Bull. Austral. Math. Soc. 26D15
Vol. 71 (2005) [275–284]

CARLSON TYPE INEQUALITIES FOR FINITE SUMS AND INTEGRALS ON BOUNDED INTERVALS

Leo Larsson, Zsolt Páles and Lars-Erik Persson

We investigate Carlson type inequalities for finite sums, that is, inequalities of the form

$$\sum_{k=1}^{m} a_k < C\Big(\sum_{k=1}^{m} k^{\alpha_1} a_k^{r+1}\Big)^{\mu}\Big(\sum_{k=1}^{m} k^{\alpha_2} a_k^{r+1}\Big)^{\lambda},$$

to hold for some constant C independent of the finite, non-zero set $a_1, \dots, a_m$ of non-negative numbers. We find constants C which are strictly smaller than the sharp constants in the corresponding infinite series case. Moreover, corresponding results for integrals over bounded intervals are given and a case with any finite number of factors on the right-hand side is proved.

1. Introduction

Carlson [6] showed in 1935 that the multiplicative inequality

$$\sum_{k=1}^{\infty} a_k < \sqrt{\pi}\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/4}\Big(\sum_{k=1}^{\infty} k^2 a_k^2\Big)^{1/4} \tag{1}$$

holds whenever $\{a_k\}_{k=1}^{\infty}$ is a non-zero sequence of non-negative numbers, provided that the series on the right-hand side are finite. The constant $\sqrt{\pi}$ is sharp. The inequality (1) and the corresponding integral version, as well as their generalisations and variations, turn out to have applications in various branches of mathematics, such as Fourier analysis (see for example, [5] and [9] as well as the Ph.D. thesis [10]), interpolation theory and embeddings of function spaces (see for example, [21, 12, 15], and the Ph.D. thesis [13]), the theory of optimal sampling (see [3]), and moment problems (see [11]), and have also become a popular object to study for its own interest (see for example, [14] and the Ph.D. thesis [13], and the references given there). See also the book by Mitrinović [20, pp. 370–372].

Among the many authors who have given generalisations of (1), we mention Bellman [2], who in 1943 proved the inequality

$$\sum_{k=1}^{\infty} a_k < C\Big(\sum_{k=1}^{\infty} k^{\alpha_1} a_k^{r+1}\Big)^{(\alpha_2-r)/((\alpha_2-\alpha_1)(r+1))}\Big(\sum_{k=1}^{\infty} k^{\alpha_2} a_k^{r+1}\Big)^{(r-\alpha_1)/((\alpha_2-\alpha_1)(r+1))}, \tag{2}$$

Received 28th October, 2004

when $\alpha_1 > \alpha_2 \geqslant 0$, $\lambda, \mu > 0$, with the sharp constant

$$C = 2^{1/(r+1)}\Big[\frac{1}{\alpha_2-\alpha_1}B\Big(\frac{\alpha_2-r}{(\alpha_2-\alpha_1)r},\frac{r-\alpha_1}{(\alpha_2-\alpha_1)r}\Big)\Big]^{r/(r+1)}, \tag{3}$$

where $B(\cdot,\cdot)$ denotes the Beta function, as well as the continuous counterpart

$$\int_0^\infty f(x)\,dx \leqslant C\Big(\int_0^\infty x^{\alpha_1}f^{r+1}(x)\,dx\Big)^{\frac{\alpha_2-r}{(\alpha_2-\alpha_1)(r+1)}}\Big(\int_0^\infty x^{\alpha_2}f^{r+1}(x)\,dx\Big)^{\frac{r-\alpha_1}{(\alpha_2-\alpha_1)(r+1)}}, \tag{4}$$

with the same sharp constant.

In Section 2 of this paper, we investigate inequalities of Carlson–Bellman type (2), where the infinite series are replaced by finite sums, as well as of the type (4), where the positive real line is replaced by a bounded interval of integration. We prove that the constants C in these cases can be chosen strictly smaller than the sharp constants in the classical cases. In particular, (1) may be replaced by the inequalities

$$\sum_{k=1}^m a_k < \sqrt{2\arctan m}\Big(\sum_{k=1}^m a_k^2\Big)^{1/4}\Big(\sum_{k=1}^m k^2a_k^2\Big)^{1/4}, \quad m=1,2,\dots \tag{5}$$

(for $m=\infty$, we get (1)). In Section 3, we generalise in a similar way some results of Levin and Godunova [**17**] by proving some inequalities of the type (2) with any finite number of factors on the right-hand side with finite sums. A continuous version of this is presented as well. Finally, Section 4 is reserved for some concluding remarks and open questions.

2. Two Inequalities of Carlson–Bellman Type

For $\alpha > 0$, $\beta > 0$, and $0 < u < t < 1$, we define two types of *truncated Beta functions* by

$$B(\alpha,\beta;t) = \int_0^t (1-s)^\alpha s^\beta \frac{ds}{(1-s)s}$$

and

$$B(\alpha,\beta;u,t) = \int_u^t (1-s)^\alpha s^\beta \frac{ds}{(1-s)s},$$

respectively.

2.1. The Discrete Case. Suppose that m is a positive integer. The inequality (2) holds, in particular, for sequences $\{a_k\}_{k=1}^\infty$ for which $a_k = 0$ whenever $k > m$. Thus

$$\sum_{k=1}^m a_k < C\Big(\sum_{k=1}^m k^{\alpha_1}a_k^{r+1}\Big)^{(\alpha_2-r)/((\alpha_2-\alpha_1)(r+1))}\Big(\sum_{k=1}^m k^{\alpha_2}a_k^{r+1}\Big)^{(r-\alpha_1)/((\alpha_2-\alpha_1)(r+1))}, \tag{6}$$

where C is given by (3). However, if we consider fixed-length finite sequences only, the constant is no longer sharp but can be made strictly smaller. We prove the following inequality.

THEOREM 1. *Suppose that m is a positive integer, and that $\alpha_2 > r > \alpha_1 \geqslant 0$. If $a_1, \dots, a_m$ are non-negative numbers, not all zero, then (6) holds, where we may choose*

$$C = 2^{1/(r+1)}\left[\frac{1}{\alpha_2-\alpha_1}B\left(\frac{\alpha_2-r}{(\alpha_2-\alpha_1)r}, \frac{r-\alpha_1}{(\alpha_2-\alpha_1)r}; \frac{m^{\alpha_2-\alpha_1}}{1+m^{\alpha_2-\alpha_1}}\right)\right]^{r/(r+1)}. \tag{7}$$

REMARK 1. The constant C defined by (7), which depends on m, tends to the sharp constant (3) when $m \to \infty$. In the special case $r = 1$, $\alpha_1 = 0$, $\alpha_2 = 2$, we have

$$B\left(\frac{1}{2}, \frac{1}{2}; \frac{m^2}{1+m^2}\right) = 2\arcsin\frac{m}{\sqrt{1+m^2}} = 2\arctan m,$$

showing that the inequality (5) holds. When $m \to \infty$, this constant tends to the sharp constant $\sqrt{\pi}$ in (1).

PROOF OF THEOREM 1: Let

$$S = \sum_{k=1}^{m} k^{\alpha_1} a_k^{r+1} \quad \text{and} \quad T = \sum_{k=1}^{m} k^{\alpha_2} a_k^{r+1},$$

and let λ be any positive number. If we write

$$a_k = \left(\lambda k^{\alpha_1} + \frac{1}{\lambda}k^{\alpha_2}\right)^{-(1/(r+1))} \cdot \left(\lambda k^{\alpha_1} + \frac{1}{\lambda}k^{\alpha_2}\right)^{1/(r+1)} a_k,$$

we get by Hölder's inequality

$$\begin{aligned}\left(\sum_{k=1}^{m} a_k\right)^{r+1} &\leqslant \left(\sum_{k=1}^{m}(\lambda k^{\alpha_1} + \lambda^{-1}k^{\alpha_2})^{-(1/r)}\right)^{r}\left(\sum_{k=1}^{m}(\lambda k^{\alpha_1} + \lambda^{-1}k^{\alpha_2})a_k^{r+1}\right) \\ &= \left(\sum_{k=1}^{m}(\lambda k^{\alpha_1} + \lambda^{-1}k^{\alpha_2})^{-(1/r)}\right)^{r}(\lambda S + \lambda^{-1}T).\end{aligned} \tag{8}$$

Since the function

$$x \mapsto (\lambda x^{\alpha_1} + \lambda^{-1}x^{\alpha_2})^{-(1/r)}$$

is non-increasing for positive x, the last sum in (8) can be estimated by an integral. In fact, by making the substitution

$$x = \lambda^{2/(\alpha_2-\alpha_1)}\left(\frac{s}{1-s}\right)^{1/(\alpha_2-\alpha_1)},$$

we find that

$$\begin{aligned}\sum_{k=1}^{m}(\lambda k^{\alpha_1} + \lambda^{-1}k^{\alpha_2})^{-(1/r)} &< \int_0^m \frac{dx}{(\lambda x^{\alpha_1} + \lambda^{-1}x^{\alpha_2})^{1/r}} \\ &= \frac{\lambda^{\frac{2r-(\alpha_1+\alpha_2)}{(\alpha_2-\alpha_1)r}}}{\alpha_2-\alpha_1}\int_0^{\frac{m^{\alpha_2-\alpha_1}}{\lambda^2+m^{\alpha_2-\alpha_1}}}(1-s)^{\frac{\alpha_2-r}{(\alpha_2-\alpha_1)r}}s^{\frac{r-\alpha_1}{(\alpha_2-\alpha_1)r}}\frac{ds}{(1-s)s} \\ &= \frac{\lambda^{\frac{r-(\alpha_1+\alpha_2)/2}{(\alpha_2-\alpha_1)/2r}}}{\alpha_2-\alpha_1}B\left(\frac{\alpha_2-r}{(\alpha_2-\alpha_1)r}, \frac{r-\alpha_1}{(\alpha_2-\alpha_1)r}; \frac{m^{\alpha_2-\alpha_1}}{\lambda^2+m^{\alpha_2-\alpha_1}}\right).\end{aligned} \tag{9}$$

The function $t \mapsto B(\alpha, \beta; t)$ is obviously increasing, and hence

$$\lambda \mapsto B\Big(\alpha, \beta; \frac{m^{\alpha_2-\alpha_1}}{\lambda^2 + m^{\alpha_2-\alpha_1}}\Big)$$

is decreasing. Moreover, we always have $S \leqslant T$, so if we put

$$\lambda = \sqrt{\frac{T}{S}},$$

then

$$B\Big(\alpha, \beta; \frac{m^{\alpha_2-\alpha_1}}{\lambda^2 + m^{\alpha_2-\alpha_1}}\Big) \leqslant B\Big(\alpha, \beta; \frac{m^{\alpha_2-\alpha_1}}{1 + m^{\alpha_2-\alpha_1}}\Big).$$

Hence, in view of (8) and (9), it follows that

$$\Big(\sum_{k=1}^{m} a_k\Big)^{r+1} < 2\Big(\frac{1}{\alpha_2-\alpha_1} B\Big(\frac{\alpha_2-r}{(\alpha_2-\alpha_1)r}, \frac{r-\alpha_1}{(\alpha_2-\alpha_1)r}; \frac{m^{\alpha_2-\alpha_1}}{1+m^{\alpha_2-\alpha_1}}\Big)\Big)^r S^{\frac{\alpha_2-r}{\alpha_2-\alpha_1}} T^{\frac{r-\alpha_1}{\alpha_2-\alpha_1}}.$$

This yields the inequality (6) with C given by (7). □

2.2. THE CONTINUOUS CASE. By analogy with the discrete case, the inequality (4) holds, in particular, for a function f supported on the interval $[0, m]$. As opposed to the discrete case, however, the constant cannot be altered. The reason for this is that it is possible to find a maximising function with mass concentrated arbitrarily close to 0. We therefore have to modify the continuous case slightly.

THEOREM 2. *Suppose that $m > 1$ and $\alpha_2 > r > \alpha_1 \geqslant 0$. Then the inequality*
(10)

$$\int_{m^{-1}}^{m} f(x)\,dx \leqslant C\Big(\int_{m^{-1}}^{m} x^{\alpha_1} f^{r+1}(x)\,dx\Big)^{\frac{\alpha_2-r}{(\alpha_2-\alpha_1)(r+1)}} \Big(\int_{m^{-1}}^{m} x^{\alpha_2} f^{r+1}(x)\,dx\Big)^{\frac{r-\alpha_1}{(\alpha_2-\alpha_1)(r+1)}}$$

holds for all non-negative, measurable functions f on (m^{-1}, m). We may choose
(11)

$$C = \Big[\frac{2^{1/r}}{\alpha_2-\alpha_1} B\Big(\frac{\alpha_2-r}{r(\alpha_2-\alpha_1)}, \frac{r-\alpha_1}{r(\alpha_2-\alpha_1)}; \frac{m^{2(\alpha_2-r)}-1}{m^{2(\alpha_2-\alpha_1)}-1}, \frac{m^{2(\alpha_2-\alpha_1)}-m^{2(r-\alpha_1)}}{m^{2(\alpha_2-\alpha_1)}-1}\Big)\Big]^{r/(r+1)}.$$

REMARK 2. As in the discrete case, the constant $C = C(m)$ approaches the sharp constant in (4) as $m \to \infty$.

PROOF OF THEOREM 2: For any $\lambda > 0$, we have by Hölder's inequality

$$\begin{aligned}\int_{m^{-1}}^{m} f(x)\,dx &= \int_{m^{-1}}^{m} (\lambda x^{\alpha_1} + \lambda^{-1} x^{\alpha_2})^{-1/(r+1)} (\lambda x^{\alpha_1} + \lambda^{-1} x^{\alpha_2})^{1/(r+1)} f(x)\,dx \\ &\leqslant \Big(\int_{m^{-1}}^{m} \frac{dx}{(\lambda x^{\alpha_1} + \lambda^{-1} x^{\alpha_2})^{1/r}}\Big)^{r/(r+1)} \Big(\int_{m^{-1}}^{m} (\lambda x^{\alpha_1} + \lambda^{-1} x^{\alpha_2}) f^{r+1}(x)\,dx\Big)^{1/(r+1)} \\ &= \Big(\frac{1}{\alpha_2-\alpha_1} B\Big)^{r/(r+1)} \Big(\lambda^{(2r-(\alpha_1+\alpha_2))/(\alpha_2-\alpha_1)} (\lambda S_1 + \lambda^{-1} S_2)\Big)^{1/(r+1)},\end{aligned}$$

https://doi.org/10.1017/S0004972700038247 Published online by Cambridge University Press

where

$$S_i = \int_{m^{-1}}^{m} x^{\alpha_i} f^{r+1}(x)\,dx, \quad i = 1, 2,$$

and

$$B = \int_{(m^{\alpha_2-\alpha_1})/(\lambda^2+m^{\alpha_2-\alpha_1})}^{(m^{-(\alpha_2-\alpha_1)})/(\lambda^2+m^{-(\alpha_2-\alpha_1)})} (1-s)^{(\alpha_2-r)/(r(\alpha_2-\alpha_1))} s^{(r-\alpha_1)/(r(\alpha_2-\alpha_1))} \frac{ds}{(1-s)s}.$$

Now, we can maximise this truncated Beta function with respect to λ, simply by differentiating. We find that the maximum is

$$B = \left(\frac{\alpha_2 - r}{r(\alpha_2-\alpha_1)}, \frac{r-\alpha_1}{r(\alpha_2-\alpha_1)}; \frac{m^{2(\alpha_2-r)}-1}{m^{2(\alpha_2-\alpha_1)}-1}, \frac{m^{2(\alpha_2-\alpha_1)}-m^{2(r-\alpha_1)}}{m^{2(\alpha_2-\alpha_1)}-1}\right).$$

We may now put

$$\lambda = \sqrt{\frac{S_2}{S_1}},$$

which yields (10) with C given by (11). ∎

3. Some Generalisations

By combining our ideas with those of Levin and Godunova [**17**], we can in some cases extend Theorems 1 and 2 to hold for more than two factors on the right-hand side of the inequality.

3.1. The Discrete Case.

Theorem 3. *Let $N \geqslant 2$ be an integer, and suppose that $\alpha_1 \leqslant \cdots \leqslant \alpha_N$. Moreover, let*

$$r = \frac{1}{N}\sum_{j=1}^{N} \alpha_j.$$

Then, if $a_1, \ldots, a_m$ are non-negative numbers, not all zero, it holds that

$$\sum_{k=1}^{m} a_k < C \prod_{j=1}^{N} \left(\sum_{k=1}^{m} k^{\alpha_j} a_k^{r+1}\right)^{1/(N(r+1))}. \tag{12}$$

We may choose

$$C = \min_{1\leqslant n<N} (KN)^{1/(r+1)} B\left(\frac{n}{Nr}, \frac{N-n}{Nr}; \frac{m^{r_2-r_1}}{n/(N-n)+m^{r_2-r_1}}\right)^{r/(r+1)}, \tag{13}$$

where

$$K = \frac{n^{-(n/N)}(N-n)^{-(N-n)/N}}{(r_2-r_1)^{1/r}} \tag{14}$$

and $r_i = r_i(n)$ are defined by

$$r_1 = \frac{1}{n}\sum_{j=1}^{n} \alpha_j \quad \text{and} \quad r_2 = \frac{1}{N-n}\sum_{j=n+1}^{N} \alpha_j. \tag{15}$$

https://doi.org/10.1017/S0004972700038247 Published online by Cambridge University Press

REMARK 3. We see that when $N = 2$, this reduces to the special case of Theorem 1 where $r = (\alpha_1 + \alpha_2)/2$. However, for $N > 2$, the constant does not necessarily approach the sharp constant for the corresponding infinite series inequality when $m \to \infty$.

PROOF OF THEOREM 3: Let

$$S_i = \sum_{k=1}^{m} k^{\alpha_i} a_k^{r+1}, \quad i = 1, \dots, N.$$

If $\lambda_1, \dots, \lambda_N$ are any positive numbers, we write

$$a_k = \left(\lambda_1 k^{\alpha_1} + \dots + \lambda_N k^{\alpha_N}\right)^{-(1/(r+1))} \cdot \left(\lambda_1 k^{\alpha_1} + \dots + \lambda_N k^{\alpha_N}\right)^{1/(r+1)} a_k$$

and apply Hölder's inequality with parameters

$$\frac{r+1}{r} \quad \text{and} \quad r+1,$$

which yields

$$\left(\sum_{k=1}^{m} a_k\right)^{r+1} < \left(\sum_{k=1}^{m} \left(\lambda_1 k^{\alpha_1} + \dots + \lambda_N k^{\alpha_N}\right)^{-(1/r)}\right)^{r} \sum_{i=1}^{N} \lambda_i S_i.$$

The first sum on the right-hand side can be estimated by the integral

$$I = \int_0^m \left(\lambda_1 x^{\alpha_1} + \dots + \lambda_N x^{\alpha_N}\right)^{-(1/r)} dx.$$

Let $1 \leqslant n < N$, and let r_1 and r_2 be as defined by (15). Moreover, put

$$K_1 = n(\lambda_1 \cdots \lambda_n)^{1/n} \quad \text{and} \quad K_2 = (N-n)(\lambda_{n+1} \cdots \lambda_N)^{1/(N-n)}.$$

By the arithmetic–geometric mean inequality, it holds that

$$\begin{aligned} \lambda_1 x^{\alpha_1} + \dots + \lambda_N x^{\alpha_N} &= \left(\lambda_1 x^{\alpha_1} + \dots + \lambda_n x^{\alpha_n}\right) + \left(\lambda_{n+1} x^{\alpha_{n+1}} + \dots + \lambda_N x^{\alpha_N}\right) \\ &= n \frac{\lambda_1 x^{\alpha_1} + \dots + \lambda_n x^{\alpha_n}}{n} + (N-n) \frac{\lambda_{n+1} x^{\alpha_{n+1}} + \dots + \lambda_N x^{\alpha_N}}{N-n} \\ &\geqslant n\left(\lambda_1 x^{\alpha_1} \cdots \lambda_n x^{\alpha_n}\right)^{1/n} + (N-n)\left(\lambda_{n+1} x^{\alpha_{n+1}} \cdots \lambda_N x^{\alpha_N}\right)^{1/(N-n)} \\ &= K_1 x^{r_1} + K_2 x^{r_2}, \end{aligned}$$

and therefore, also using the relations

$$\frac{r_2 - r}{r_2 - r_1} = \frac{n}{N} \quad \text{and} \quad \frac{r - r_1}{r_2 - r_1} = \frac{N-n}{N}$$

we get

$$
\begin{aligned}
I &\leqslant \int_0^m (K_1 x^{r_1} + K_2 x^{r_2})^{-(1/r)}\,dx \\
&= \frac{\left(K_1^{-(r_2-r)} K_2^{-(r-r_1)}\right)^{1/(r(r_2-r_1))}}{r_2 - r_1} B\left(\frac{r_2-r}{r(r_2-r_1)}, \frac{r-r_1}{r(r_2-r_1)}; \frac{m^{r_2-r_1}}{(K_1)/(K_2) + m^{r_2-r_1}}\right) \\
&= \frac{\left(K_1^{-(n/N)} K_2^{-(N-n)/N}\right)^{1/r}}{r_2 - r_1} B\left(\frac{n}{Nr}, \frac{N-n}{Nr}; \frac{m^{r_2-r_1}}{(K_1)/(K_2) + m^{r_2-r_1}}\right) \\
&= \frac{n^{-(n/(Nr)}(N-n)^{-(N-n)/(Nr)}(\lambda_1\cdots\lambda_N)^{-(1/Nr)}}{r_2 - r_1} B\left(\frac{n}{Nr}, \frac{N-n}{Nr}; \frac{m^{r_2-r_1}}{(K_1)/(K_2) + m^{r_2-r_1}}\right).
\end{aligned}
$$

Now, choose the numbers λ_j so that

$$\lambda_j S_j = \left(\prod_{i=1}^N S_i\right)^{1/N}, \quad j = 1, \ldots, N. \tag{16}$$

Thus

$$\lambda_1 \cdots \lambda_N = 1,$$

and if K is defined by (14) it follows that

$$\left(\sum_{k=1}^m a_k\right)^{r+1} < KN \cdot B\left(\frac{n}{Nr}, \frac{N-n}{Nr}; \frac{m^{r_2-r_1}}{(K_1)/(K_2) + m^{r_2-r_1}}\right)^r \left(\prod_{i=1}^N S_i\right)^{1/N}.$$

Now, we note that

$$\frac{K_1}{K_2} = \frac{n}{N-n}(\lambda_1 \cdots \lambda_n)^{N/(n(N-n))}.$$

Since S_j increases with j, it follows from (16) that λ_j decreases with j, and since the product of all of them equals 1, it must hold that

$$\lambda_1 \cdots \lambda_n \geqslant 1.$$

Thus

$$B\left(\cdot, \cdot; \frac{m^{r_2-r_1}}{(K_1)/(K_2) + m^{r_2-r_1}}\right) \leqslant B\left(\cdot, \cdot; \frac{m^{r_2-r_1}}{n/(N-n) + m^{r_2-r_1}}\right).$$

Since this can be done for any n, the conclusion follows upon taking $(r+1)$th roots. ∎

3.2. The Continuous Case. We can now combine the ideas from Theorems 2 and 3 to get the following.

Theorem 4. *Suppose that N is an integer $\geqslant 2$, $m > 1$ and $\alpha_1 \leqslant \cdots \leqslant \alpha_N$, and put*

$$r = \frac{1}{N}\sum_{j=1}^N \alpha_j.$$

Then, for all non-negative, measurable functions f on (m^{-1}, m), it holds that

$$\int_{m^{-1}}^{m} f(x)\,dx \leqslant C\prod_{j=1}^{N}\left(\int_{m^{-1}}^{m} x^{\alpha_j} f^{r+1}(x)\,dx\right)^{1/(N(r+1))}.$$

We may choose

$$C = \min_{1\leqslant n<N}(KN)^{1/(r+1)}B\left(\frac{n}{Nr},\frac{N-n}{Nr};\frac{m^{2(r-r_1)}-1}{m^{2(r_2-r_1)}-1},\frac{m^{2(r_2-r_1)}-m^{2(r_2-r)}}{m^{2(r_2-r_1)}-1}\right)^{r/(r+1)},$$

where K and r_i, $i = 1, 2$, are as in Theorem 3.

4. Concluding Remarks

REMARK 4. The constants given by (7) and (13) increase with m. This is obviously the case also for the sharp constant. However, although our constants are strictly smaller than the sharp constants for the corresponding infinite series cases, they are not sharp in the sense of finite sums. For example, as can be seen by homogeniety, the sharp constant is 1 when $m = 1$. It would be interesting to find the sharp constant for the setting of this paper.

REMARK 5. The result in Levin **[18]** suggests that we should look for an inequality of the type

$$\sum_{k=1}^{m} a_k < C\left(\sum_{k=1}^{m} k^{p-1-\lambda}a_k^p\right)^s\left(\sum_{k=1}^{m} k^{q-1+\mu}a_k^q\right)^t. \tag{17}$$

The proof of Theorem 1 cannot be directly extended to cover also this case. We therefore leave it as an open problem to prove the inequality (17) with a constant which is strictly smaller than the sharp constant in the corresponding infinite series inequality.

REMARK 6. As mentioned in the introduction, the Carlson inequalities have been generalised in various directions, and many different types of applications have been found. For instance, Barza, Burkenov, Pečarić and Persson [1] considered an infinite cone Ω in $\mathbb{R}^n$ as the domain of integration (rather than the interval $(0,\infty)$), and they achieved the sharp constant in the inequality

$$\left(\int_\Omega |f(x)w(x)|^p\,dx\right)^{1/p} \leqslant C\left(\int_\Omega |f(x)w_0(x)|^{p_0}\,dx\right)^{(1-\theta)/p_0}\left(\int_\Omega |f(x)w_1(x)|^{p_1}\,dx\right)^{\theta/p_1},$$

where the *weights* w, w_0 and w_1 are homogeneous functions. Obviously, the results and ideas in this paper can be developed to derive a similar result but with the infinite cone Ω replaced by a bounded one and with a strictly smaller constant.

REMARK 7. We can view the inequality (1) and many of its versions and successors as special cases of the inequality

$$\|f\|_X \leqslant C\,\|f\|_{A_0}^{1-\theta}\,\|f\|_{A_1}^{\theta} \tag{18}$$

between norms in Banach spaces X, A_0 and A_1. With this viewpoint, the inequalities under consideration become closely related to interpolation theory. The inequality (18) is, in fact, equivalent to the statement that the Banach space X is of Peetre class $C_J(\theta; A_0, A_1)$ (see for example, [4] for details on interpolation theoretical matters). We mention that the article [**15**] uses general inequalities of Carlson type to get embeddings of real interpolation spaces into weighted L_p spaces over general measure spaces. We refer to the book manuscript [**16**] for a more complete exposition of the various ways in which the inequalities have been extended and applied.

REMARK 8. (On the mathematician Fritz David Carlson.) Fritz David Carlson (1888–1952) was a Swedish professor who made several contributions to mathematics besides the discovery of Carlson's inequalities, described in this paper, among which the following are well-known: Carlson's theorem in complex analysis, the Pólya–Carlson theorem for rational functions and Carlson's theorem on Dirichlet series. For a more detailed description of the scientific work of Carlson, we refer to the book by Gårding [**7**]. Some complementary information about the remarkable mathematician and man F. Carlson can also be found in the historical note [**19**] (see also [**16**]).

REMARK 9. Carlson's original proof of (1) was based on his deep knowledge in complex analysis. Obviously, he strongly believed that his inequality was a limiting case of the Hölder inequality, but which could not be derived from it. Hence, it must have been a surprise for him when Hardy [**8**] in 1936 presented two new proofs of (1), one of which in fact shows that the inequality follows even from the Schwarz inequality. More details and discussions around these and related facts can be found in the new book manuscript [**16**] and the references pointed out there.

REFERENCES

[1] S. Barza, V. Burenkov, J. Pečarić and L.-E. Persson, 'Sharp multidimensional multiplicative inequalities for weighted L_p spaces with homogeneous weights', *Math. Inequal. Appl.* **1** (1998), 53–67.

[2] R. Bellman, 'An integral inequality', *Duke Math. J.* **10** (1943), 547–550.

[3] J. Bergh, 'An optimal reconstruction of sampled signals', *J. Math. Anal. Appl.* **115** (1986), 574–577.

[4] J. Bergh and J. Löfström, *Interpolation spaces* (Springer-Verlag, Berlin, Heidelberg, New York, 1976).

[5] A. Beurling, 'Sur les intégrales de Fourier absolument convergentes et leur application à une transformation fonctionelle', *C. R. Neuvième Congrès Math. Scandinaves 1938, Helsinki* (1939), 345–366.

[6] F. Carlson, 'Une inégalité', *Ark. Mat. Astr. Fysik* **25B** (1935).

[7] L. Gårding, *Mathematics and mathematicians*, (in Swedish) (Lund University Press, Lund, 1994).

[8] G.H. Hardy, 'A note on two inequalities', *J. London Math. Soc.* **11** (1936), 167–170.

[9] A. Kamaly, 'Fritz Carlson's inequality and its application', *Math. Scand.* **86** (2000), 100–108.

[10] A. Kamaly, *Type-inequality problems in Fourier analysis*, (PhD Thesis) (Department of Electromagnetic Theory, Royal Institute of Technology, Stockholm, 1998).

[11] B. Kjellberg, 'Ein momentenproblem', *Ark. Mat. Astr. Fysik* **29A** (1942).

[12] N.Ya. Kruglyak, L. Maligranda and L.-E. Persson, 'A Carlson type inequality with blocks and interpolation', *Studia Math.* **104** (1993), 161–180.

[13] L. Larsson, *Carlson type inequalities and their applications* (Uppsala Dissertations in Mathematics, Sweden, 2003).

[14] L. Larsson, 'A New Carlson type inequality', *Math. Inequal. Appl.* **6** (2003), 55–79.

[15] L. Larsson, 'Carlson type inequalities and embeddings of interpolation spaces', *Proc. Amer. Math. Soc.* **134** (2004), 2351–2356.

[16] L. Larsson, J. Pečarić and L.-E. Persson, *Multiplicative inequalities of Carlson type and interpolation*, (book manuscript).

[17] V.I. Levin and E.K. Godunova, 'Generalizations of Carlson's inequalities', (in Russian), *Mat. Sb.* **67** (1965), 643–646.

[18] V.I. Levin, 'Sharp constants in Carlson type inequalities', (in Russian), *Dokl. Akad. Nauk, SSSR* **59** (1948), 635–638.

[19] L. Maligranda, *Carlson* `http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Carlson.html`.

[20] D.S. Mitrinović, *Analytic inequalities* (Springer-Verlag, Berlin, Heidelberg, New York, 1979).

[21] J. Peetre, 'Sur le nombre de paramètres dans la définition de certains espaces d'interpolation', *Ricerche Mat.* **12** (1963), 248–261.

Department of Mathematics
Uppsala University
Box 480
SE-791 06 Uppsala
Sweden
e-mail: leo@math.uu.se

Institute of Mathematics and Informatics
University of Debrecen
H-4010 Debrecen Pf. 12
Hungary
e-mail: pales@math.klte.hu

Department of Mathematics
Luleå University
SE-971 87 LULEÅ
Sweden
and
Department of Mathematics
Uppsala University
Box 480
SE-791 06 UPPSALA
Sweden
e-mail: larserik@sm.luth.se