z}_-$ is an irreducible $2n+2$ dimensional locally complete intersection scheme.*

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Proof. First we notice that

$$\mathfrak{Z}_- = S^{[n-1,n]} \times_{S^{[n-1]}} S^{[n-1,n]}$$

has the expected dimension also equal to $2n+2$. Thus, $\mathfrak{Z}_-$ is a locally complete intersection scheme and, thus, is Cohen-Macaulay.

We recall the Serre's criterion [34, Tag 033P] for the normality of a variety: to prove that a variety is normal, we only need to show that it satisfies the Serre's condition S2 (which is always satisfied if the variety is Cohen-Macaulay), and R1 (i.e. the singular locus has at least codimension 1). By Lemma 4.5, $\alpha_-^{-1}(U) \cong \theta^{-1}(U)$ is a $2n+2$-dimensional smooth open subscheme and the complement is the $2n$-dimensional closed subscheme $S^{[n-1,n]}$ by (4.4). Hence, $\mathfrak{Z}_-$ satisfies R1 and is normal. □

Proposition 4.7. *Let $K_{\mathfrak{Z}_-}$ and $K_{\mathfrak{Y}}$ be the canonical divisors of $\mathfrak{Z}_-$ and $\mathfrak{Y}$, respectively. Then*

$$\alpha_-^* K_{\mathfrak{Z}_-} = K_{\mathfrak{Y}} + \mathcal{O}(\Delta_{\mathfrak{Y}})$$

and $\mathfrak{Z}_-$ is a canonical singularity. Moreover, we have the formula

$$\mathbf{R}\alpha_{-*}(\mathcal{O}_{\mathfrak{Y}}) = \mathcal{O}_{\mathfrak{Z}_-}.$$

Proof. The complement of $\theta^{-1}(U)$ in $\mathfrak{Y}$ is $\Delta_{\mathfrak{Y}} = S^{[n-1,n,n+1]}$, and, thus, there exists $a \in \mathbb{Q}$, such that

$$\alpha_-^* K_{\mathfrak{Z}_-} = K_{\mathfrak{Y}} + a\mathcal{O}(\Delta_{\mathfrak{Y}}).$$

Given two closed points $x,y \in S$, let $\mathcal{I}_x$ and $\mathcal{I}_y$ be the ideal sheaf of closed point x,y. We consider

$$V_2 := \{(\mathcal{I}_{n-1},x,y) \in S^{[n]} \times S \times S | (\mathcal{I}_{n-1},x) \notin \mathcal{Z}_{n-1} \text{ and } (\mathcal{I}_{n-1},y) \notin \mathcal{Z}_{n-1}\}$$

and regard V_2 as an open subvariety of $\mathfrak{Z}_-$ through the embedding

$$(\mathcal{I}_{n-1},x,y) \to (\mathcal{I}_{n-1},\mathcal{I}_{n-1} \cap \mathcal{I}_x,\mathcal{I}_{n-1} \cap \mathcal{I}_y).$$

Let $V_1 := \alpha_-^{-1}(V_2)$. We denote $\mathfrak{Y}_1$ the quadruple moduli space $\mathfrak{Y}$ when $n=1$. By Example 4.1, $\mathfrak{Y}_1 = Bl_{\Delta_S}(S \times S)$ and we have the Cartesian diagram:

$$\begin{array}{ccc} V_1 & \longrightarrow & \mathfrak{Y}_1 \\ \downarrow{\scriptstyle \alpha_-|_{V_2}} & & \downarrow \\ V_2 & \longrightarrow & S \times S \end{array}$$

while the right vertical arrow of the above diagram is the projection morphism of the blow up. Thus, by Lemma B.2, $a=1$ and $\mathfrak{Z}_-$ is a canonical singularity and, hence, a rational singularity by Theorem B.7. □

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4.3. The geometry of $\mathfrak{Z}_+$

The geometry of $\mathfrak{Z}_+$ is more complicated, as it is no longer irreducible. We define W_0 as a closed subscheme of $\mathfrak{Z}_+$ by

$$W_0 := \{(\mathcal{I}_n,\mathcal{I}'_n,\mathcal{I}_{n+1},x,y) \in \mathfrak{Z}_+ | (\mathcal{I}'_n,y) \in \mathcal{Z}_n \text{ and } (\mathcal{I}_n,x) \in \mathcal{Z}_n\}.$$

W_0 is the closure of $\beta_+^{-1}(U)$ in $\mathfrak{Z}_+$. We define $W_1 := S^{[n,n+1]}$.

Proposition 4.8. *The scheme $\mathfrak{Z}_+$ is a locally complete intersection scheme (and hence, Cohen-Macaulay) of dimension $2n+2$, with two irreducible components W_0 and W_1. $W_0 \cap W_1 = p_n^{-1}\mathcal{Z}_n$.*

Proof. W_1 is $2n+2$ dimensional and the complement of W_1 in $\mathfrak{Z}_+$ is $\beta_+^{-1}(U) = \theta^{-1}(U)$, which is also $2n+2$ dimensional by Lemma 4.5. Thus, $\mathfrak{Z}_+$ is $2n+2$ dimensional and

$$\mathfrak{Z}_+ = S^{[n,n+1]} \times_{S^{[n+1]}} S^{[n,n+1]}$$

has expected dimension also equal to $2n+2$. Hence, $\mathfrak{Z}_+$ is a locally complete intersection scheme.

Any closed point of $W_0 \cap W_1$ corresponds to $(\mathcal{I}_n,\mathcal{I}_{n+1},x) \in S^{[n,n+1]}$, such that x has length ≥ 1 in $\mathcal{O}/\mathcal{I}_n$ and, thus, is in $p_n^{-1}\mathcal{Z}_n$. □

Example 4.9. When $n = 2$, by Example 3.1, $\mathcal{Z}_2 = S^{[1,2]}$. Thus, for any point $(\mathcal{I}'_2,\mathcal{I}_2,\mathcal{I}_3,x,y) \in W_0$, there exists a unique ideal sheaf $\mathcal{I}_1$ with two short exact sequences

$$0 \to \mathcal{I}'_2 \to \mathcal{I}_1 \to k_y \to 0 \quad 0 \to \mathcal{I}_2 \to \mathcal{I}_1 \to k_x \to 0.$$

Thus, $W_0 \cong \mathfrak{Y}$.

$W_0 \cap W_1 = S^{[1,2,3]}$ by Example 3.1 and is smooth. So $(\mathfrak{Z}_+,0)$ is a semi-snc pair.

Consider the closed subscheme $W_2 \subset p_n^{-1}\mathcal{Z}_n$

$$W_2 := \{(\mathcal{I}_n,\mathcal{I}_{n+1},x) \in W_1 | \text{ the length of } k_x \text{ in } \mathcal{O}/\mathcal{I}_n \geq 3.\}.$$

Lemma 4.10. *The schemes $q_n^{-1}(W_2)$ and W_2 have dimension less or equal to $2n-1$.*

Proof. The scheme

$$S^{[n-2,n-1,n,n+1]} := S^{[n-2,n-1,n]} \times_{S^{[n-1,n]}} S^{[n-1,n,n+1]}$$

has dimension $2n-1$ by (5.21) of [26]. The image of the projection morphism

$$S^{[n-2,n-2,n,n+1]} \to S^{[n,n+1]}$$

is W_2, and the image of the projecton morphism

$$S^{[n-2,n-2,n,n+1]} \to S^{[n-1,n,n+1]}$$

is $q_n^{-1}(W_2)$. Hence, $q_n^{-1}(W_2)$ and W_2 have dimension less or equal to $2n-1$. □

As the morphism $\alpha_+ : \mathfrak{Y} \to \mathfrak{Z}_+$ factors through W_0, we will abuse the notation to denote the morphism $\alpha_+ : \mathfrak{Y} \to W_0$.

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Lemma 4.11. *The morphism $\alpha_+ : \mathfrak{Y} \to W_0$ is an isomorphism when restricting to $W_0 - W_2$.*

Proof. We denote $\mathfrak{Y}_2$ the quadruple moduli space when $n = 2$. Let

$$V_3 := \{(\mathcal{I}_{n-2},(\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_2',\mathcal{I}_3,x,y)) \in S^{[n-1]} \times \mathfrak{Y}_2 | \mathcal{I}_{n-2} + \mathcal{I}_3 = \mathcal{O}\}$$

and $V_4 := \alpha_+ V_3$. The morphism

$$(\mathcal{I}_{n-2},(\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_2',\mathcal{I}_3,x,y)) \to (\mathcal{I}_{n-2} \cap \mathcal{I}_1, \mathcal{I}_{n-2} \cap \mathcal{I}_2, \mathcal{I}_{n-2} \cap \mathcal{I}_2', \mathcal{I}_{n-2} \cap \mathcal{I}_3, x, y)$$

induces an open emebedding $V_3 \subset \mathfrak{Y}$. By Example 4.9,

$$V_3 \cong V_4.$$

By Lemma 4.5, α_+ is also an isomorphism when restricting to $\beta_+^{-1}(U)$. W_2 is the complement of $\beta_+^{-1}(U) \cup V_4$ in W_0. □

Let $\overline{W_0}$ be the normalisation of W_0 and $\overline{W_0 \cap W_1}, \overline{W_2}$ the preimage of $W_0 \cap W_1$ and W_2 in the normalisation. The morphism $\alpha_+ : \mathfrak{Y} \to W_0$ will factor through $\overline{\alpha_+} : \mathfrak{Y} \to \overline{W_0}$.

Lemma 4.12. *The scheme $\overline{W}_0$ is a canonical singularity, and the pair $(\overline{W_0}, \overline{W_0 \cap W_1})$ is plt.*

Proof. The codimension of $\overline{W_2}$ in $\overline{W_0}$ is 3, and $\overline{\alpha_+}$ is an isomorphism outside of $\overline{W_2}$. The preimage of $\overline{W_0 \cap W_1}$ in $\mathfrak{Y}$ is $\Delta_{\mathfrak{Y}}$, which is a smooth divisor of $\mathfrak{Y}$. Hence, $\overline{W}_0$ is a canonical singularity and the pair $(\overline{W_0}, \overline{W_0 \cap W_1})$ is plt. □

Proposition 4.13. *The pair $(\mathfrak{Z}_+, 0)$ is semi-dlt, and W_0 is normal.*

Proof. Let

$$V_5 := \mathcal{I}_{n-2},(\mathcal{I}_2,\mathcal{I}_2',\mathcal{I}_3,x,y)) \in S^{[n-1]} \times \mathfrak{Z}_2^+ | \mathcal{I}_{n-2} + \mathcal{I}_3 = \mathcal{O}\},$$

where $\mathfrak{Z}_2^+ = S^{[2,3]} \times_{S^{[2]}} S^{[2,3]}$ is the triple moduli space when $n = 2$. V_5 is an open subscheme of $\mathfrak{Z}_+$. The pair$(V_5, 0)$ is a semi-snc by Example 4.9 and so is $(V_5 \cup \beta_+^{-1}(U), 0)$. W_2 is the complement of $V_5 \cup \beta_+^{-1}(U)$ in $\mathfrak{Z}_+$, and

$$codim_{W_2} \mathfrak{Z}_+ = 3.$$

By Definition B.13, $W_0 \cap W_1$ is the conductor subscheme of $\mathfrak{Z}_+$. By Example B.9, $W_0 \cap W_1$ is a canonical singularity. By the inversion of adjunction theorem Theorem B.4, the pair $(W_1, W_0 \cap W_1)$ is plt and, thus, dlt.

By Lemma 4.12, $(\overline{W_0}, \overline{W_0 \cap W_1})$ is also a plt pair and, thus, a dlt pair.

By Proposition B.17, $(\mathfrak{Z}_+, 0)$ is a semi-dlt pair and W_0 is normal. □

Corollary 4.14. *We have the formula*

$$\mathbf{R}\alpha_{+*}(\mathcal{O}_{\mathfrak{Y}}) = \mathcal{O}_{W_0}.$$

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5. The Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Given an integer n, we denote $\iota : S^{[n]} \times S^{[n]} \times S_1 \times S_2 \to S^{[n]} \times S^{[n]} \times S_2 \times S_1$ as the morphism which is identity on $S^{[n]} \times S^{[n]}$ but change the order of two copies of S.

Theorem 5.1. *Given two integers m and r and let*

$$e_{m-r} = \mathcal{L}_1^{m-r} \mathcal{O}_{S^{[n,n+1]}} \in D^b(S^{[n]} \times S^{[n+1]} \times S)$$
$$f_r = \mathcal{L}_1^{k-1} \mathcal{O}_{S^{[n,n+1]}}[1] \in D^b(S^{[n+1]} \times S^{[n]} \times S).$$

Then

(1) *If $m > 0$, then there are $b^k_{m,r} \in D^b(S^{[n]} \times S^{[n]} \times S \times S)$ for $0 \leq k \leq m$, such that $b^m_{m,r} = \iota_* e_{m-r} f_r$ and $b^0_{m,r} = f_r e_{m-r}$, with exact triangles*

$$\mathfrak{B}^k_{m,r} : \mathbf{R}\Delta_*(h^+_{m,k})[-1] \to b^k_{m,r} \to b^{k+1}_{m,r} \to \mathbf{R}\Delta_*(h^+_{m,k})$$

(2) *If $m < 0$, then there are $c^r_{m,r} \in D^b(S^{[n]} \times S^{[n]} \times S \times S)$ for $m \leq k \leq 0$, such that $c^m_{m,r} = \iota_* e_{m-r} f_r$ and $c^0_{m,r} = f_r e_{m-r}$, and exact triangles*

$$\mathfrak{C}^k_{m,r} : \mathbf{R}\Delta_*(h^-_{m,k})[-1] \to c^{k-1}_{m,r} \to c^k_{m,r} \to \mathbf{R}\Delta_*(h^-_{m,k})$$

(3) *If $m = 0$, then there is an explicit isomorphism $f_r e_{-r} = e_r f_{-r} \oplus \mathcal{O}_\Delta[1]$.*

Theorem 1.1 follows from Theorems 3.2 and 5.1. We will prove Theorem 5.1 later in this section.

5.1. $\iota_* e_{m-r} f_r$ and $f_r e_{m-r}$ revisited

As $\mathfrak{Z}_-$ and $\mathfrak{Z}_+$ are both Cohen-Macaulay of expected dimension, we have the following formula:

$$\iota_* e_{m-r} f_r = \mathbf{R}\beta_{-*}(\mathcal{L}_2'^{m-r} \mathcal{L}_2^{r-1} \mathcal{O}_{\mathfrak{Z}_-})[1] \in D^b(S^{[n]} \times S^{[n]} \times S \times S)$$
$$f_r e_{m-r} = \mathbf{R}\beta_{+*}(\mathcal{L}_1^{m-r} \mathcal{L}_1'^{r-1} \mathcal{O}_{\mathfrak{Z}_+})[1] \in D^b(S^{[n]} \times S^{[n]} \times S \times S).$$

By Proposition 4.6, $\mathbf{R}\alpha_{-*}\mathcal{O}_{\mathfrak{Y}} = \mathcal{O}_{\mathfrak{Z}_-}$, and by Theorem 4.3

$$\begin{aligned}\iota_* e_{m-r} f_r &= \mathbf{R}\theta_*(\mathcal{L}_2'^{m-r} \mathcal{L}_2^{r-1} \mathcal{O}_{\mathfrak{Y}})[1] \\ &= \mathbf{R}\theta_*(\mathcal{L}_1'^{m-r} \mathcal{L}_1^{r-1} \mathcal{O}((m-1)\Delta_{\mathfrak{Y}}))[1]. \end{aligned} \tag{5.1}$$

5.2. $a^k_{m,r}$ and $\mathfrak{A}^k_{m,r}$

We recall the short exact sequence

$$0 \to \mathcal{O}_{\mathfrak{Y}}(-\Delta_{\mathfrak{Y}}) \to \mathcal{O}_{\mathfrak{Y}} \to \mathcal{O}_{\Delta_{\mathfrak{Y}}} \to 0. \tag{5.2}$$

Definition 5.2. For any integer r, we define

$$a^k_{m,r} = \mathbf{R}\theta_*(\mathcal{L}_1'^{m-r} \mathcal{L}_1^{r-1} \mathcal{O}(k\Delta_{\mathfrak{Y}}))[1].$$

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We define the exact triangles

$$\mathfrak{A}^k_{m,r} : a^{k-1}_{m,r} \to a^k_{m,r} \to \mathbf{R}\Delta_*(\mathbf{R}(p_n \circ q_n)_* \mathcal{L}_1^{m-1-k}\mathcal{L}_2^k)[1] \to a^{k-1}_{m,r}[1] \tag{5.3}$$

by applying the functor $\mathbf{R}\theta_*(- \otimes \mathcal{L}_1'^{m-r}\mathcal{L}_1^{r-1}\mathcal{O}(k\Delta_{\mathfrak{Y}}))[1]$ to (5.2).

By (5.1), $\iota_* e_{m-r} f_r \cong a^{m-1}_{m,r}$. Moreover, when $m<0$, by (3.14), $\mathbf{R}(p_n \circ q_n)_* \mathcal{L}_1^{-1}\mathcal{L}_2^m = 0$ and $\iota_* e_{m-r} f_r \cong a^{m-1}_{m,r} \cong a^m_{m,r}$.

5.3. $\mathfrak{B}'^0_{m,r}$ and $\mathfrak{C}'^0_{m,r}$

By Example B.9, $\mathbf{R}\alpha_{+*}\mathcal{O}_{\Delta_{\mathfrak{Y}}} = \mathcal{O}_{W_0 \cap W_1}$. Taking the $\mathbf{R}\alpha_{+*}$ functor to the short exact sequence (5.2), we have the short exact sequence

$$0 \to \mathbf{R}\alpha_{+*}\mathcal{O}(-\Delta_{\mathfrak{Y}}) \to \mathcal{O}_{W_0} \to \mathcal{O}_{W_0 \cap W_1} \to 0.$$

Recalling the short exact sequence in Corollary 3.6:

$$0 \to \mathcal{L} \otimes \mathcal{O}_{W_1} \to \mathcal{O}_{W_1} \to \mathcal{O}_{W_0 \cap W_1} \to 0.$$

We have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbf{R}\alpha_{+*}\mathcal{O}(-\Delta_{\mathfrak{Y}}) & \xrightarrow{\cong} & \mathbf{R}\alpha_{+*}\mathcal{O}(-\Delta_{\mathfrak{Y}}) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{L} \otimes \mathcal{O}_{W_1} & \longrightarrow & \mathcal{O}_{\mathfrak{Z}_+} & \longrightarrow & \mathcal{O}_{W_0} & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{L} \otimes \mathcal{O}_{W_1} & \longrightarrow & \mathcal{O}_{W_1} & \longrightarrow & \mathcal{O}_{W_0 \cap W_1} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where all rows and columns are short exact sequences. Thus, we have short exact sequences:

$$0 \to \mathbf{R}\alpha_{+*}\mathcal{O}(-\Delta_{\mathfrak{Y}}) \to \mathcal{O}_{\mathfrak{Z}_+} \to \mathcal{O}_{W_1} \to 0 \tag{5.4}$$

$$0 \to \mathcal{L}_1 \mathcal{O}_{S^{[n,n+1]}} \to \mathcal{O}_{\mathfrak{Z}_+} \to \mathcal{O}_{W_0} \to 0. \tag{5.5}$$

Taking the functor $\mathbf{R}\beta_{+*}(- \otimes \mathcal{L}_1'^{m-r}\mathcal{L}_1^{r-1})[1]$ to (5.4) and (5.5), respectively, we get the exact triangles

$$\mathfrak{C}'^0_{m,r} : a^{-1}_{m,r} \to f_r e_{m-r} \to \mathbf{R}\Delta_*(\mathbf{R}p_{n*}\mathcal{L}^{m-1})[1] \to a^{-1}_{m,r}[1] \tag{5.6}$$

$$\mathfrak{B}'^0_{m,r} : \mathbf{R}\Delta_*(\mathbf{R}p_{n*}\mathcal{L}^m)[1] \to f_r e_{m-r} \to a^0_{m,r} \to \mathbf{R}\Delta_*(\mathbf{R}p_{n*}\mathcal{L}^m)[2]. \tag{5.7}$$

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5.4. $\mathfrak{B}^k_{m,r}$ and $\mathfrak{C}^k_{m,r}$

Definition 5.3. When $m>0$, we define $b^k_{m.r} \in D^b(S^{[n]}\times S^{[n]}\times S\times S)$ and natural transforms $\mathfrak{B}^k_{m,r}: b^k_{m,r} \to b^{k+1}_{m,r}$ by

$$b^k_{m,r} := \begin{cases} f_r e_{m-r} & k=0 \\ a^{k-1}_{m,r} & 1\le k\le m \end{cases} \tag{5.8}$$

$$\mathfrak{B}^k_{m,r} := \begin{cases} \mathfrak{B}'^0_{m,r} & k=0 \\ \mathfrak{A}^k_{m,r} & 1\le k\le m-1. \end{cases} \tag{5.9}$$

We have $b^0_{m,r} = f_r e_{m-r}$ and $b^m_{m,r} = \iota_* e_{m-r} f_r$.

When $m<0$, we define $c^k_{m,r} \in D^b(S^{[n]}\times S^{[n]}\times S\times S)$ and $\mathfrak{C}^k_{m,r}: c^{k-1}_{m,r} \to c^k_{m,r}$ by

$$c^k_{m,r} = \begin{cases} f_r e_{m-r} & k=0 \\ a^k_{m,r} & m\le k\le -1 \end{cases} \tag{5.10}$$

$$\mathfrak{C}^k_{m,r} = \begin{cases} \mathfrak{C}'^0_{m,r} & k=0 \\ \mathfrak{A}^k_{m,r} & m+1\le k\le -1. \end{cases} \tag{5.11}$$

We have $f_r e_{m-r} = c^0_{m,r}$ and $\iota_* e_{m-r} f_r = c^m_{m,r}$.

Proof of Theorem 5.1. When $m>0$, we only need to prove that the cone of $\mathfrak{B}^k_{m,r}$ is $\mathbf{R}\Delta_*(h^+_{m,k})$ and the cone of $\mathfrak{C}^k_{m,r}$ is $\mathbf{R}\Delta_*(h^-_{m,k})$. It follows from (5.3), (5.6) and (5.7).

When $m=0$,

$$e_{-r} f_r = \mathbf{R}\beta_{+*}(\mathbf{R}\alpha_{+*}(\mathcal{L}_1'^{-r}\mathcal{L}_1^{r-1}\mathcal{O}(-\Delta_{\mathfrak{Y}}))).$$

We have the short exact sequence:

$$0 \to \mathcal{L}_1'^{-r}\mathcal{L}_1^{r-1}\mathcal{O}_{\mathfrak{Z}_+} \to \mathcal{L}_1'^{-r}\mathcal{L}_1^{r-1}(\mathcal{O}_{W_0}\oplus\mathcal{O}_{W_1}) \to \mathcal{L}^{-1}\mathcal{O}_{W_0\cap W_1} \to 0, \tag{5.12}$$

and

$$\begin{aligned} \mathbf{R}\beta_{+*}(\mathcal{L}^{-1}\mathcal{O}_{W_0\cap W_1}) &= \{\mathbf{R}\beta_{+*}(\mathcal{O}_{W_1} \to \mathcal{L}^{-1}\mathcal{O}_{W_1})\} \\ &= \mathbf{R}\Delta_*\{\mathcal{O}_{S^{[n]}\times S} \to \mathcal{O}_{S^{[n]}\times S}\} \text{ by Lemma 3.8} \\ &= 0. \end{aligned}$$

By (5.4) and (5.12), we have isomorphisms

$$\begin{aligned} f_r e_{-r} &\cong \mathbf{R}\beta_{+*}\mathcal{L}_1'^{-r}\mathcal{L}_1^{r-1}(\mathcal{O}_{W_0})[1] \oplus \mathcal{O}_\Delta[1] \\ e_{-r} f_r &\cong \mathbf{R}\beta_{+*}\mathcal{L}_1'^{-r}\mathcal{L}_1^{r-1}(\mathcal{O}_{W_0})[1]. \end{aligned}$$

□

6. Extension classes between $h^{\pm}_{m,k}$

When $m>0$, the composition of $\mathfrak{B}^k_{m,r}$ and $\mathfrak{B}^{k+1}_{m,r}$ in Theorem 5.1 induces an extension of $\mathbf{R}\Delta_* h^+_{m,k-1}$ and $\mathbf{R}\Delta_* h^+_{m,k}$, that is an extension class of

$$Hom(\mathbf{R}\Delta_* h^+_{m,k}, \mathbf{R}\Delta_* h^+_{m,k-1}[1]).$$

Similarly, when $m<0$, there is also an extension class of

$$Hom(\mathbf{R}\Delta_*h^-_{m,k},\mathbf{R}\Delta_*h^-_{m,k-1}[1])$$

which relies on m,k,r. The purpose of this section is to study the above extension classes. First we recall the Hochschild-Kostant-Rosenberg theorem (short for HKR theorem). Let $f:X\to Y$ be a regular embedding of smooth varieties. On X, we have the following short exact sequence

$$0\to N^\vee_{Y/X}\xrightarrow{\iota_{Y/X}}T^*Y|_X\to T^*X\to 0.$$

Let $g:T^*Y\to N^\vee_{Y/X}$ be a splitting of the above short exact sequence, that is $g\circ\iota_{Y/X}=id$.

Theorem 6.1 (HKR isomorphism, Theorem 1.4 and Section 1.11 of [1], see Theorem C.10 for the precise formulation). *The splitting g induces a canonical isomorphism*

$$\tau^{F,G}_{N^\vee_{Y/X}}:\mathbf{R}Hom_X(F\otimes^{\mathbf{L}}(\bigoplus_{j=0}^{\infty}\wedge^jN^\vee_{Y/X}),G)\cong\mathbf{R}Hom_Y(\mathbf{R}f_*F,\mathbf{R}f_*G)$$

which is functorial respect to $F,G\in D(X)$.

Remark 6.2. The isomorphism $\tau^{F,G}_{N^\vee_{X/Y}}$ depends on the choice of the splitting g. For a different choice of g, the difference of the isomorphism $\tau^{F,G}_{N^\vee_{X/Y}}$ is represented by a transition matrix in Proposition 3.7 of [13].

Now we consider the diagonal embedding $\Delta:S^{[n]}\times S\to S^{[n]}\times S^{[n]}\times S\times S$. Then $T^*(S^{[n]}\times S^{[n]}\times S\times S)|_\Delta=T^*(S^{[n]}\times S)\oplus T^*(S^{[n]}\times S)$, where the conormal bundle is

$$T^*(S^{[n]}\times S)\cong\{(x,-x)\in T^*(S^{[n]}\times S)\oplus T^*(S^{[n]}\times S)\}.$$

We take the split

$$(T^*\Delta)^{-1}:T^*(S^{[n]}\times S)\oplus T^*(S^{[n]}\times S)\to T^*(S^{[n]}\times S) \tag{6.1}$$

which maps (a,b) to $(\frac{1}{2}(a-b),\frac{1}{2}(b-a))$ and, hence, induce isomorphisms

$$Hom(\mathbf{R}\Delta_*h^+_{m,k},\mathbf{R}\Delta_*h^+_{m,k-1}[1])=\bigoplus_{j=0}^{\infty}Hom_{S^{[n]}\times S}(h^+_{m,k}\otimes\wedge^jT^*(S^{[n]}\times S),h^+_{m,k-1}[1-j]),$$

$$Hom(\mathbf{R}\Delta_*h^-_{m,k},\mathbf{R}\Delta_*h^-_{m,k-1}[1])=\bigoplus_{j=0}^{\infty}Hom_{S^{[n]}\times S}(h^-_{m,k}\otimes\wedge^jT^*(S^{[n]}\times S),h^-_{m,k-1}[1-j]).$$

In this section, we will be explicitly computing the above extension classes and prove that

Proposition 6.3. *The extension class in $Hom(\mathbf{R}\Delta_*h^+_{m,k},\mathbf{R}\Delta_*h^+_{m,k-1}[1])$ is*

$$(\mathfrak{F}^+_{m,k,r},\mathfrak{H}^+_{m,k},0,0,\cdots)$$

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and the extension class in $Hom(\mathbf{R}\Delta_* h^-_{m,k}, \mathbf{R}\Delta_* h^-_{m,k-1}[1])$ *is*

$$(\mathfrak{F}^-_{m,k,r}, \mathfrak{H}^-_{m,k}, 0, 0, \cdots),$$

where $\mathfrak{F}^+_{m,k,r}$, $\mathfrak{F}^-_{m,k,r}$, $\mathfrak{H}^+_{m,k}$ *and* $\mathfrak{H}^-_{m,k}$ *will be defined in Definition 6.7. The classes* $\mathfrak{H}^+_{m,k}$ *and* $\mathfrak{H}^-_{m,k}$ *only depend on* m,k *but not* r.

Proposition 6.3 will be proved later in this section.

6.1. A morphism $s : \mathcal{L}_1^{-1}\mathcal{L}_2 \to (p_n \circ q_n)^* T(S^{[n]} \times S)$

By restricting $\theta : \mathfrak{Y} \to S^{[n]} \times S^{[n]} \times S \times S$ to the diagonal of $S^{[n]} \times S^{[n]} \times S \times S$, we get the Cartesian diagram:

$$\begin{array}{ccc} S^{[n-1,n,n+1]} & \xrightarrow{\Delta_{\mathfrak{Y}}} & \mathfrak{Y} \\ \downarrow{\scriptstyle p_n \circ q_n} & & \downarrow{\scriptstyle \theta} \\ S^{[n]} \times S & \xrightarrow{\Delta} & S^{[n]} \times S^{[n]} \times S \times S \end{array} \tag{6.2}$$

and the normal bundle of $S^{[n]} \times S$ in $S^{[n]} \times S^{[n]} \times S \times S$ is $T(S^{[n]} \times S)$. Thus, the diagram (6.2) induces a morphism of short exact sequences:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ TS^{[n-1,n,n+1]} & \xrightarrow{s'} & (p_n \circ q_n)^* T_{S^{[n]} \times S} \\ \downarrow & & \downarrow \\ T\mathfrak{Y}|_{S^{[n-1,n,n+1]}} & \longrightarrow & (p_n \circ q_n)^* T_{S^{[n]} \times S^{[n]} \times S \times S}|_{\Delta} \\ \downarrow & & \downarrow \\ \mathcal{L}_1^{-1}\mathcal{L}_2 & \longrightarrow & (p_n \circ q_n)^* T_{S^{[n]} \times S} \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array} \tag{6.3}$$

where $N_{\mathfrak{Y}/\Delta_{\mathfrak{Y}}} \cong \mathcal{L}_1^{-1}\mathcal{L}_2$. The purpose of this subsection is to construct a split $\lambda : \mathcal{L}_1^{-1}\mathcal{L}_2 \to T\mathfrak{Y}|_{S^{n-1,n,n+1}}$ which is compatible with the splitting of tangent bundle (or cotangent bundle) in (6.1) and give an explicit formulation of s' in Proposition 6.6. First we recall the description of the tangent space of $S^{[n-1,n,n+1]}$ and $\mathfrak{Y}$ in Proposition 5.28 of [26]: we consider a closed point $t \in S^{[n-1,n,n+1]}$ corresponds to two short exact sequences of coherent sheaves on S:

$$0 \to \mathcal{I}_n \xrightarrow{i_n} \mathcal{I}_{n-1} \xrightarrow{j_n} k_x \to 0 \quad 0 \to \mathcal{I}_{n+1} \xrightarrow{i_{n+1}} \mathcal{I}_n \xrightarrow{j_{n+1}} k_x \to 0.$$

Let V be the vector space of pairs $\{(w_0, w_1) \in Ext^1(\mathcal{I}_{n-1}, \mathcal{I}_{n-1}) \oplus Ext^1(\mathcal{I}_n, \mathcal{I}_n)\}$, such that w_0, w_1 map to the same element of $Ext^1(\mathcal{I}_n, \mathcal{I}_{n-1})$. By the proof of Proposition 5.28 of [26], elements in V are in 1-1 correspondence with the commutative diagrams of short exact sequences:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_n & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{I}_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i_n} & & \downarrow & & \downarrow{\scriptstyle i_n} & & \\ 0 & \longrightarrow & \mathcal{I}_{n-1} & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{I}_{n-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j_n} & & \downarrow & & \downarrow{\scriptstyle j_n} & & \\ 0 & \longrightarrow & k_x & \longrightarrow & \mathcal{C} & \longrightarrow & k_x & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

and, thus, induce a morphism $dp_S^1 : V \to Ext^1(k_x,k_x)$. Let V' be the vector space of pairs

$$\{(w_1',w_2) \in Ext^1(\mathcal{I}_n,\mathcal{I}_n) \oplus Ext^1(\mathcal{I}_{n+1},\mathcal{I}_{n+1})\},$$

such that w_1',w_2 map to the same element of $Ext^1(\mathcal{I}_n,\mathcal{I}_{n+1})$. Then elements in V' are in 1-1 correspondence with the commutative diagrams of short exact sequences:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_{n+1} & \longrightarrow & \mathcal{A}' & \longrightarrow & \mathcal{I}_{n+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_n & \longrightarrow & \mathcal{B}' & \longrightarrow & \mathcal{I}_n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & k_x & \longrightarrow & \mathcal{C}' & \longrightarrow & k_x & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

and, thus, also induces a morphism $dp_S^2 : V' \to Ext^1(k_x,k_x)$.

Consider the natural morphism:

$$\hat{s}_1 : Hom(\mathcal{I}_n,k_x) \otimes Ext^1(k_x,\mathcal{I}_n) \to Ext^1(\mathcal{I}_n,\mathcal{I}_n)$$
$$\hat{s}_2 : Hom(\mathcal{I}_n,k_x) \otimes Ext^1(k_x,\mathcal{I}_n) \to Ext^1(k_x,k_x),$$

and the short exact sequences

$$Hom(\mathcal{I}_n,k_x) \xrightarrow{\hat{s}_1(-\otimes r_2)} Ext^1(\mathcal{I}_n,\mathcal{I}_n) \to Ext^1(\mathcal{I}_n,\mathcal{I}_{n-1})$$
$$Ext^1(k_x,\mathcal{I}_n) \xrightarrow{\hat{s}_1(r_1\otimes -)} Ext^1(\mathcal{I}_n,\mathcal{I}_n) \to Ext^1(\mathcal{I}_{n+1},\mathcal{I}_n).$$

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For any element $u \in Hom(\mathcal{I}_n,k_x)$, $(\hat{s}_1(u \otimes r_2),0) \in V$. Moreover, by diagram chasing (we left it to interested readers)

$$dp_S^1(\hat{s}_1(u \otimes r_2),0) = \hat{s}_2(u \otimes r_2). \tag{6.4}$$

Similarly, for any element $v \in Ext^1(k_x,\mathcal{I}_n)$, $(0,\hat{s}_1(r_1 \otimes v)) \in V'$ and

$$dp_S^2(0,\hat{s}_1(r_1 \otimes v)) = \hat{s}_2(r_1 \otimes v). \tag{6.5}$$

Lemma 6.4 (Proposition 5.28 of [26]). *The tangent space $T_t\mathfrak{Y}$ is the space of (w_0,w_1,w_1',w_2,u,v) in*

$$Ext^1(\mathcal{I}_{n-1},\mathcal{I}_{n-1}) \oplus Ext^1(\mathcal{I}_n,\mathcal{I}_n) \oplus Ext^1(\mathcal{I}_n,\mathcal{I}_n) \oplus Ext^1(\mathcal{I}_{n+1},\mathcal{I}_{n+1}) \oplus Ext^1(k_x,k_x) \oplus Ext^1(k_x,k_x),$$

such that

(1) (w_0,w_1) *and* (w_0,w_1') *are in* V.
(2) (w_1,w_2) *and* (w_1',w_2) *are in* V'.
(3) $dp_S^1(w_0,w_1) = dp_S^2(w_1',w_2) = u$,
(4) $dp_S^1(w_0,w_1') = dp_S^2(w_1,w_2) = v$.

The tangent space $T_tS^{[n-1,n,n+1]}$ consists of subspace of $T_t\mathfrak{Y}$, such that $w_1 = w_1'$ and $u = v$.

Corollary 6.5. *The morphism $\lambda_t : T_t\mathfrak{Y} \to T_t\mathfrak{Y} : (w_0,w_1,w_1',w_2,u,v) \to (0,w_1'-w_1,w_1'-w_1,0,u-v,v-u)$ induces a splitting $\lambda : \mathcal{L}_1^{-1}\mathcal{L}_2 \to T\mathfrak{Y}|_{S^{[n-1,n,n+1]}}$ which is compatible with (6.1). Moreover, under the splitting λ, $\mathcal{L}_1\mathcal{L}_2^{-1}|_t$ is*

$$\{(w,v) \in Ext^1(\mathcal{I}_n,\mathcal{I}_n) \oplus Ext^1(k_x,k_x) | (0,w) \in V, (w,0) \in V', dp_S^1(0,w) = dp_S^2(w,0) = u\}.$$

Proof. If (w_0,w_1,w_1',w_2,u,v) is in the tangent space, then so is

$$\frac{(2w_0,w_1+w_1',w_1+w_1',2w_2,u+v,u+v)}{2} \text{ and } (0,w_1'-w_1,w_1-w_1',0,u-v,v-u).$$

Hence, the tangent space $T_t\mathfrak{Y}$ decomposes into a direct sum of two subspaces: the subspace $w_1 = w_1'$, which is $T_tS^{[n-1,n,n+1]}$, and the subspace N, which consists of elements $(w_1,u) \in Ext^1(\mathcal{I}_n,\mathcal{I}_n) \oplus Ext^1(k_x,k_x)$, such that

(1) w_1 maps to 0 in $Ext^1(\mathcal{I}_n,\mathcal{I}_{n-1})$ and $Ext^1(\mathcal{I}_{n+1},\mathcal{I}_n)$,
(2) $dp_S^1(w_1,0) = dp_S^2(0,w_1) = u$.

We define the morphism λ_t just to be the projection to N, and it induces a splitting $\lambda : \mathcal{L}_1^{-1}\mathcal{L}_2 \to T\mathfrak{Y}|_{S^{[n-1,n,n+1]}}$. □

Now we give an explicit description of s'. We first construct two canonical morphisms:

$$s_1 : \mathcal{L}_1^{-1}\mathcal{L}_2 \to (p_n \circ q_n)^*TS^{[n]} \quad s_2 : \mathcal{L}_1^{-1}\mathcal{L}_2 \to (p_n \circ q_n)^*(TS). \tag{6.6}$$

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Let $\pi : S^{[n-1,n,n+1]} \times S \to S^{[n-1,n,n+1]}$ be the projection morphism and Γ be the graph of the projection map to S. Then there exists short exact sequences:

$$0 \to \mathcal{I}_{n+1} \to \mathcal{I}_n \to \pi^*\mathcal{L}_1 \otimes \mathcal{O}_\Gamma \to 0 \quad 0 \to \mathcal{I}_n \to \mathcal{I}_{n-1} \to \pi^*\mathcal{L}_2 \otimes \mathcal{O}_\Gamma \to 0$$

which induce two global sections:

$$r_1 : \mathcal{L}_1^{-1} \to \mathcal{H}om_\pi(\mathcal{I}_n,\mathcal{O}_\Gamma), \quad r_2 : \mathcal{L}_2 \to \mathcal{E}xt^1_\pi(\mathcal{O}_\Gamma,\mathcal{I}_n).$$

Let $\mathrm{proj}_n : S^{[n]} \times S \to S^{[n]}$ be the projection map and Γ_n be the graph of the projection map from $S^{[n]} \times S \to S$. Then

$$\begin{aligned}(p_n \circ q_n)^* TS^{[n]} &= (p_n \circ q_n)^* \mathcal{E}xt^1_{\mathrm{proj}_n}(\mathcal{I}_n,\mathcal{I}_n) \\ &= \mathcal{E}xt^1_\pi(\mathcal{I}_n,\mathcal{I}_n) \\ (p_n \circ q_n)^* TS &= \mathcal{E}xt^1_\pi(\mathcal{O}_\Gamma,\mathcal{O}_\Gamma)\end{aligned}$$

by the flat base change theorem. With the composition of the following two natural homomorphisms:

$$\begin{aligned}\mathcal{E}xt^1_\pi(\mathcal{O}_\Gamma,\mathcal{I}_n) \otimes \mathcal{H}om_\pi(\mathcal{I}_n,\mathcal{O}_\Gamma) \to \mathcal{E}xt^1_\pi(\mathcal{I}_n,\mathcal{I}_n) = (p_n \circ q_n)^* TS^{[n]} \\ \mathcal{H}om_\pi(\mathcal{I}_n,\mathcal{O}_\Gamma) \otimes \mathcal{E}xt^1_\pi(\mathcal{O}_\Gamma,\mathcal{I}_n) \to \mathcal{E}xt^1_\pi(\mathcal{O}_\Gamma,\mathcal{O}_\Gamma) = (p_n \circ q_n)^* TS,\end{aligned}$$

we get the two canonical morphisms in (6.6)

$$s_1 : \mathcal{L}_1^{-1}\mathcal{L}_2 \to (p_n \circ q_n)^* TS^{[n]} \quad s_2 : \mathcal{L}_1^{-1}\mathcal{L}_2 \to (p_n \circ q_n)^*(TS).$$

Let

$$s = (s_1,s_2) : \mathcal{L}_1^{-1}\mathcal{L}_2 \to (p_n \circ q_n)^* T(S^{[n]} \times S).$$

Proposition 6.6. *The morphism s' in (6.3) coincides with s in (6.6).*

Proof. A closed point t on $S^{[n-1,n,n+1]}$ corresponds to two short exact sequences of coherent sheaves on S:

$$0 \to \mathcal{I}_n \xrightarrow{i_n} \mathcal{I}_{n-1} \xrightarrow{j_n} k_x \to 0 \quad 0 \to \mathcal{I}_{n+1} \xrightarrow{i_{n+1}} \mathcal{I}_n \xrightarrow{j_{n+1}} k_x \to 0,$$

which induces $r_1 \in Hom(\mathcal{I}_n,k_x)$ and $r_2 \in Ext^1(k_x,\mathcal{I}_n)$. The image of s is $(\hat{s}_1(r_1 \otimes r_2),\hat{s}_2(r_1 \otimes r_2))$, and we only need to prove that it is also the image of N in the proof of Corollary 6.5. It follows from the fact that

$$dp^1_S(\hat{s}_1(r_1 \otimes r_2),0) = dp^2_S(0,\hat{s}_1(r_1 \otimes r_2)) = \hat{s}_2(r_1 \otimes r_2)$$

by (6.4), (6.5). □

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6.2. The definition of $\mathfrak{F}^{+}_{m,k,r}$,$\mathfrak{H}^{\pm}_{m,k}$

We recall the Cartesian diagram (6.2)

$$\begin{array}{ccc} S^{[n-1,n,n+1]} & \xrightarrow{\Delta_{\mathfrak{Y}}} & \mathfrak{Y} \\ \downarrow{\scriptstyle p_n\circ q_n} & & \downarrow{\scriptstyle \theta} \\ S^{[n]}\times S & \xrightarrow{\Delta} & S^{[n]}\times S^{[n]}\times S\times S. \end{array}$$

Consider the dual of s

$$s^{\vee} : (p_n\circ q_n)^{*}T^{*}(S^{[n]}\times S)\to \mathcal{L}_1\mathcal{L}_2^{-1}$$

and the splitting $\lambda^{\vee}$ in Corollary 6.5. We define

$$\tau^k_m : \mathbf{R}(p_n\circ q_n)_*\mathcal{L}_1^{m-1-k}\mathcal{L}_2^{k}\otimes^{L} T^{*}(S^{[n]}\times S)\to \mathbf{R}(p_n\circ q_n)_*\mathcal{L}_1^{m-k}\mathcal{L}_2^{k-1} \tag{6.7}$$

as $\mathbf{R}(p_n\circ q_n)_*(s^{\vee}\otimes\mathcal{L}_1^{m-k-1}\mathcal{L}_2^{k})$.

On the other hand, we recall the definition of Definition 5.2

$$a^k_{m,r}=\mathbf{R}\theta_*(\mathcal{L}_1'^{m-r}\mathcal{L}_1^{r-1}\mathcal{O}(k\Delta_{\mathfrak{Y}}))[1].$$

By Lemma C.8, the splitting λ identify an isomorphism between the first order infinitesimal neighborhood of $\Delta_{\mathfrak{Y}}$ in $\mathfrak{Y}$ with $Spec_{S^{[n-1,n,n+1]}}(\mathcal{O}_{S^{[n-1,n,n+1]}}\oplus\mathcal{L}_1\mathcal{L}_2^{-1})$. We denote $\alpha^k_{m,r}$ as the Bass-Quillen class (see Appendix C.3 for the definition) of $a^k_{m,r}$ when restricting to $Spec_{S^{[n-1,n,n+1]}}(\mathcal{O}_{S^{[n-1,n,n+1]}}\oplus\mathcal{L}_1\mathcal{L}_2^{-1})$.

By Lemma 3.9,

$$Rq_{n*}(\mathcal{L}_1^{m-1})=\{\mathcal{L}^m\to\mathcal{L}^{m-1}\}$$

is the cone of $\mathcal{L}^m$ to $\mathcal{L}^{m-1}$. Hence, we have exact triangles

$$Rp_{n*}\mathcal{L}^m\to Rp_{n*}\mathcal{L}^{m-1}\xrightarrow{\mu_m} R(p_n\circ q_n)_*(\mathcal{L}_1^{m-1})\xrightarrow{\nu_m} Rp_{n*}\mathcal{L}^m[1]. \tag{6.8}$$

Definition 6.7. When $m\geq k>0$, we define

$$\mathfrak{H}^{+}_{m,k} : h^{+}_{m,k}\otimes T^{*}(S^{[n]}\times S)\to h^{+}_{m,k-1}$$

by

$$\mathfrak{H}^{+}_{m,k}=\begin{cases}\tau_{m,k} & k>1\\ \nu_m\circ\tau_{m,0} & k=1,\end{cases} \tag{6.9}$$

and when $m<k\leq 0$, we define

$$\mathfrak{H}^{-}_{m,k} : h^{-}_{m,k}\otimes T^{*}(S^{[n]}\times S)\to h^{-}_{m,k-1}$$

by

$$\mathfrak{H}^{-}_{m,k}=\begin{cases}\tau_{m,k} & k<0\\ \tau_{m,0}\circ\mu_m & k=0.\end{cases} \tag{6.10}$$

Definition 6.8. When $m \geq k > 0$ and r is an integer, we define

$$\mathfrak{F}^{+}_{m,k} : h^{+}_{m,k} \otimes T^{*}(S^{[n]} \times S) \to h^{+}_{m,k-1}$$

by

$$\mathfrak{F}^{+}_{m,k} = \begin{cases} \alpha_{m,k} & k > 1 \\ \nu_m \circ \alpha_{m,0} & k = 1, \end{cases} \tag{6.11}$$

and when $m < k \leq 0$, we define

$$\mathfrak{F}^{-}_{m,k} : h^{-}_{m,k} \otimes T^{*}(S^{[n]} \times S) \to h^{-}_{m,k-1}$$

by

$$\mathfrak{F}^{-}_{m,k} = \begin{cases} \alpha_{m,k} & k < 0 \\ \alpha_{m,0} \circ \mu_m & k = 0. \end{cases} \tag{6.12}$$

Proof of Proposition 6.3. We only compute the postive part of the extension and only consider the $k > 1$ and the case $k = 1$ follows from the (5.5). The extension class is induced from the short exact sequence on $\mathfrak{Y}$:

$$0 \to \mathcal{L}_1 \mathcal{L}_2^{-1} a^r_{m,k} \mathcal{O}_{\Delta_{\mathfrak{Y}}} \to a^r_{m,k} \mathcal{O}_{2\Delta_{\mathfrak{Y}}} \to a^r_{m,k} \mathcal{O}_{\Delta_{\mathfrak{Y}}} \to 0,$$

where $\mathcal{O}_{2\Delta_{\mathfrak{Y}}}$ is the cokernel of $\mathcal{O}(-2\Delta_{\mathfrak{Y}}) \to \mathcal{O}_{\mathfrak{Y}}$. Hence, we obtain the extension formula through Theorem C.11. □

Appendix A. Exterior powers and formal series

Definition A.1. Let V be a locally free sheaf over X, we define the exterior powers of V by

$$\wedge^{\bullet}(xV) = \sum_{i=0}^{\infty} (-x)^i [\wedge^i V], \quad \wedge^{\bullet}(-xV) = \sum_{i=0}^{\infty} x^i [S^i V]$$

as elements in $K(X)[[x]]$.

Definition A.2. For a two term complex of locally free sheaves

$$U := \{W \xrightarrow{\mathfrak{u}} V\},$$

we define

$$\wedge^{\bullet}(-xU) = \wedge^{\bullet}(xW) \wedge^{\bullet}(-xV) \qquad \wedge^{\bullet}(xU) = \wedge^{\bullet}(xV) \wedge^{\bullet}(-xW).n$$

$[S^k(U)]$ and $[\wedge^k(U)]$ are the x^k coefficients of $\wedge^{\bullet}(-xU)$ and $\wedge^{\bullet}(xU)$, respectively.

Lemma A.3 (Exercise II.5.16 of [12]). *For a two term complex of locally free sheaves*

$$U := \{W \xrightarrow{\mathfrak{u}} V\},$$

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we have

$$\wedge^\bullet(xU)\wedge^\bullet(-xU)=1. \tag{A.1}$$

Definition A.4 (Section 3.8 of [24]). For any nonnegative integer m, we define $h_m^\pm \in K(S^{[n]}\times S)$

$$h_m^+ := [\omega_S]\sum_{i=0}^{m}(-1)^{m-i}[\omega_S^{-i}][S^i\mathcal{I}_n][\wedge^{m-i}\mathcal{I}_n] \tag{A.2}$$

$$h_m^- := \sum_{i=0}^{m}(-1)^{i}[\omega_S^{i}][\wedge^i\mathcal{I}_n^\vee][S^{m-i}\mathcal{I}_n^\vee], \tag{A.3}$$

where we abuse the notation to denote

$$\mathcal{I}_n := \{W_n \xrightarrow{s} V_n\}$$

in the short exact sequence (2.1).

Lemma A.5. *Definitions A.4 and 2.4 are equivalent.*

Proof. First note that

$$\sum_{i=0}^{m}[\wedge^i\mathcal{I}_n][S^{m-i}\mathcal{I}_n]=0 \tag{A.4}$$

by (A.1). Thus

$$\begin{aligned}
&[\omega_S]\sum_{i=0}^{m}(-1)^{m-i}[\omega_S^{-i}][S^i\mathcal{I}_n][\wedge^{m-i}\mathcal{I}_n]\\
&=[\omega_S]\sum_{i=1}^{m}(-1)^{m-i}([\omega_S^{-i}]-1)[S^i\mathcal{I}_n][\wedge^{m-i}\mathcal{I}_n] && \text{by (A.4)}\\
&=([\omega_S]-1)\sum_{i=1}^{m}(-1)^{m-i}[S^i\mathcal{I}_n][\wedge^{m-i}\mathcal{I}_n]\sum_{j=1}^{i}[\omega_S^{-j+1}]\\
&=([\omega_S]-1)\sum_{j=1}^{m}[\omega_S^{-j+1}]\sum_{i=j}^{m}(-1)^{m-i}[S^i\mathcal{I}_n][\wedge^{m-i}\mathcal{I}_n]\\
&=(1-[\omega_S])\sum_{j=0}^{m-1}[\omega_S^{-j}]\sum_{i=0}^{j}(-1)^{i}[S^{m-i}\mathcal{I}_n][\wedge^{i}\mathcal{I}_n]. && \text{by (A.4)}
\end{aligned}$$

Hence, h_m^+ are equivalent in two definitions.

Then we note that

$$\sum_{i=0}^{m}(-1)^i[S^i\mathcal{I}_n^\vee][\wedge^{m-i}\mathcal{I}_n^\vee]=0 \tag{A.5}$$

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by (A.1). Thus

$$\begin{aligned}
&\sum_{i=0}^{m}(-1)^i[\omega_S^i][\wedge^i\mathcal{I}_n^\vee][S^{m-i}\mathcal{I}_n^\vee] \\
&=\sum_{i=1}^{m}(-1)^i([\omega_S^i]-1)[\wedge^i\mathcal{I}_n^\vee][S^{m-i}\mathcal{I}_n^\vee] && \text{by (A.5)}\\
&=([\omega_S]-1)\sum_{i=1}^{m}(-1)^i[\wedge^i\mathcal{I}_n^\vee][S^{m-i}\mathcal{I}_n^\vee]\sum_{j=1}^{i}[\omega_S^{j-1}] \\
&=([\omega_S]-1)\sum_{j=1}^{m}[\omega_S^{j-1}]\sum_{i=j}^{m}(-1)^i[\wedge^i\mathcal{I}_n^\vee][S^{m-i}\mathcal{I}_n^\vee] \\
&=(1-[\omega_S])\sum_{j=0}^{m-1}(-1)^j[\omega_S^j]\sum_{i=0}^{j}(-1)^i[\wedge^{m-i}\mathcal{I}_n^\vee][S^i\mathcal{I}_n^\vee]. && \text{by (A.5)}
\end{aligned}$$

Hence, h_m^- are equivalent in two definitions. □

Appendix B. Singularity of the minimal model program

In this section, we will review the singularities in the minimal model program from [17, 18]. We use the notation D to replace Δ in [17, 18], as Δ is already used to denote the diagonal embedding in our paper. For a normal variety, we will denote K_X the canonical Weil divisor. We will denote by ω_X the dualising sheaf when X is Cohen-Macaulay. They coincide when X is Gorenstein.

B.1. Discrepancy and classification of singularities

Definition B.1 (Definition 2.25 of [18] or Definition 2.4 of [17], Discrepancy). Let (X,D) be a pair, where X is a normal variety and $D=\sum a_iD_i, a_i\in\mathbb{Q}$ is a sum of distinct prime divisors. Assume that $m(K_X+D)$ is Cartier for some $m>0$. Suppose $f:Y\to X$ is a birational morphism from a normal variety Y. Let $E\subset Y$ denote the exceptional locus of f and $E_i\subset E$ the irreducible exceptional divisors. The two line bundles

$$\mathcal{O}(m(K_Y+f^{-1}D))|_{Y-E} \text{ and } f^*\mathcal{O}_X(m(K_X+D))|_{Y-E}$$

are naturally isomorphic. Thus, there are rational numbers $a(E_i,X,D)$, such that $ma(E_i,X,D)$ are integers and

$$\mathcal{O}_Y(m(K_Y+f^{-1}D))\cong f^*\mathcal{O}_X(m(K_X+D))\otimes\mathcal{O}_Y(\sum_i ma(E_i,X,D)E_i),$$

$a(E_i,X,D)$ is called the discrepancy of E_i with respect to (X,D). We define the centre of E in X by

$$centre_X(E):=f(E).$$

When $D=0$, then $a(E_i,X,D)$ depends only on E_i but not on f.

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Lemma B.2 (Lemma 2.29 of [18]). *Let X be a smooth variety and $D = \sum a_i D_i$ a sum of distinct prime divisors. Let $Z \subset X$ be a closed subvariety of codimension k. Let $p: Bl_Z X \to X$ be the blow up of Z and $E \subset Bl_Z X$ the irreducible component of the exceptional divisor which dominates Z, (if Z is smooth, then this is the only component). Then*

$$a(E,X,D) = k - 1 - \sum_i a_i mult_Z D_i,$$

where $mult_Z D_i$ is the multiplicity of D_i in Z.

Definition B.3 (Definition 2.34 and 2.37 of [18], or Definition 2.8 of [17]). Let (X,D) be a pair, where X is a normal variety and $D = \sum a_i D_i$ is a sum of distinct prime divisors, where $a_i \in \mathbb{Q}$ and $a_i \leq 1$. Assume that $m(K_X + D)$ is Cartier for some $m > 0$. We say that (X,D) is

$$\begin{cases} terminal \\ canonical \\ klt \\ plt \\ dlt \\ lc \end{cases} \quad \text{if } a(E,X,D) \text{ is } \begin{cases} > 0, \text{ for every exceptional } E \\ \geq 0, \text{ for every exceptional } E \\ > -1, \text{ for every } E \\ > -1, \text{ for every exceptional } E \\ > -1, \text{ if } centre_X E \subset \text{non-snc}(X,D) \\ \geq -1. \text{ for every } E. \end{cases}$$

Here klt is short for 'Kawamata log terminal', plt for 'pure log terminal' and lc for 'log canonical'. Above, non-snc(X,D) denotes the set of points where (X,D) is not simple normal crossing (snc for short). We say that X is terminal (canonical, etc.) if and only if $(X,0)$ is terminal (canonical, etc.).

Each class contains the previous one, except canonical does not imply klt if D contains a divisor with coefficient 1.

Theorem B.4 (Theorem 5.50 of [18], or Theorem 4.9 of [17], Inversion of adjunction). *Let X be normal and $S \subset X$ a normal Weil divisor which is Cartier in codimension 2. Let B be an effective $\mathbb{Q}$-divisor, and assume that $K_X + S + B$ is $\mathbb{Q}$-Cartier. Then $(X, S+B)$ is plt near S iff $(S, B|_S)$ is klt.*

B.2. Rational singularities

Definition B.5 (Definition 5.8 of [18]). Let X be a variety over a field of characteristic 0. We say that X is a rational singularity if there exists a resolution of singularities $f: Y \to X$, such that

(1) $f_* \mathcal{O}_Y = \mathcal{O}_X$ (equivalently, X is normal) and
(2) $\mathbf{R}^i f_* \mathcal{O}_Y = 0$ for $i > 0$.

Remark B.6. By Theorem 5.10 of [18], if X is a rational singularity, then for all resolution of singularities $f: Y \to X$,

(1) $f_* \mathcal{O}_Y = \mathcal{O}_X$ and
(2) $\mathbf{R}^i f_* \mathcal{O}_Y = 0$ for $i > 0$.

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Theorem B.7 (Theorem 5.22 of [18]). *Let X be a normal variety over a field of characteristic* 0*. If X is a canonical singularity, then X is a rational singularity. If X is Gorenstein, then X is a canonical singularity if X is a rational singularity.*

Here are also some examples of rational singularities:

Example B.8. By [33], the universal closed subscheme $\mathcal{Z}_n$ is a rational singularity.

Example B.9. The morphism $q_n : S^{[n-1,n,n+1]} \to S^{[n,n+1]}$ factors through $p_n^{-1}\mathcal{Z}_n$ and induces a resolution of $p_n^{-1}\mathcal{Z}_n$. By Lemma 3.9 and Corollary 3.6, $p_n^{-1}\mathcal{Z}_n$ is a Gorenstein rational singularity and, thus, is a canonical singularity by Theorem B.7.

B.3. Semi-dlt pairs

Definition B.10 (Definition 1.10 of [17], Semi-snc pairs). Let W be a regular scheme and $\sum_{i\in I} E_i$ a snc divisor on W. Write $I = I_Y \cup I_D$ as a disjoint union. Set $Y := \sum_{i\in I_V} E_i$ as a subscheme of W and $D_Y := \sum_{i\in I_D} a_i E_i|_Y$ as a divisor on Y for some $a_i \in \mathbb{Q}$. We call (Y,D_Y) an embedded semi-snc pair. A pair (X,D) is called semi-snc if it is Zariski locally isomorphic to an embedded semi-snc pair.

Example B.11. We have the following three examples of semi-snc pairs (X,D):

(1) $X = \{z = 0\} \subset \mathbb{A}^3$ and $D = a_x(x|_X = 0) + a_y(y|_X = 0)$.
(2) $X = \{yz = 0\} \subset \mathbb{A}^3$ and $D = a_x(x|_X = 0)$.
(3) $X = \{xyz = 0\} \subset \mathbb{A}^3$ and $D = 0$.

Definition B.12 (Definition 5.1 of [17], Demi-normal schemes). A scheme X is called demi-normal if it satisfies the Serre condition S_2 (see [34, Tag 033P] for the definition of the Serre condition) and codimension 1 points are either regular points or nodes. Here we say a scheme X has a node at a point $x \in X$ if its local ring $\mathcal{O}_{x,X}$ can be written as $R/(f)$, where (R,m) is a regular local ring of dimension 2, $f \in m^2$ and f is not a square in m^2/m^3.

Definition B.13 (Section 5.2 of [17], conductor). Let X be a reduced scheme and $\pi : \bar{X} \to X$ its normalisation. The conductor ideal

$$cond_X := \mathcal{H}om(\pi_*\mathcal{O}_{\bar{X}}, \mathcal{O}_X) \subset \mathcal{O}_X$$

is the largest ideal sheaf on X that is also an ideal sheaf on $\bar{X}$. We write it as $cond_{\bar{X}}$ when we view the conductor as an ideal sheaf on $\bar{X}$. The conductor subschemes are defined as

$$T := Spec_X(\mathcal{O}_X/cond_X) \qquad \bar{T} := Spec_{\bar{X}}(\mathcal{O}_{\bar{X}}/cond_{\bar{X}}).$$

Definition B.14 (Definition-Lemma 5.10 of [17]). Let X be a demi-normal scheme with normalisation $\pi : \bar{X} \to X$ and conductors $T \subset X$ and $\bar{T} \subset \bar{X}$. Let D be an effective $\mathbb{Q}$-divisor whose support does not contain any irreducible components of T and $\bar{D}$ the divisorial part of $\pi^{-1}(D)$. The pair (X,D) is called semi log canonical or slc if

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(1) $K_X + D$ is $\mathbb{Q}$-Cartier and
(2) $(\bar{X}, \bar{D} + \bar{T})$ is lc.

Definition B.15 (Definition 5.19 of [17]). An slc pair (X,D) is semidivisorial log terminal or semi-dlt if $a(E,X,D) > -1$ for every exceptional divisor E over X, such that (X,D) is not snc at the generic point of $centre_X E$.

Example B.16. A semi-snc pair (X,D) is always semi-dlt.

Proposition B.17 (Proposition 5.20 of [17]). *Let (X,D) be a demi-normal pair over a field of characteristic 0. Assume that the normalisation $(\bar{X}, \bar{T} + \bar{D})$ is dlt and there is a codimension 3 set $W \subset X$, such that $(X \backslash W, D|_{X \backslash W})$ is semi-dlt. Then*

(1) *the irreducible components of X are normal,*
(2) *$K_X + D$ is $\mathbb{Q}$-Cartier and*
(3) *(X,D) is semi-dlt.*

Appendix C. The Bass-Quillen class and HKR isomorphism

C.1. The extension of a scheme by a coherent sheaf

Given a scheme X, let $Coh(X)$ be the abelian category of coherent sheaves on X. Let X_{fi} be the category of schemes with finite morphisms to X. Given a coherent sheaf $M \in Coh(X)$, we consider the scheme $X_M := Spec_X(\mathcal{O}_X \oplus M)$, where $\mathcal{O}_X \oplus M$ is regarded as an $\mathcal{O}_X$ algebra, such that the multiplication on M is 0. We have the contravariant functor:

$$\begin{aligned} X_{-} : Coh(X) &\to X_{fi} \\ M &\to X_M := Spec_X(\mathcal{O}_X \oplus M). \end{aligned}$$

The functor X_{-} maps surjective morphisms to closed embeddings. For every coherent sheaf M, we denote the closed embedding $i_M : X \to X_M$ as the image of $M \to 0$ under the functor X_{-} and the projection morphism $pr_M : X_M \to X$ as the image of $0 \to M$ under the functor X_{-}. We have $pr_M \circ i_M = id$.

Remark C.1. One should notice that the scheme X_M and X have the same topological space, and pr_M and j_M are identity at the level of topological spaces. The difference between X_M and X are their structure sheaves, where $\mathcal{O}_{X_M} \cong \mathcal{O}_X \oplus M$.

For the category of coherent sheaves, we have

$$Coh(X_M) = \{(F,d) | F \in Coh(X), d \in Hom_X(F \otimes M, F), (M \otimes d) \circ d = 0\}. \tag{C.1}$$

The ideal sheaf of X in X_M is $(M,0)$, which is square zero. The functors j_{M*}, j_M^*, i_{M*}, i_M^* between the category of coherent sheaves could be represented by:

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$$\begin{aligned} j_{M*}: Coh(X_M) \to Coh(X) &\quad (F,d) \to F \\ i_{M*}: Coh(X) \to Coh(X_M) &\quad F \to (F,0) \\ j_M^*: Coh(X) \to Coh(X_M) &\quad F \to (F \oplus F \otimes M, d_F) \\ i_M^*: Coh(X_M) \to Coh(X) &\quad (F,d) \to coker(d), \end{aligned}$$

where d_F is induced by the following transition matrix from $F \otimes M^2 \oplus F \otimes M$ to $F \otimes M \oplus F$

$$d_F := \begin{pmatrix} 0 & id \\ 0 & 0 \end{pmatrix}.$$

Lemma C.2. *Given a morphism $f: X \to Y$ and $M \in Coh(X)$, we have*

$$\{g \in Hom(X_M, Y) | g \circ i_M = f\} = Hom_X(f^*\Omega_Y, M),$$

where Ω_Y is the sheaf of differentials on Y.

Proof. At the level of topological spaces, f and g are the same. Let $f^\#: \mathcal{O}_Y \to f_*\mathcal{O}_X$ be the induced sheaf of rings homomorphism of structure sheaves. Then any $g \in Hom(X_M, Y)$, such that $g \circ i_M = f$ is in one-to-one correspondence with morphisms of $\mathcal{O}_Y$-algebras $(f^\#, h^\#): \mathcal{O}_Y \to f_*\mathcal{O}_X \oplus f_*M$. Thus, $h^\# \in Der_{\mathcal{O}_Y}(\mathcal{O}_Y, f_*M) = Hom_Y(\Omega_Y, f_*M) = Hom_X(f^*\Omega_Y, M)$ by the adjunction formula. □

C.2. The HKR isomorphism on the first order neighborhood

Let M be a locally free sheaf on X. Thus, X_M is the first order infinitesimal neighborhood of X in the total space of $M^\vee$. The projection morphism j_M is flat, since $\mathcal{O}_X \oplus M$ is also locally free as a $\mathcal{O}_X$ module. We recall the description of $Coh(X_M)$ in terms of coherent sheaves on X by (C.1). Then $\mathcal{O}_{X_M} \cong (\mathcal{O}_X \oplus M, d_{\mathcal{O}_X})$ and $j_M^* M \cong (M \oplus M \otimes M, d_M)$. Moreover, $d_{\mathcal{O}_X}$ induces a $\mathcal{O}_{X_M}$ morphism between $j_M^* M$ and $\mathcal{O}_{X_M}$, which we denote as s_M. We consider the following complex:

$$K_\bullet^{\otimes M}: \cdots \to j_M^* M^{\otimes 2} \to j_M^* M \to \mathcal{O}_{X_M} \to 0$$

induced by the contraction morphism of s_M. The complex $K_\bullet^{\otimes M}$ is a resolution of $i_{M*}(\mathcal{O}_X)$ and, hence, induces a canonical isomorphism

$$t_M^F: \mathbf{R}i_{M*}F \cong \mathbf{L}j_M^* F \otimes_{X_M}^{\mathbf{L}} K_\bullet^{\otimes M}$$

for any element $F \in D^u(X)$. The isomorphism t_M^F is functorial with respect to $F \in D^u(X)$. We notice that

$$\mathbf{L}i_M^*(\mathbf{L}j_M^* F \otimes_{X_M}^{\mathbf{L}} K_\bullet^{\otimes M}) \cong F \otimes^{\mathbf{L}} \bigoplus_{k=0}^{\infty} M^{\otimes k}[k].$$

Hence, by the adjunction formula, we induce a canonical morphism:

$$b_M^F: \mathbf{L}j_M^* F \otimes_{X_M} K_\bullet^{\otimes M} \to \mathbf{R}i_{M*}(F \otimes_X^{\mathbf{L}} \bigoplus_{k=0}^{\infty} M^{\otimes k}[k]),$$

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such that the composition of the following morphisms

$$\mathbf{R}Hom_X(F\otimes^{\mathbf{L}}\bigoplus_{k=0}^{\infty}M^{\otimes k}[k],G)\xrightarrow{\mathbf{R}i_{M*}}\mathbf{R}Hom_{X_M}(\mathbf{R}i_{M*}(F\otimes^{\mathbf{L}}\bigoplus_{k=0}^{\infty}M^{\otimes k})[k],\mathbf{R}i_{M*}G)$$
$$\xrightarrow{\mathbf{R}Hom(b_M^F,-)}\mathbf{R}Hom_{X_M}(\mathbf{L}j_M^*F\otimes^{\mathbf{L}}K_\bullet^{\otimes M},\mathbf{R}i_{M*}G)$$

is an isomorphism for any $G\in D^u(X)$. It induces a canonical isomorphism

$$t_M^{F,G}:\mathbf{R}Hom_X(F\otimes^{\mathbf{L}}\bigoplus_{k=0}^{\infty}M^{\otimes k}[k],G)\cong\mathbf{R}Hom_{X_M}(\mathbf{R}i_{M*}F,\mathbf{R}i_{M*}G).$$

Now we prove that the isomorphism $t_M^{F,G}$ is compatible with the morphisms between schemes:

Lemma C.3. *Let $f:X\to Y$ be a projective morphism of schemes. Let M be a locally free sheaf on Y and $M':=f^*M$. Let $f_M:X_{M'}\to Y_M$ be the induced morphism from f. Then we have the following commutative diagram:*

$$\begin{array}{ccc}\mathbf{R}Hom_X(F\otimes^{\mathbf{L}}\bigoplus_{k=0}^{\infty}M'^{\otimes k}[k],G) & \xrightarrow{t_{M'}^{F,G}} & \mathbf{R}Hom_{X_{M'}}(\mathbf{R}i_{M'*}F,\mathbf{R}i_{M'*}G)\\ \downarrow{\mathbf{R}f_*} & & \downarrow{\mathbf{R}f_{M*}}\\ \mathbf{R}Hom_Y(\mathbf{R}f_*F\otimes^{\mathbf{L}}\bigoplus_{k=0}^{\infty}M^{\otimes k}[k],\mathbf{R}f_*G) & \xrightarrow{t_M^{\mathbf{R}f_*F,\mathbf{R}f_*G}} & \mathbf{R}Hom_{Y_M}(\mathbf{R}i_{M*}\mathbf{R}f_*F,\mathbf{R}i_{M*}\mathbf{R}f_*G).\end{array} \tag{C.2}$$

Proof. To prove the diagram (C.2), we need to prove that for any $F,G\in D^u(X)$,

$$\mathbf{R}f_{M*}\circ\mathbf{R}Hom(b_{M'}^F,-)\circ\mathbf{R}i_{M'*}=\mathbf{R}Hom(b_M^{\mathbf{R}f_*F},-)\circ\mathbf{R}i_{M*}\circ\mathbf{R}f_*. \tag{C.3}$$

from $\mathbf{R}Hom_X(F\otimes^{\mathbf{L}}\bigoplus_{k=0}^{\infty}M'^{\otimes k}[k],G)$ to $\mathbf{R}Hom_{Y_M}(\mathbf{R}i_{M*}\mathbf{R}f_*F,\mathbf{R}i_{M*}\mathbf{R}f_*G)$. We notice that Cartesian diagram:

$$\begin{array}{ccc}X_{M'} & \xrightarrow{j_{M'}} & X\\ \downarrow{f_M} & & \downarrow{f}\\ Y_M & \xrightarrow{j_M} & Y.\end{array}$$

Since j_M is flat, we have a canonical isomorphism $\mathbf{L}j_{M*}\circ\mathbf{R}f_*\cong\mathbf{R}f_{M*}\circ\mathbf{L}j_{M'}^*$ by the flat base change theorem, and, hence, $\mathbf{R}f_{M*}(b_{M'}^F)=b_M^{\mathbf{R}f_*F}$. By the Yoneda lemma, we have

$$\mathbf{R}f_{M*}\circ\mathbf{R}Hom(b_{M'}^F,-)=\mathbf{R}Hom(b_M^{\mathbf{R}f_*F},-)\circ\mathbf{R}f_*.$$

As $f_M\circ i_{M'}=i_M\circ f$, we have $\mathbf{R}i_{M*}\circ\mathbf{R}f_*\cong\mathbf{R}f_{M*}\circ\mathbf{R}i_{M'*}$ and, hence, get (C.3). □

Let $f:N\to M$ be a surjective morphism of locally free sheaves on X, and we denote $f^{\otimes k}:N^{\otimes k}\to M^{\otimes k}$. The morphism f induces a closed embedding $X_M\to X_N$, which we denote as $i_{N,M}$ (it is denoted as X_f in Appendix C.1).

Lemma C.4. *For any two elements $F,G \in D^u(X)$, we have the following commutative diagram:*

$$\begin{array}{ccc} \mathbf{R}Hom_X(F\otimes^{\mathbf{L}}\bigoplus_{k=0}^{\infty}M^{\otimes k}[k],G) & \xrightarrow{t_M^{F,G}} & \mathbf{R}Hom_{X_M}(\mathbf{R}i_{M*}F,\mathbf{R}i_{M*}G) \\ \big\downarrow \mathbf{R}Hom(F\otimes^{\mathbf{L}}(\bigoplus_{k=0}^{\infty}f^{\otimes k}),-) & & \big\downarrow \mathbf{R}i_{N,M*} \\ \mathbf{R}Hom_X(F\otimes^{\mathbf{L}}\bigoplus_{k=0}^{\infty}N^{\otimes k}[k],G) & \xrightarrow{t_N^{F,G}} & \mathbf{R}Hom_{X_N}(\mathbf{R}i_{N*}F,\mathbf{R}i_{N*}G). \end{array} \tag{C.4}$$

Proof. The morphism $f^{\otimes k}$ for all k canonically induces a morphism $\iota_{M,N}^F : \mathbf{L}j_M^* F\otimes K_\bullet^M \to \mathbf{R}i_{N,M*}(\mathbf{L}j_N^* F\otimes K_\bullet^N)$, which is a quasi-isomorphism. To prove (C.6), we need to prove that for and $F,G \in D^u(X)$,

$$\mathbf{R}Hom(\iota_{M,N}^F,-)\circ\mathbf{R}i_{N,M*}\circ\mathbf{R}Hom(b_M^F,-)\circ\mathbf{R}i_{M*} = \mathbf{R}Hom(b_N^F,-)\circ\mathbf{R}i_{N*}\circ\mathbf{R}Hom(F\otimes^{\mathbf{L}}(\bigoplus_{k=0}^{\infty}f^{\otimes k}),-), \tag{C.5}$$

as morphisms from $\mathbf{R}Hom_X(F\otimes^{\mathbf{L}}\bigoplus_{k=0}^{\infty}M^{\otimes k}[k],G)$ to $\mathbf{R}Hom_{X_N}(\mathbf{R}i_{N*}F,\mathbf{R}i_{N*}G)$. We notice that $i_{N,M}\circ i_M = i_N$, and, hence

$$\begin{aligned} &\mathbf{R}Hom(\iota_{M,N}^F,-)\circ\mathbf{R}i_{N,M*}\circ\mathbf{R}Hom(b_M^F,-)\circ\mathbf{R}i_{M*} = \\ &\mathbf{R}Hom(\iota_{M,N}^F,-)\circ\mathbf{R}Hom(\mathbf{R}i_{N,M*}(b_M^F),-)\circ\mathbf{R}i_{N,M*}\circ\mathbf{R}i_{M*} = \\ &\mathbf{R}Hom(\mathbf{R}i_{N,M*}(b_M^F)\circ\iota_{M,N}^F,-)\circ\mathbf{R}i_{N*} \end{aligned}$$

and

$$\begin{aligned} &\mathbf{R}Hom(b_N^F,-)\circ\mathbf{R}i_{N*}\circ\mathbf{R}Hom(F\otimes^{\mathbf{L}}(\bigoplus_{k=0}^{\infty}f^{\otimes k}),-) = \\ &\mathbf{R}Hom(b_N^F,-)\circ\mathbf{R}Hom(\mathbf{R}i_{N*}(F\otimes^{\mathbf{L}}(\bigoplus_{k=0}^{\infty}f^{\otimes k})),-)\circ\mathbf{R}i_{N*} = \\ &\mathbf{R}Hom(\mathbf{R}i_{N*}(F\otimes^{\mathbf{L}}(\bigoplus_{k=0}^{\infty}f^{\otimes k})\circ b_N^F,-)\circ\mathbf{R}i_{N*}. \end{aligned}$$

Hence, (C.5) follows from the identity that

$$\mathbf{R}i_{N*}(F\otimes^{\mathbf{L}}(\bigoplus_{k=0}^{\infty}f^{\otimes k})\circ b_N^F = \mathbf{R}i_{N,M*}(b_M^F)\circ\iota_{M,N}^F. \qquad \square$$

Combining Lemmas C.3 and C.4, we have

Corollary C.5. *Let $f : X \to Y$ be a projective morphism of varieties. Let M and N be locally free sheaves on X and Y, respectively, with a surjective morphism $g : f^*N \to M$. The morphism g induces a canonical morphism $g_{N,M} : X_M \to Y_N$. Then for any elements*

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$F,G \in D^u(X)$, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{R}Hom_X(F\otimes^{\mathbf{L}}\bigoplus_{k=0}^{\infty} M^{\otimes k}[k],G) & \xrightarrow{t_M^{F,G}} & \mathbf{R}Hom_{X_M}(\mathbf{R}i_{M*}F,\mathbf{R}i_{M*}G) \\ \Big\downarrow{\scriptstyle \mathbf{R}f_*\circ\mathbf{R}Hom(F\otimes^{\mathbf{L}}(\bigoplus_{k=0}^{\infty} g^{\otimes k}),-)} & & \Big\downarrow{\scriptstyle \mathbf{R}g_{N,M*}} \\ \mathbf{R}Hom_Y(\mathbf{R}f_*F\otimes^{\mathbf{L}}\bigoplus_{k=0}^{\infty} N^{\otimes k}[k],\mathbf{R}f_*G) & \xrightarrow{t_N^{\mathbf{R}f_*F,\mathbf{R}f_*G}} & \mathbf{R}Hom_{Y_N}(\mathbf{R}i_{N*}\mathbf{R}f_*F,\mathbf{R}i_{N*}\mathbf{R}f_*G). \end{array} \tag{C.6}$$

C.3. The Bass-Quillen classes and the HKR isomorphism

The Bass-Quillen class comes from the Bass-Quillen conjecture, which asks the following question: let M be a locally free sheaf on X and V be a locally free sheaf on $Tot_X(M^\vee)$. When is V the pullback of a locally free sheaf on X? While this question is always true if X is affine, an obstruction class would appear when X is not an affine variety.

In this subsection, we consider a similar but easier question: given a locally free sheaf V on X, classify all the locally free sheaves L on X_M, such that $i_{M*}L \cong V$.

Theorem C.6. *The locally free sheaves L on X_M, such that $i_M^*L \cong V$ are in one-to-one correspondence with elements in $Ext_X^1(V,V\otimes M)$.*

Proof. An extension class $h \in Ext_X^1(V,V\otimes M)$ induces a short exact sequence:

$$0 \to V\otimes M \xrightarrow{\alpha_h} L_0 \xrightarrow{\beta_h} V \to 0. \tag{C.7}$$

We define $d_L : L_0\otimes M \to L_0$ as $\alpha_h\circ(\beta_h\otimes M)$. Then $d_L\circ(d_L\otimes M) = 0$ and, hence, $L := (L_0,d_L)$ is also locally free on X_M.

On the other hand, given L on X_M, such that $i_{M*}L \cong V$, we have a short exact sequence:

$$0 \to i_{M*}(V\otimes M) \xrightarrow{\alpha_h'} L \xrightarrow{\beta_h'} i_{M*}V \to 0 \tag{C.8}$$

and, hence, a short exact sequence

$$0 \to V \otimes M \to j_{M*}L \to V \to 0$$

which induces the extension class $h \in Ext_X^1(V,V\otimes M)$. □

We say that the extension class $h \in Ext_X^1(V,V\otimes M)$ is the Bass-Quillen class of L, following the notation of [13]. Let M be a locally free sheaf on X. The short exact sequence (C.8) induces an extension class $h' \in Ext_{X_M}^1(i_{M*}V,i_{M*}(V\otimes M))$.

Lemma C.7. *The image of h' under the isomorphism*

$$t_M^{V\otimes M,V} : Ext_{X_M}^1(i_{M*}V,i_{M*}(V\otimes M)) \cong Hom_X(V,V)\oplus Ext_X^1(V,V\otimes M)$$

is (id,h).

Proof. We recall the description of $Coh(X_M)$ in the term of (C.1). We denote $L = (L_0,d_L)$ as the locally free sheaf on X_M generated by the extension class h. The morphism

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$(t_M^{V\otimes M,V})^{-1}(id,h)$ is induced by the composition of following morphisms of complexes (up to the inverse of quasi-isomorphisms)

$$\begin{array}{ccccc} 0 & \xleftarrow{0} & (V\otimes M,0) & \xrightarrow{\alpha_h} & (L_0,0) \\ \downarrow & & \downarrow (0,id) & & \downarrow \beta_h \\ (V,0) & \xleftarrow[id\oplus 0]{} & (V\oplus V\otimes M,d_V) & \xrightarrow{id\oplus 0} & (V,0) \end{array} \tag{C.9}$$

and the quasi-isomorphism

$$\begin{array}{ccc} (L_0,0) & \xleftarrow[\alpha_h]{} & (V\otimes M,0) \\ \downarrow \beta_h & & \downarrow \\ (V,0) & \longleftarrow & 0. \end{array}$$

We consider the morphism $d:(V\oplus L_0)\otimes M \to V\oplus L_0$ by the matrix

$$\begin{pmatrix} 0 & \alpha_h \\ 0 & 0 \end{pmatrix},$$

which makes $(V\oplus L_0,d)$ a coherent sheaf on X_M. We consider the following morphism of complexes (up to the inverse of quasi-isomorphisms):

$$\begin{array}{ccccc} 0 & \longrightarrow & (V\oplus L_0,d) & \xleftarrow[(0,id)]{} & (L_0,d) \\ \downarrow & & \downarrow (id\oplus 0,0\oplus\beta_h) & & \downarrow \beta_h \\ (V,0) & \xrightarrow{(id,0)} & (V,0)\oplus(V,0) & \xleftarrow[(0,id)]{} & (V,0) \end{array} \tag{C.10}$$

We recall the definition of morphism spaces in $D^u(X_M)$ by [34, Tag 04VB] and [34, Tag 05RN] and see that the diagrams in (C.9) and (C.10) represent the same morphism, as their compositions are in the same homotopy class. On the other hand, the morphism h' is induced by the following morphism of complexes (up to inverse of quasi-isomorphisms)

$$\begin{array}{ccccc} 0 & \longrightarrow & (L_0,d_L) & \xleftarrow[\alpha'_h]{} & (V\otimes M,0) \\ \downarrow & & \downarrow \beta'_h & & \downarrow \\ (V,0) & \xrightarrow{id} & (V,0) & \longleftarrow & 0. \end{array} \tag{C.11}$$

Hence, the equality of h' and $(t_M^{V\otimes M,V})^{-1}(id,h)$ follows from the following quasi-isomorphism

$$\begin{array}{ccc} (L_0,d_L) & \xrightarrow{(\beta_h,id)} & (V\oplus L_0,d) \\ \downarrow \beta'_h & & \downarrow (id\oplus 0,0\oplus\beta_h) \\ (V,0) & \xrightarrow{(0,id)} & (V,0)\oplus(V,0) \end{array}$$

□

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C.4. The splitting of the first order neighborhood

Let $f: X \to Y$ be a closed embedding of smooth varieties, and let $N_{Y/X}$ be the normal bundle of X in Y. On X, we have the short exact sequence:

$$0 \to N_{Y/X}^{\vee} \xrightarrow{\iota_{Y/X}} \Omega_Y|_X \to \Omega_X \to 0.$$

By Lemma C.2, the space

$$\{g \in Hom(X_{N_{Y/X}^{\vee}}, Y) | g \circ i_{N_{Y/X}^{\vee}} = f\} = Hom_X(\Omega_Y|_X, N_{Y/X}^{\vee}). \tag{C.12}$$

Let $\mathcal{I}_X$ be the ideal sheaf of X in Y and $X_{(1)}^Y := Spec_Y(\mathcal{O}_Y/\mathcal{I}_X^2)$ be the first order infinitesimal neighborhood of X in Y.

Lemma C.8. *Every $g \in Hom_X(\Omega_Y|_X, N_{Y/X}^{\vee})$, regarding as a homomorphism from $X_{N_{Y/X}^{\vee}}$ to Y, factors through a homomorphism in $Hom(X_{N_{Y/X}^{\vee}}, X_{(1)}^Y)$. Moreover, g induces an isomorphism between $X_{N_{Y/X}^{\vee}}$ and $X_{(1)}^Y$ if and only if $g \circ \iota_{Y/X}$ is an isomorphism of $N_{Y/X}^{\vee}$.*

Proof. The morphism g induces a homomorphism of $\mathcal{O}_Y$-algebras $g^{\#} = (f^{\#}, h^{\#}): \mathcal{O}_Y \to \mathcal{O}_X \oplus N_{Y/X}^{\vee}$, where $f^{\#}$ is the quotient morphism and $h^{\#}$ is the derivative induced by g. As $\mathcal{I}_X$ is the kernel of $f^{\#}$, the image of $\mathcal{I}_X^2$ in $g^{\#}$ is 0 and, hence, induces a homomorphism in $Hom(X_{N_{Y/X}^{\vee}}, X_{(1)}^Y)$. Moreover, we have the short exact sequence of $\mathcal{O}_Y$ modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_{Y/X}^{\vee} & \longrightarrow & \mathcal{O}_Y/\mathcal{I}_X^2 & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g \circ \iota_{Y/X}} & & \downarrow{\scriptstyle g^{\#}} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & N_{Y/X}^{\vee} & \longrightarrow & \mathcal{O}_X \oplus N_{Y/X}^{\vee} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0.
\end{array}$$

Hence, $g^{\#}$ induces an isomorphism between $\mathcal{O}_Y/\mathcal{I}_X^2$ and $\mathcal{O}_X \oplus N_{Y/X}^{\vee}$ if and only if $g \circ \iota_{Y/X}$ is an isomorphism. □

Lemma C.8 could be generalised to a relative version. We consider a Cartesian diagram of smooth varieties: Let $f': X \to Y$ be a regular embedding of smooth varieties. Let $\pi: Y' \to Y$ be a proper morphism of smooth varieties. Let $X' = Y' \times_Y X$ and $f: X' \to Y'$ and $\pi_X: X' \to X$ be the respective fibre morphisms. Moreover, we assume that X' is also smooth.

Lemma C.9. *There exists a canonical morphism of short exact sequences:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_X^* N_{Y/X}^{\vee} & \xrightarrow{\iota_{Y/X}} & \pi_X^* \Omega_Y|_X & \longrightarrow & \pi_X^* \Omega_X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle s_{f,\pi}} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_{Y'/X'}^{\vee} & \xrightarrow{\iota_{Y'/X'}} & \Omega_{Y'}|_{X'} & \longrightarrow & \Omega_{X'} & \longrightarrow & 0,
\end{array} \tag{C.13}$$

where the middle and right rows are the canonical morphisms of differentials. Moreover, a splitting of $\iota_{Y'/X'}$ and $\iota_{Y/X}$, which is compatible with (C.13), which we denote as g'

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and g, respectively, would induce a commutative diagram

$$\begin{array}{ccc} X'_{N^\vee_{Y'/X'}} & \xrightarrow{g'} & {X'}^{Y'}_{(1)} \\ \downarrow{\scriptstyle s_{f,\pi}} & & \downarrow \\ X_{N^\vee_{Y/X}} & \xrightarrow{g} & X^{Y}_{(1)} \end{array} \tag{C.14}$$

where g' and g are isomorphisms.

Finally, we explain the relation between the isomorphism $t_M^{F,G}$ and the HKR isomorphism. Given a locally free sheaf M on a scheme X and a positive integer k, there is a canonical shuffle morphism

$$sh_k : M^{\otimes k} \to \wedge^k(M)$$
$$v_1\otimes\cdots\otimes v_k \to v_1\wedge\cdots\wedge v_k.$$

Given a regular embedding of smooth varieties $f : X \to Y$, we denote $M = N^\vee_{Y/X}$. We fix a splitting $g \in Hom_X(\Omega_Y|_X, N^\vee_{Y/X})$, such that $g\circ\iota_{Y/X} = id$ and abuse the notation to denote g as the closed embedding from X_M to Y.

Theorem C.10 (HKR isomorphism, Theorem 1.4 and Section 1.11 of [1]). *Let $\tau_M^{F,G}$ be the composition of the following morphisms:*

$$\begin{aligned} \mathbf{R}Hom_X(F\otimes^{\mathbf{L}}\bigoplus_{k=0}^{\infty}\wedge^k M[k],G) &\xrightarrow{\mathbf{R}Hom_X(F\otimes^{\mathbf{L}} sh_k,-)} \mathbf{R}Hom_X(F\otimes^{\mathbf{L}}\bigoplus_{k=0}^{\infty}M^{\otimes k}[k],G) \\ &\xrightarrow{t_M^{F,G}} \mathbf{R}Hom_{X_M}(\mathbf{R}i_{M*}F,\mathbf{R}i_{M*}G) \\ &\xrightarrow{\mathbf{R}g_*} \mathbf{R}Hom_Y(\mathbf{R}f_*F,\mathbf{R}f_*G). \end{aligned}$$

Then $\tau_M^{F,G}$ is an isomorphism.

Now we combine Lemmas C.7 and C.9 and Theorem C.10 to induce an extension formula. We recall the schemes of X,Y,X',Y' and morphisms in the setting of Lemma C.9 and assume that $s_{f,\pi}$ in (C.13) is surjective. We fix splittings g' and g in Lemma C.9. Let L be a locally free sheaf on ${X'}^{Y'}_{(1)}$ and $V := L|_{X'}$. Then L induces extension classes $h'_L \in Ext^1_{X'}(V,V\otimes M)$ and $h_L \in Ext^1_{Y'}(f'_*V,f'_*V\otimes M)$, such that h_L induces the short exact sequence:

$$0 \to f'_*V\otimes M \to L \to f'_*V \to 0, \tag{C.15}$$

where we abuse the notation to denote L as a coherent sheaf on Y' through the pushforward of the closed embedding ${X'}^{Y'}_{(1)} \to Y'$. By applying then functor $\mathbf{R}\pi_*$ to (C.15), we induce a triangle:

$$\cdots \to \mathbf{R}\pi_*(f'_*V\otimes M) \to \mathbf{R}\pi_*L \to \mathbf{R}\pi_*(f'_*V) \to \mathbf{R}\pi_*(f'_*V\otimes M)[1] \to \cdots$$

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and thus induce an extension class

$$e_L = \mathbf{R}\pi_* h_L \in \mathbf{R}Hom_Y(\mathbf{R}\pi_*(f'_* V), \mathbf{R}\pi_*(f'_* V \otimes M)[1]).$$

Theorem C.11. *The image of e_L under the HKR isomorphism in Theorem C.10 is*

$$(\mathbf{R}\pi_{X*} h_L, \mathbf{R}\pi_{X*}(V \otimes s_{f,\pi}), 0, \cdots, 0).$$

Proof. It follows directly from Lemmas C.7 and C.9, Theorem C.10 and (C.6). □

Remark C.12. One should notice that while the Bass-Quillen class h_L depends on L, $\mathbf{R}\pi_{X*}(V \otimes s_{f,\pi})$ only depends on V but not the line bundle L.

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