BULL. AUSTRAL. MATH. SOC. 16S40, 57T05
VOL. 72 (2005) [109-127]

FINITE PRESENTABILITY OF SOME METABELIAN HOPF ALGEBRAS

DESSISLAVA H. KOCHLOUKOVA

We classify the Hopf algebras $U(L)\#kQ$ of homological type FP_2 where L is a Lie algebra and Q an Abelian group such that L has an Abelian ideal A invariant under the Q-action via conjugation and $U(L/A)\#kQ$ is commutative. This generalises the classification of finitely presented metabelian Lie algebras given by J. Groves and R. Bryant.

INTRODUCTION

The purpose of this paper is to try to unite some existing methods used in the classification results of metabelian Lie algebras and metabelian discrete groups of homological type FP_2 via the language of Hopf algebras. This sheds more light on the similarities between the Lie and group cases and explains partially the differences. Still some of the results in the group case have homotopical flavour, using methods from covering spaces to establish that having homological type FP_2 imposes strong condition on the first Σ-invariant of the group ([**4**]). These methods do not have a purely algebraic counterpart. The Lie case was treated in [**5, 6**] with algebraic methods, and a Lie invariant (with a valuation flavour) for metabelian Lie algebras was proposed. This plays the same role in the Lie theory as the Bieri-Strebel Σ-invariant for metabelian groups. In this paper we do not suggest a new invariant but establish that the main result of [**5**] holds for some metabelian Hopf algebras. It is interesting to note that in both the Lie and group cases calculations with the second homology group of Abelian objects (Lie algebras or Abelian groups) viewed as modules over a commutative ring via the corresponding diagonal action was always quite helpful. The definition of the diagonal Lie and group actions can be united via the comultiplication map of Hopf algebras, and this was the starting point of our considerations.

We study Hopf algebras $H = U(L)\#kG$ over a field k, that is, smash products of universal enveloping algebras $U(L)$ of Lie algebras L over k by group rings kG, where G acts via conjugation on L and write $\mathcal{X}$ for the category of such Hopf algebras. This category is quite important. If $\mathrm{char}(k) = 0$ it coincides with the category of cocommutative,

Received 21st March, 2005
Partially supported by "bolsa de produtividade de pesquisa" from CNPq, Brazil.

pointed Hopf algebras over k [**8**, 5.6.4, 5.6.5]. Still in this paper we do not impose any condition on the characteristic of k.

By the main results of [**5**, **6**] for metabelian Lie algebras the homological property FP_2 is equivalent to finite presentability (in the Lie sense). Finite presentability in the category $\mathcal{X}$ was defined in [**7**] and is further explained in Section 1. The following theorem classifies some metabelian Hopf algebras in $\mathcal{X}$ and shows that again finite presentability and the property FP_2 coincide. It is well known that the properties FP_2 and finite presentability are not equivalent for general groups ([**2**]), but the same problem is open for Lie algebras.

THEOREM 1. *Let $H = U(L)\#kQ$ be a finitely generated Hopf algebra with A an Q-invariant (via conjugation) Abelian ideal in the Lie algebra L such that $R = U(L/A)\#kQ$ is commutative. Then the following are equivalent:*

1. *H is finitely presented in the category $\mathcal{X}$;*
2. *H as associative ring is of homological type FP_2;*
3. *$A \wedge A$ is finitely generated right R-module, where R acts on A via the adjoint action and R acts on $A \otimes A$ via the comultiplication $\Delta : R \to R \otimes R$;*
4. *$A \otimes A$ is finitely generated right R-module, where R acts on $A \otimes A$ via the comultiplication Δ.*

The proof of the above theorem will be given by showing that every condition implies the following and that 4 implies 1. The most difficult part of the proof is 4. implies 1. and is done in Theorem 4. The proof of Theorem 4 is quite long. It partially follows the method introduced in [**5**]. Still our set of relations that will show that H is finitely presented is much larger than the one considered in [**5**] (even if we are in the Lie case $Q = 1$), but by considering more relations we manage to simplify the argument from [**5**]. In fact a blind translation of the method of [**5**] in the Hopf case does not work because the comultiplication Δ sends group-like elements g to $g \otimes g$, thus increases the length of elements and a big part of the proofs in [**5**] is based on induction on length.

Finally we observe that Theorem 1 is an extension of the main results of [**5**] and [**6**] but the main result about metabelian groups of [**4**] does not follow from Theorem 1. It will be interesting to find a description of the Hopf algebras $H = U(L)\#kG$ of type FP_2, where H is a Hopf extension in the category $\mathcal{X}$ of two commutative smash products (in particular this implies that G is metabelian) and find a new invariant that generalises simultaneously the Bieri–Strebel invariant [**4**] and the Bieri-Groves invariant [**6**].

1. Preliminaries on the category $\mathcal{X}$ and finite presentability in $\mathcal{X}$

The category $\mathcal{X}$ is a subcategory of the category of Hopf algebras over k. An object of $\mathcal{X}$ is a smash product

$$H = U(L)\#kG \simeq U(L) \otimes kG,$$
$$L = \{a \in H \mid \Delta(a) = a \otimes 1 + 1 \otimes a\},\ G = \{g \in H \mid \Delta(g) = g \otimes g\}$$
and $\Delta : H \to H \otimes H$ is the comultiplication of H.

If not stated otherwise, the tensor products are over the field k. Here $U(L)$ is the universal enveloping algebra of the Lie algebra L and kG is the group algebra of G with coefficients in k. The group G acts on L via right conjugation: for $a \in L, g \in G$ we have $a^g := g^{-1}ag \in L$. The elements of L are called Lie elements and the elements of G group-like elements.

Let $Z = X \cup Y$ be a disjoint union of sets, F the free group with basis Y, X_0 a set on which the group F acts freely with $X_0/F \simeq X$ (that is, X_0 is the disjoint union of free F-orbits $\bigcup_{x \in X} x^F$) and L_0 the free Lie algebra over the field k with basis X_0. As defined in [7]

$$H(X \mid Y) = U(L_0)\#kF,$$

where the action of F on L_0 via right conjugation is induced by the action of F on X_0.

Let H be a Hopf algebra from the category $\mathcal{X}$. We say that H is finitely generated if it is finitely generated as an associative k-algebra. Note this is equivalent to the existence of a disjoint union $X \cup Y$ of finite sets and an epimorphism of Hopf algebras $\pi : H(X \mid Y) \to H$ that is, $H(X \mid Y)$ is a Hopf extension of the Hopf kernel H_1 of π by H.

We say that H is finitely presented as a Hopf algebra (or finitely presented in the category $\mathcal{X}$) if it is finitely generated and there is a finite set $X \cup Y$ such that for the Hopf kernel $H_1 = hker(\pi)$ defined in the previous paragraph there is a finite subset R of $\Omega(H_1)$ such that the orbits generated by the elements of R via the right adjoint action of $H(X \mid Y)$ on H_1 generate H_1 as an associative k-algebra. Here $\Omega(H_1)$ denotes the kernel of the counit map $H_1 \to k$, the right adjoint action of the group-like elements is conjugation, and the right adjoint action of the Lie elements is given by the Lie bracket. The following results can be viewed as a generalisation of the main results of [**1**] and will be used later on in the proof of Theorem 4. It shows that being finitely presented in the category $\mathcal{X}$ does not depend on the choice of generators.

Theorem 2. [**7**, Theorem B, Cororollary 2]

(a) *An element of $\mathcal{X}$ is finitely presented in the category $\mathcal{X}$ if and only if it is finitely presented as an associative k-algebra.*

(b) *Let H be a Hopf algebra in the category $\mathcal{X}$ and $\pi : H(X \mid Y) \to H$ be an epimorphisms of Hopf algebras where $X \cup Y$ is a disjoint union of finite sets. Then H is finitely presented in the category $\mathcal{X}$ if and only if there is* *a finite subset R of the Hopf kernel H_1 of π such that the orbits generated by the elements of R via the adjoint action of $H(X \mid Y)$ generate H_1 as an associative k-algebra and R contains only Lie elements and group-like elements - 1.*

2. Tensor and exterior squares

Let A be a finitely generated module over a commutative Hopf algebra $H = U(L_1)\#kQ$. We view $A \otimes A$ and its quotient $A \wedge A$ as right H-modules via the comultiplication $\Delta : H \to H \otimes H$.

Theorem 3. *The module $A \otimes A$ is finitely generated over H if and only if $A \wedge A$ is finitely generated over H.*

Proof: The above theorem is known in the case when $L_1 = 0$ [**3**, Theorem 4.3] or $Q = 1$ [**6**, Proposition 2.4]. In fact the proof of [**6**, Proposition 2.4] is for any finitely generated commutative k-algebra with unity H that acts on $A \wedge A$ via some homomorphism of k-algebras $\Delta : H \to H \otimes H$ such that for some generating set (as commutative k-algebra with unity) Y of H we have $\Delta(y) = y \otimes 1 + 1 \otimes y$ for every $y \in Y$. A close observation of the proof of [**6**, Proposition 2.4] shows that in fact it uses only that for every $y \in Y$ the element $\Delta(y)$ is invariant under the action of the symmetric group on two elements that permutes the factors of $H \otimes H$. In particular the proof of [**6**, Propisition 2.4] holds in our case where $Y = L_0 \cup Q_0$, L_0 is a basis of L_1 over k, Q_0 is a generating set of the Abelian group Q. ∎

3. Finite generation of tensor squares and finite presentability

The purpose of this rather long section is to develop the techniques of the proof of Theorem 4. The proof of Theorem 4 will be completed in Section 4.

Theorem 4. *Let $H = U(L)\#kQ$ be a finitely generated Hopf algebra with A an Q-invariant (via conjugation) Abelian ideal in the Lie algebra L such that $R = U(L/A)\#kQ$ is commutative. If $A \otimes A$ is finitely generated over R via the comultiplication $\Delta : R \to R \otimes R$, where A is viewed as a right R-module via the adjoint action, then H is finitely presented in $\mathcal{X}$.*

3.1. More about generators and relations for the Hopf algebra of Theorem 4.

Let $H = U(L)\#kQ$ be a Hopf algebra with A an Q-invariant (via conjugation) Abelian ideal in the Lie algebra L, let $R = U(L/A)\#kQ$ be commutative, and let H be finitely generated as an associative k-algebra. We view A as a right R-module via the adjoint action; that is, the elements of Q act by right conjugation, the elements of L act via the adjoint Lie action. For $r \in R$, $a \in A$ we denote by $a \circ r$ the image of the action of r on a, hence for $a \in A$

$$a \circ q = q^{-1}aq \text{ where } q \in Q \text{ and } a \circ l = [a, l_0] \text{ where } l_0 \in L, l = l_0 + A \in L/A.$$

Under the assumptions of Theorem 4 $A \otimes A$ is finitely generated over R via the comultiplication Δ.

Lemma 1. *The right R-module A is finitely generated.*

Proof: Since H is finitely generated there is a free resolution of the trivial right H-module k

$$\mathcal{F} : \ldots \to F_1 \to F_0 = H \xrightarrow{\varepsilon_H} k \to 0$$

with F_1 finitely generated. Here ε_H is the counit map. We use this resolution to calculate $H_1(A) = Tor_1^{U(A)}(k,k) \simeq A$ as $H_1(\mathcal{F} \otimes_{U(A)} k)$. Note that $H_1(\mathcal{F} \otimes_{U(A)} k)$ is a section of $F_1 \otimes_{U(A)} k$ and $F_1 \otimes_{U(A)} k$ is a finitely generated module over $H \otimes_{U(A)} k \simeq R$. As R is Noetherian we deduce that $H_1(A) \simeq A$ is finitely generated over R. The fact that the action of R on $A \simeq H_1(\mathcal{F} \otimes_{U(A)} k)$ induced by right multiplication on $\mathcal{F} \otimes_{U(A)} k$ is the right adjoint action on A is proved in [7, Theorem C]. ∎

Lemma 2. *The Lie algebra L/A is finite dimensional and for every $q \in Q$, $l \in L$ we have $q^{-1}lq - l \in A$.*

Proof: Note that $R = U(L/A)\#kQ$ is a finitely generated commutative ring, hence L/A is finite dimensional and $q^{-1}lq - l \in \operatorname{Ker}(H \to R) = H\Omega(U(A))$, where $\Omega(U(A)) = AU(A)$ is the kernel of the counit map $\varepsilon_A : U(A) \to k$. As $q^{-1}lq - l \in L$ and $L \cap HAU(A) = A$ we deduce that $q^{-1}lq - l \in A$. ∎

We observe that in general Q is a finitely generated Abelian group, hence the direct product of a torsion-free subgroup Q_0 and a finite subgroup. If Theorem 4 is known in the case when Q is torsion-free, we can deduce that in the general case, if $A \otimes A$ is a finitely generated R-module then $A \otimes A$ is a finitely generated $U(L)\#kQ_0$-module, and hence $U(L)\#kQ_0$ is finitely presented in $\mathcal{X}$. As Q_0 has finite index in Q this implies easily that $U(L)\#kQ$ is finitely presented in $\mathcal{X}$. Then to show Theorem 4 it is sufficient to consider the case when Q is torsion-free.

From now on we assume that Q is a free Abelian group of rank m with basis $q_1, \ldots, q_m$. Let $x_1, \ldots, x_n$ be elements of L such that $\{x_i' = x_i + A\}_{1 \leqslant i \leqslant n}$ is a basis of L/A as a vector space over k. By Lemma 1 there is a finite set $\{w_1, \ldots, w_z\}$ that generates A as R-module, that is,

$$A = w_1 \circ R + \cdots + w_z \circ R$$

By enlarging the set $\{w_1, \ldots, w_z\}$ if necessary we can assume that the following relations hold in H:

$$\begin{aligned} x_i x_j - x_j x_i = [x_i, x_j] &= w_{\alpha(i,j)} \text{ for } 1 \leqslant i < j \leqslant n, \\ x_i^{q_j^\varepsilon} - x_i &= w_{\beta(i,j,\varepsilon)} \text{ for } 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m, \varepsilon = \pm 1, \\ q_i q_j &= q_j q_i \text{ for } 1 \leqslant i < j \leqslant m. \end{aligned} \tag{1}$$

3.2. Some notation. Let $X = \{W_1, \ldots, W_z, X_1, \ldots, X_n\}$ and $Y = \{Y_1, \ldots, Y_m\}$. We define F as the quotient of $H(X \mid Y)$ by the two-sided ideal generated by all commutators $Y_i Y_j Y_i^{-1} Y_j^{-1} - 1$ for $1 \leqslant i < j \leqslant m$. Then $H(X \mid Y) = U(L_0)\#kF_0$, where F_0 is the free group with basis Y and L_0 is the free Lie algebra with basis the disjoint union of free F_0-orbits $\bigcup_{x \in X} x^{F_0}$ and

$$F = U(L_1)\#kQ,$$

where $Q \simeq F_0/[F_0, F_0]$ and L_1 is the free Lie algebra with basis the disjoint union of free Q-orbits $\bigcup_{x \in X} x^Q$. Thus F is a Hopf algebra in the category $\mathcal{X}$. We identify the image of Y_i in Q with the element q_i defined at the end of section 3.1. Now we consider a surjective homomorphism of Hopf algebras

$$\pi : F \to H,$$

sending W_i to w_i, X_i to x_i and q_i to q_i. Denote by $\widetilde{R}$ the quotient of F by the associative two sided ideal generated by all W_i's. Note that $\widetilde{R}$ is a Hopf algebra in the category $\mathcal{X}$ and π induces a surjective homomorphism of Hopf algebras

$$\rho : \widetilde{R} \to R.$$

There is a k-linear map

$$\eta : R \to \widetilde{R}$$

sending $x_1^{z_1} \ldots x_m^{z_m} q_1^{i_1} \ldots q_n^{i_n}$ to $X_1^{z_1} \ldots X_m^{z_m} q_1^{i_1} \ldots q_n^{i_n}$. Note that η is not a homomorphism of k-algebras. Denote by $\widetilde{\Delta} : \widetilde{R} \to \widetilde{R} \otimes \widetilde{R}$ the comultiplication of $\widetilde{R}$ and by $\Delta : R \to R \otimes R$ the comultiplication of R. As in [5] if not otherwise stated $f, g, \ldots$ denote elements of R and $\widetilde{f}, \widetilde{g}, \ldots$ denote elements of $\widetilde{R}$. We note that by the definitions of $\widetilde{R}$ and R

$$\widetilde{R} \simeq k[X_1, \ldots, X_n, {q_1}^{\pm 1}, \ldots, {q_m}^{\pm 1}]$$

and

$$R \simeq k[x_1, \ldots, x_n, {q_1}^{\pm 1}, \ldots, {q_m}^{\pm 1}],$$

where both rings are associative polynomial rings where the variables ${q_1}^{\pm 1}, \ldots, {q_m}^{\pm 1}$ commute with each other and the variables $x_1, \ldots, x_n$ are central elements. We call

$$w = t_1^{\varepsilon_1} \ldots t_s^{\varepsilon_s},$$

where

$$\varepsilon_i = \pm 1, \ t_i \in \{X_1, \ldots, X_n, q_1^{\pm 1}, \ldots, q_m^{\pm 1}\}$$

(respectively, $t_i \in \{x_1, \ldots, x_n, q_1^{\pm 1}, \ldots, q_m^{\pm 1}\}$), monomial on

$$\{X_1, \ldots, X_n, q_1^{\pm 1}, \ldots, q_m^{\pm 1}\}$$

https://doi.org/10.1017/S0004972700034900 Published online by Cambridge University Press

(respectively, monomial on $\{x_1, \ldots, x_n, q_1^{\pm 1}, \ldots, q_m^{\pm 1}\}$) if w does not have a subword $q_i q_i^{-1}$ and $q_i^{-1} q_i$ for some $i \leqslant m$. By definition the length $|w|$ of w is s. For $f \in R \cup \widetilde{R}$ we define $\operatorname{supp}(f)$ to be the set of all monomials in f and the length $|f|$ of f is

$$|f| = \max\{|w| : w \in \operatorname{supp}(f)\}.$$

By definition a monomial in $R \otimes R$ is $f \otimes g$ for some monomials $f, g \in R$ and has length $|f| + |g|$. A monomial in $\widetilde{R} \otimes \widetilde{R}$ is $\widetilde{f} \otimes \widetilde{g}$ for some monomials $\widetilde{f}, \widetilde{g} \in \widetilde{R}$ and has length $|\widetilde{f}| + |\widetilde{g}|$. If S is a k-linear subspace of any of the following rings $R \otimes 1$, $1 \otimes R$, $\widetilde{R} \otimes 1$, $1 \otimes \widetilde{R}$, $R \otimes R$, $\widetilde{R} \otimes \widetilde{R}$ then S_t denotes the linear subspace of all elements of S of length at most t.

Lemma 3. *For non-negative integers s and t we have*

$$\Delta(R_s)(R \otimes R)_t = (R \otimes R)_t \Delta(R_s)$$

Proof: Note that $R \otimes R$ is a commutative ring. ∎

3.3. Some properties of the map ρ. The following lemma is a generalisation of [**5**, Lemma 2.2].

Lemma 4. *Let $t \geqslant 2$ be a natural number. Then*

1. *$\operatorname{Ker}(\rho)_t$ is generated as a vector space by the elements of $\widetilde{R}$ of the form*

$$\widetilde{p}_1(X_j X_i - X_i X_j)\widetilde{p}_2 \text{ and } \widetilde{p}_1(X_i q_k^{\varepsilon} - q_k^{\varepsilon} X_i)\widetilde{p}_2,$$

where $1 \leqslant i, j \leqslant n$, $1 \leqslant k \leqslant m$, $\varepsilon = \pm 1$, $\widetilde{p}_1, \widetilde{p}_2$ are monomials in $\widetilde{R}$ and $|\widetilde{p}_1| + |\widetilde{p}_2| \leqslant t - 2$.

2. *$\operatorname{Ker}(\rho \otimes \rho)_t$ is generated as a vector space by the elements of $\widetilde{R} \otimes \widetilde{R}$ of the form*

$$\widetilde{p}_1\big((X_j X_i - X_i X_j) \otimes 1\big)\widetilde{p}_2,\ \widetilde{p}_1\big(1 \otimes (X_j X_i - X_i X_j)\big)\widetilde{p}_2,$$
$$\widetilde{p}_1\big((X_i q_k^{\varepsilon} - q_k^{\varepsilon} X_i) \otimes 1\big)\widetilde{p}_2, \widetilde{p}_1\big(1 \otimes (X_i q_k^{\varepsilon} - q_k^{\varepsilon} X_i)\big)\widetilde{p}_2,$$

where $1 \leqslant i, j \leqslant n$, $1 \leqslant k \leqslant m$, $\varepsilon = \pm 1$, $\widetilde{p}_1, \widetilde{p}_2$ are monomials in $\widetilde{R} \otimes \widetilde{R}$ and $|\widetilde{p}_1| + |\widetilde{p}_2| \leqslant t - 2$.

Proof: We give a proof of the first part of the lemma following the proof of [**5**, Lemma 2.2]. We omit the proof of the second part as it is similar. Let $\widetilde{f} \in \widetilde{R}_t$, then we can write $\widetilde{f}$ in the form $\widetilde{f} = \widetilde{f}_1 + \widetilde{f}_2$ where $\widetilde{f}_1$ is a linear combination of monomials of the form $X_1^{z_1} \ldots X_m^{z_m} q_1^{i_1} \ldots q_n^{i_n}$ and $\widetilde{f}_2$ is a linear combination of the elements specified in part 1 of the lemma. Note that $\widetilde{f}_2 \in \operatorname{Ker}(\rho)_t$ and $\widetilde{f}_1 = \eta(f_1)$ for some $f_1 \in R$. If $\widetilde{f} \in \operatorname{Ker}(\rho)_t$ then $\widetilde{f}_1 \in \operatorname{Ker}(\rho)$ and $f_1 = \rho\eta(f_1) = \rho(\widetilde{f}_1) = 0$, hence $\widetilde{f} = \widetilde{f}_2$. ∎

The following lemma is an obvious corollary of the definition of $\operatorname{Ker}(\rho)_t$.

Lemma 5. *Let $\widetilde{p}$ be a monomial in $\widetilde{R}$ of length $t \geqslant 1$. Thus $\widetilde{p}$ is a product of t elements of $X \cup \{q_1^{\pm 1}, \dots, q_m^{\pm 1}\}$ and let $\widetilde{p}_1$ be a product of the same t entries in $\widetilde{p}$ but possibly in a different order. Then*

$$\widetilde{p} - \widetilde{p}_1 \in \operatorname{Ker}(\rho)_t.$$

3.4. The choice of the number e_0. Since R is a Noetherian ring there are elements $g_{r1}, \dots, g_{rc}$ of R such that

$$\operatorname{Ann}_R(w_r) = g_{r1}R + \cdots + g_{rc}R.$$

Since the set of generators $\{w_1, \dots, w_z\}$ of A as a right R-module is finite we may assume that c is independent of r. As in [5] we have

$$\operatorname{Ann}_{R\otimes R}(w_r \otimes w_s) = (g_{r1}\otimes 1)R\otimes R + \cdots + (g_{rc}\otimes 1)R\otimes R + (1\otimes g_{s1})R\otimes R + \cdots + (1\otimes g_{sc})R\otimes R.$$

Note that the $\Delta(R)$-submodule $M_{r,s,k}$ of $A\otimes A$ generated by $\{(w_r\otimes w_s)\circ(x_k^i\otimes 1)\}_{i\geqslant 0}$, is a submodule of the finitely generated $\Delta(R)$-module $A\otimes A$. Since $\Delta(R) \simeq R$ is Noetherian there exists $l \in \mathbb{N}$ such that $M_{r,s,k}$ is generated by the elements $(w_r \otimes w_s) \circ (x_k^i \otimes 1)$ for $0 \leqslant i \leqslant l$. We may assume that l is the same for all $1 \leqslant r, s \leqslant z, 1 \leqslant k \leqslant n$. Then there are elements $f_{rsk0}, f_{rsk1}, \dots, f_{rskl} \in R$ such that

$$x_k^{l+1} \otimes 1 + \sum_{i=0}^{l} (x_k^i \otimes 1)\Delta(f_{rski}) \in \operatorname{Ann}_{R\otimes R}(w_r \otimes w_s). \tag{2}$$

Thus there are elements $\phi_{rsk1}, \dots, \phi_{rskc}, \psi_{rsk1}, \dots, \psi_{rskc} \in R \otimes R$ such that

$$x_k^{l+1} \otimes 1 + \sum_{i=0}^{l} (x_k^i \otimes 1)\Delta(f_{rski}) + \sum_{j=1}^{c} (g_{rj} \otimes 1)\phi_{rskj} + \sum_{j=1}^{c} (1 \otimes g_{sj})\psi_{rskj} = 0 \tag{3}$$

for $1 \leqslant k \leqslant n$ and $1 \leqslant r, s \leqslant z$.

Similarly considering the $\Delta(R)$-submodule of $A\otimes A$ generated by $(w_r\otimes w_s)\circ(q_k^i\otimes 1)$ for $i \geqslant 0$ we get the existence of $\widehat{l} \in \mathbb{N}$, $\widehat{f}_{rsk0}, \widehat{f}_{rsk1}, \dots, \widehat{f}_{rsk\widehat{l}} \in R$, $\widehat{\phi}_{rsk1}, \dots, \widehat{\phi}_{rskc}$ and $\widehat{\psi}_{rsk1}, \dots, \widehat{\psi}_{rskc}$ of $R \otimes R$ such that

$$q_k^{\widehat{l}+1} \otimes 1 + \sum_{i=0}^{\widehat{l}} (q_k^i \otimes 1)\Delta(\widehat{f}_{rski}) \in \operatorname{Ann}_{R\otimes R}(w_r \otimes w_s),$$

$$q_k^{\widehat{l}+1} \otimes 1 + \sum_{i=0}^{\widehat{l}} (q_k^i \otimes 1)\Delta(\widehat{f}_{rski}) + \sum_{j=1}^{c} (g_{rj} \otimes 1)\widehat{\phi}_{rskj} + \sum_{j=1}^{c} (1 \otimes g_{sj})\widehat{\psi}_{rskj} = 0 \tag{4}$$

for $1 \leqslant k \leqslant m$ and $1 \leqslant r, s \leqslant z$.

Similarly considering the $\Delta(R)$-submodule of $A\otimes A$ generated by $(w_r\otimes w_s)\circ(q_k^i\otimes 1)$ for $i \leqslant 0$ we get the existence of

$$\underline{l} \in \mathbb{N},\ \underline{f}_{rsk0}, \underline{f}_{rsk(-1)}, \dots, \underline{f}_{rsk(-\underline{l})} \in R,\ \underline{\phi}_{rsk1}, \dots, \underline{\phi}_{rskc}$$

and $\underline{\psi}_{rsk1}, \dots, \underline{\psi}_{rskc}$ of $R \otimes R$ such that

$$q_k^{-\underline{l}-1} \otimes 1 + \sum_{i=-\underline{l}}^{0} (q_k^i \otimes 1)\Delta(\underline{f}_{rski}) \in \mathrm{Ann}_{R\otimes R}(w_r \otimes w_s),$$

$$(5) \qquad q_k^{-\underline{l}-1} \otimes 1 + \sum_{i=-\underline{l}}^{0} (q_k^i \otimes 1)\Delta(\underline{f}_{rski}) + \sum_{j=1}^{c} (g_{rj} \otimes 1)\underline{\phi}_{rskj} + \sum_{j=1}^{c} (1 \otimes g_{sj})\underline{\psi}_{rskj} = 0$$

for $1 \leqslant k \leqslant m$ and $1 \leqslant r, s \leqslant z$.

Furthermore as $l+1$, $\widehat{l}+1$ and $\underline{l}+1$ correspond to the cardinality of the corresponding finite generating sets of $\Delta(R)$-modules we can assume that $l = \widehat{l} = \underline{l}$. Finally we define

$$(6) \quad e_0 = \max_{0\leqslant i \leqslant l, 1\leqslant r,s\leqslant z, 1\leqslant j\leqslant c, 1\leqslant k\leqslant m, 1\leqslant t\leqslant n} \Big\{ l(n+m)+1, 2\,|f_{rsti}|, \\ 2\left|\widehat{f}_{rski}\right|, 2\left|\underline{f}_{rsk(-i)}\right|, \left|(g_{rj}\otimes 1)\phi_{rstj}\right|, \left|(g_{rj}\otimes 1)\widehat{\phi}_{rskj}\right|, \left|(g_{rj}\otimes 1)\underline{\phi}_{rskj}\right|, \\ \left|(1\otimes g_{sj})\psi_{rstj}\right|, \left|(1\otimes g_{sj})\widehat{\psi}_{rskj}\right|, \left|(1\otimes g_{sj})\underline{\psi}_{rskj}\right| \Big\}.$$

Now we make a simple but important remark. We have defined e_0 as depending on a choice of a generating set $w_1, \dots, w_z$ of A as R-module. We show that if we extend this generating set to $w_1, \dots, w_z, \dots, w_b$ then we still can keep the value of e_0 the same provided the newly added generators are of the type $w_i \circ \lambda$ for some old generator w_i and some $\lambda \in R$. Note that $\mathrm{Ann}_{R\otimes R}(w_r \otimes w_s) \subset \mathrm{Ann}_{R\otimes R}\big((w_r \circ \lambda_r) \otimes (w_s \circ \lambda_s)\big)$ for $\lambda_r, \lambda_s \in R$ and by (2),(4), (5) we can use the values of $l, f, \widehat{f}, \underline{f}, \phi, \widehat{\phi}, \underline{\phi}, \psi, \widehat{\psi}, \underline{\psi}$ for the old generators w_i's and not introduce new ones for the new generator $w_i \circ \lambda$. That is, for $w_\alpha = w_r \circ \lambda_r$ and $w_\beta = w_s \circ \lambda_s$ with both r and s at most z, α or β (or both) in $\{z+1, \dots, b\}$, furthermore if $\alpha \leqslant z$ assume that $\alpha = r$, $\lambda_r = 1$ and if $\beta \leqslant z$ assume that $\lambda_s = 1$, $\beta = s$ we define

$$\begin{array}{ll} f_{\alpha\beta ti} = f_{rsti}, \widehat{f}_{\alpha\beta ki} = \widehat{f}_{rski}, \underline{f}_{\alpha\beta ki} = \underline{f}_{rski} & \text{for } 1 \leqslant t \leqslant n, 1 \leqslant k \leqslant m, 1 \leqslant i \leqslant l, \\ g_{\alpha j} = g_{rj}, g_{\beta j} = g_{sj} & \text{for } 1 \leqslant j \leqslant c, \\ \phi_{\alpha\beta ti} = \phi_{rsti}, \widehat{\phi}_{\alpha\beta ki} = \widehat{\phi}_{rski}, \underline{\phi}_{\alpha\beta ki} = \underline{\phi}_{rski} & \text{for } 1 \leqslant i \leqslant c, 1 \leqslant t \leqslant n, 1 \leqslant k \leqslant m, \\ \psi_{\alpha\beta ti} = \psi_{rsti}, \widehat{\psi}_{\alpha\beta ki} = \widehat{\psi}_{rski}, \underline{\psi}_{\alpha\beta ki} = \underline{\psi}_{rski} & \text{for } 1 \leqslant i \leqslant c, 1 \leqslant t \leqslant n, 1 \leqslant k \leqslant m, \\ \phi_{\alpha\beta ti} = \widehat{\phi}_{\alpha\beta ki} = \underline{\phi}_{\alpha\beta ki} = \psi_{\alpha\beta ti} = \widehat{\psi}_{\alpha\beta ki} = \underline{\psi}_{\alpha\beta ki} = 0 & \text{for } c+1 \leqslant i \leqslant c_1, 1 \leqslant t \leqslant n, 1 \leqslant k \leqslant m, \end{array}$$

where c_1 is the new value for the old parameter c. Note that by enlarging the set of generators $\{w_1, \dots, w_z\}$ to $\{w_1, \dots, w_z, \dots, w_b\}$ the value of c changes to some new natural number $c_1 \geqslant c$. This new number c_1 can be much larger than c but it is not used in the definition of e_0. We do not specify the elements $g_{\alpha j}, g_{\beta j}$ for $c+1 \leqslant j \leqslant c_1$ as we shall not use them.

3.5. Fixing a finite generating set of A. Let $\{w_1, \dots, w_z\}$ be a generating set of A as a right R-module satisfying (1) and $\pi : F \to H$ be the projection defined in subsection 3.2. We define a finite subset E_0 of the Hopf kernel $hker(\pi)$ as the set

$$E_0 = \left\{[X_i, X_j] - W_{\alpha(i,j)}, X_r^{q_k^\varepsilon} - X_r - W_{\beta(r,k,\varepsilon)}\right\}_{1 \leqslant i < j \leqslant n, 1 \leqslant r \leqslant n, 1 \leqslant k \leqslant m, \varepsilon = \pm 1},$$

where $[X_i, X_j] = X_iX_j - X_jX_i$. By reordering $\{w_1, \dots, w_z\}$ if necessary we can assume that

$$\left\{\alpha(i,j), \beta(r,k,\varepsilon)\right\}_{1 \leqslant i < j \leqslant n, 1 \leqslant r \leqslant n, 1 \leqslant k \leqslant m, \varepsilon = \pm 1} = \{1, 2, \dots, z_0\} \text{ for some } z_0 \leqslant z.$$

For any subset B of F and $f_1, f_2 \in F$ we write $f_1 \equiv_B f_2$ if $f_1 - f_2$ belongs to the associative two-sided ideal of F generated by B. We use $\circ$ to denote the adjoint action in F, that is, $a \circ f$ is the image of a under the adjoint action of f. By definition $\widetilde{R}$ is a quotient of F and $\widetilde{R}$ can be identified with the subalgebra of F generated by $\{X_1, \dots, X_n, q_1^{\pm 1}, \dots, q_m^{\pm 1}\}$.

Lemma 6. *Let $\widetilde{f}$ be a monomial in $\widetilde{R}$. Then there exist monomials*

$$t_{i,j,k}^{(1)},\ t_{i,j,k}^{(2)},\ g_{i,j,k}^{(1)},\ g_{i,j,k}^{(2)}$$

in $\widetilde{R}$ such that for $1 \leqslant i, j \leqslant n$

$$\widetilde{f}(X_iX_j - X_jX_i) \equiv_{E_0} \sum_k (\pm W_{\theta(i,j,k)} \circ t_{i,j,k}^{(1)}) g_{i,j,k}^{(1)}$$

and for $\varepsilon = \pm 1$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$

$$\widetilde{f}(X_iq_j^\varepsilon - q_j^\varepsilon X_i) \equiv_{E_0} \sum_k (\pm W_{\mu(i,j,k)} \circ t_{i,j,k}^{(2)}) g_{i,j,k}^{(2)}$$

where all $\theta(i,j,k)$, $\mu(i,j,k) \in \{1, \dots, z_0\}$ and

$$\max\left\{\left|t_{i,j,k}^{(1)}\right|, \left|g_{i,j,k}^{(1)}\right|, \left|t_{i,j,k}^{(2)}\right| - 1, \left|g_{i,j,k}^{(2)}\right| - 1\right\} \leqslant \left|\widetilde{f}\right|.$$

Proof: (1) We induct on the length $\left|\widetilde{f}\right|$ of $\widetilde{f}$, the case when $\left|\widetilde{f}\right| = 0$ that is, $\widetilde{f} = 1$ being obvious since $X_iX_j - X_jX_i \equiv_{E_0} W_{\alpha(i,j)}$ for $i < j$.

Let $\left|\widetilde{f}\right| \geqslant 1$ and $\widetilde{f} = \widetilde{a}\widetilde{f}_0$ where $\widetilde{a} \in \{X_1, \dots, X_n, q_1^{\pm 1}, \dots, q_m^{\pm 1}\}$ and $\widetilde{f}_0 \in \widetilde{R}$ has length $\left|\widetilde{f}\right| - 1$. By the inductive hypothesis

$$\widetilde{f}_0(X_iX_j - X_jX_i) \equiv_{E_0} \sum_k (\pm W_{\theta(i,j,k)} \circ t_{i,j,k}^{(1)}) g_{i,j,k}^{(1)}$$

where all $\theta(i,j,k) \leqslant z_0$ and $\max\left\{\left|t_{i,j,k}^{(1)}\right|, \left|g_{i,j,k}^{(1)}\right|\right\} \leqslant \left|\widetilde{f}_0\right|$. Then

$$\widetilde{f}(X_iX_j - X_jX_i) \equiv_{E_0} \sum_k \widetilde{a}(\pm W_{\theta(i,j,k)} \circ t_{i,j,k}^{(1)}) g_{i,j,k}^{(1)}.$$

Note that if $\widetilde{a} \in \{q_1^{\pm 1}, \ldots, q_m^{\pm 1}\}$

$$\widetilde{a}(W_{\theta(i,j,k)} \circ t_{i,j,k}^{(1)})g_{i,j,k}^{(1)} = \widetilde{a}(W_{\theta(i,j,k)} \circ t_{i,j,k}^{(1)})\widetilde{a}^{-1}\widetilde{a}g_{i,j,k}^{(1)} = \left(W_{\theta(i,j,k)} \circ (t_{i,j,k}^{(1)}\widetilde{a}^{-1})\right)\widetilde{a}g_{i,j,k}^{(1)},$$
$$\max\left\{\left|t_{i,j,k}^{(1)}\widetilde{a}^{-1}\right|, \left|\widetilde{a}g_{i,j,k}^{(1)}\right|\right\} \leqslant \left|\widetilde{f}_0\right| + 1 = \left|\widetilde{f}\right|.$$

If $\widetilde{a} \in \{X_1, \ldots, X_n\}$

$$\begin{aligned}\widetilde{a}(W_{\theta(i,j,k)} \circ t_{i,j,k}^{(1)})g_{i,j,k}^{(1)} &= \left[\widetilde{a}, W_{\theta(i,j,k)} \circ t_{i,j,k}^{(1)}\right] g_{i,j,k}^{(1)} + (W_{\theta(i,j,k)} \circ t_{i,j,k}^{(1)})\widetilde{a}g_{i,j,k}^{(1)} \\ &= -\left(W_{\theta(i,j,k)} \circ (t_{i,j,k}^{(1)}\widetilde{a})\right)g_{i,j,k}^{(1)} + (W_{\theta(i,j,k)} \circ t_{i,j,k}^{(1)})\widetilde{a}g_{i,j,k}^{(1)},\end{aligned}$$
$$\max\left\{\left|t_{i,j,k}^{(1)}\widetilde{a}\right|, \left|\widetilde{a}g_{i,j,k}^{(1)}\right|, \left|t_{i,j,k}^{(1)}\right|, \left|g_{i,j,k}^{(1)}\right|\right\} \leqslant \left|\widetilde{f}_0\right| + 1 = \left|\widetilde{f}\right|.$$

(2) We induct on the length $\left|\widetilde{f}\right|$ of $\widetilde{f}$, the case when $\left|\widetilde{f}\right| = 0$ that is, $\widetilde{f} = 1$ being obvious since

$$X_i q_j^{\varepsilon} - q_j^{\varepsilon} X_i \equiv_{E_0} q_j^{\varepsilon} W_{\beta(i,j,\varepsilon)} = (W_{\beta(i,j,\varepsilon)} \circ q_j^{-\varepsilon})q_j^{\varepsilon}$$

for $i < j$.

Let $\left|\widetilde{f}\right| \geqslant 1$ and $\widetilde{f} = \widetilde{a}\widetilde{f}_0$ where $\widetilde{a} \in \{X_1, \ldots, X_n, q_1^{\pm 1}, \ldots, q_m^{\pm 1}\}$ and $\widetilde{f}_0 \in \widetilde{R}$ has length $\left|\widetilde{f}\right| - 1$. By the inductive hypothesis

$$\widetilde{f}_0(X_i q_j^{\varepsilon} - q_j^{\varepsilon} X_i) \equiv_{E_0} \sum_k (\pm W_{\theta(i,j,k)} \circ t_{i,j,k}^{(2)})g_{i,j,k}^{(2)}$$

where all $\theta(i,j,k) \leqslant z_0$ and $\max\left\{\left|t_{i,j,k}^{(2)}\right|, \left|g_{i,j,k}^{(2)}\right|\right\} \leqslant \left|\widetilde{f}_0\right| + 1$. Then

$$\widetilde{f}(X_i q_j^{\varepsilon} - q_j^{\varepsilon} X_i) \equiv_{E_0} \sum_k \widetilde{a}(\pm W_{\theta(i,j,k)} \circ t_{i,j,k}^{(2)})g_{i,j,k}^{(2)}$$

and we can continue as in the first case. □

Now we are ready to discuss the choice of a generating set of A as a right R-module. First we start with any generating set $\{w_1, \ldots, w_z\}$ of A over R and use it to calculate the numbers e_0, z_0 and c from Sections 3.4 and 3.5. Then we enlarge the generating set to

$$\Lambda = \{w_1, \ldots, w_z\} \cup \{w_i \circ \lambda \mid 1 \leqslant i \leqslant z_0, \lambda \text{ is a monomial in } R_{e_0}\} = \{w_1, \ldots, w_{r_0}\},$$

for some $r_0 \geqslant z$. Thus for $i \leqslant z_0$ and a monomial $\lambda \in R_{e_0}$ there is some natural number $\nu(i, \lambda) \leqslant r_0$ such that

$$w_i \circ \lambda = w_{\nu(i,\lambda)}.$$

Define F with respect to this generating set Λ that is,

F is the quotient of $H(X \mid Y)$ modulo the associative two-sided ideal generated by $\{Y_i Y_j Y_i^{-1} Y_j^{-1} - 1\}_{1 \leqslant i < j \leqslant m}$, where $X = \{W_1, \ldots, W_{r_0}, X_1, \ldots, X_n\}, Y = \{Y_1, \ldots, Y_m\}$.

Now we define a subset E_1 of F using the subset E_0 of F defined at the beginning of Section 3.5

$$E_1 = E_0 \cup \left\{W_i \circ f - W_{\nu(i,\rho(f))} : i \leqslant z_0, f \text{ monomial in } \widetilde{R}_{e_0}\right\},$$

where $\rho : \widetilde{R} \to R$ is the homomorphism of k-algebras sending X_i to x_i and q_j^ε to q_j^ε.

Corollary 1. *Let $\widetilde{f}$ be a monomial in $\widetilde{R}$. Then there exist monomials*

$$t_{i,j,k}^{(1)} \in \widetilde{R}_{\max\{0,|f|-e_0\}}, t_{i,j,k}^{(2)} \in \widetilde{R}_{\max\{0,|f|-e_0+1\}}, g_{i,j,k}^{(1)} \in \widetilde{R}_{|\widetilde{f}|}, g_{i,j,k}^{(2)} \in \widetilde{R}_{|\widetilde{f}|+1}$$

such that for $1 \leqslant i \neq j \leqslant n$

$$\widetilde{f}(X_iX_j - X_jX_i) \equiv_{E_1} \sum_k (\pm W_{\theta'(i,j,k)} \circ t_{i,j,k}^{(1)}) g_{i,j,k}^{(1)}$$

and for $\varepsilon = \pm 1$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$

$$\widetilde{f}(X_iq_j^\varepsilon - q_j^\varepsilon X_i) \equiv_{E_1} \sum_k (\pm W_{\mu'(i,j,k)} \circ t_{i,j,k}^{(2)}) g_{i,j,k}^{(2)},$$

where $\mu'(i,j,k), \theta'(i,j,k) \in \{1, \ldots, r_0\}$.

Proof: 1. By Lemma 6 there exist monomials $\widetilde{t}_{i,j,k}^{(1)}, \widetilde{g}_{i,j,k}^{(1)}$ in $\widetilde{R}$ such that

$$\widetilde{f}(X_iX_j - X_jX_i) \equiv_{E_1} \sum_k (\pm W_{\theta(i,j,k)} \circ \widetilde{t}_{i,j,k}^{(1)}) \widetilde{g}_{i,j,k}^{(1)},$$

$$\max\left\{\left|\widetilde{t}_{i,j,k}^{(1)}\right|, \left|\widetilde{g}_{i,j,k}^{(1)}\right|\right\} \leqslant |\widetilde{f}| \text{ and all } \theta(i,j,k) \leqslant z_0.$$

Let $\widetilde{h}_{i,j,k}^{(1)}$ be the beginning of $\widetilde{t}_{i,j,k}^{(1)}$ of length $\min\left\{e_0, \left|\widetilde{t}_{i,j,k}^{(1)}\right|\right\}$ and $\widetilde{h}_{i,j,k}^{(2)}$ be the rest of $\widetilde{t}_{i,j,k}^{(1)}$ that is $\widetilde{h}_{i,j,k}^{(2)}$ is obtained from $\widetilde{t}_{i,j,k}^{(1)}$ after deleting the beginning $\widetilde{h}_{i,j,k}^{(1)}$. Then

$$W_{\theta(i,j,k)} \circ \widetilde{h}_{i,j,k}^{(1)} \equiv_{E_1} W_{\nu(\theta(i,j,k),\rho(\widetilde{h}_{i,j,k}^{(1)}))},$$

hence

$$(W_{\theta(i,j,k)} \circ \widetilde{t}_{i,j,k}^{(1)}) \widetilde{g}_{i,j,k}^{(1)} \equiv_{E_1} (W_{\nu(\theta(i,j,k),\rho(\widetilde{h}_{i,j,k}^{(1)}))} \circ \widetilde{h}_{i,j,k}^{(2)}) \widetilde{g}_{i,j,k}^{(1)}.$$

Note that

$$\left|\widetilde{h}_{i,j,k}^{(2)}\right| = \left|\widetilde{t}_{i,j,k}^{(1)}\right| - \left|\widetilde{h}_{i,j,k}^{(1)}\right| \leqslant \max\left\{0, \left|\widetilde{t}_{i,j,k}^{(1)}\right| - e_0\right\} \leqslant \max\left\{0, |\widetilde{f}| - e_0\right\}.$$

2. By Lemma 6 there exist monomials $\widetilde{t}_{i,j,k}^{(2)}, \widetilde{g}_{i,j,k}^{(2)}$ in $\widetilde{R}$ such that

$$\widetilde{f}(X_iq_j^\varepsilon - q_j^\varepsilon X_i) \equiv_{E_1} \sum_k (\pm W_{\theta(i,j,k)} \circ \widetilde{t}_{i,j,k}^{(2)}) \widetilde{g}_{i,j,k}^{(2)},$$

$$\max\left\{\left|\widetilde{t}_{i,j,k}^{(2)}\right|, \left|\widetilde{g}_{i,j,k}^{(2)}\right|\right\} \leqslant |\widetilde{f}| + 1 \text{ and all } \theta(i,j,k) \leqslant z_0.$$

Let $\widetilde{h}^{(1)}_{i,j,k}$ be the beginning of $\widetilde{t}^{(2)}_{i,j,k}$ of length $\min\left\{e_0, \left|\widetilde{t}^{(2)}_{i,j,k}\right|\right\}$ and $\widetilde{h}^{(2)}_{i,j,k}$ be the rest of $\widetilde{t}^{(2)}_{i,j,k}$ that is $\widetilde{h}^{(2)}_{i,j,k}$ is obtained from $\widetilde{t}^{(2)}_{i,j,k}$ after deleting the beginning $\widetilde{h}^{(1)}_{i,j,k}$. Then

$$W_{\theta(i,j,k)} \circ \widetilde{h}^{(1)}_{i,j,k} \equiv_{E_1} W_{\nu(\theta(i,j,k),\rho(\widetilde{h}^{(1)}_{i,j,k}))},$$

hence

$$(W_{\theta(i,j,k)} \circ \widetilde{t}^{(2)}_{i,j,k})\widetilde{g}^{(2)}_{i,j,k} \equiv_{E_1} (W_{\nu(\theta(i,j,k),\rho(\widetilde{h}^{(1)}_{i,j,k}))} \circ \widetilde{h}^{(2)}_{i,j,k})\widetilde{g}^{(2)}_{i,j,k}.$$

Note that $\left|\widetilde{h}^{(2)}_{i,j,k}\right| = \left|\widetilde{t}^{(2)}_{i,j,k}\right| - \left|\widetilde{h}^{(1)}_{i,j,k}\right| \leqslant \max\left\{0, \left|\widetilde{t}^{(2)}_{i,j,k}\right| - e_0\right\} \leqslant \max\left\{0, |\widetilde{f}| - e_0 + 1\right\}$. □

Finally we define

$$E = E_1 \cup \{W_i \circ \eta(g_{ij})\}_{1\leqslant i\leqslant r_0, 1\leqslant j\leqslant c} \cup \{[W_i \circ f_i, W_j \circ f_j] \mid 1 \leqslant i,j \leqslant r_0,\ f_i, f_j \text{ monomials in } \widetilde{R} \text{ such that } |f_i| + |f_j| \leqslant e_0\},$$

where $\eta : R \to \widetilde{R}$ is the k-linear map defined in Section 3.2. Thus

$$\begin{aligned} E = &\{[X_i, X_j] - W_{\alpha(i,j)}, X_r^{q_k^\varepsilon} - X_r - W_{\beta(r,k,\varepsilon)}\}_{1\leqslant i<j\leqslant n, 1\leqslant r\leqslant n, 1\leqslant k\leqslant m, \varepsilon=\pm 1} \\ &\cup \{W_i \circ f - W_{\nu(i,\rho(f))} \mid i \leqslant z_0, f \text{ monomial in } \widetilde{R}_{e_0}\} \cup \{W_i \circ \eta(g_{ij})\}_{1\leqslant i\leqslant r_0, 1\leqslant j\leqslant c} \\ &\cup \{[W_i \circ f_i, W_j \circ f_j] \mid 1 \leqslant i,j \leqslant r_0, f_i, f_j \text{ monomials in } \widetilde{R}, |f_i| + |f_j| \leqslant e_0\}. \end{aligned}$$

Note that in the definition of E we use the old value of c obtained before enlarging the generating set of A.

3.6. Some commutator calculations. Our main theorem easily follows from the following Theorem 5. The proof is split in several small lemmas. We remind the reader that we use $\circ$ to denote the adjoint action in F, that is, $a \circ f$ is the image of a under the adjoint action of f.

Theorem 5. *For any $1 \leqslant i,j \leqslant r_0$ and f_i, f_j monomials in $\widetilde{R}$ we have*

$$[W_i \circ f_i, W_j \circ f_j] \equiv_E 0.$$

An induction on $e \geqslant e_0$ will be used to show that

$$[W_i \circ f_i, W_j \circ f_j] \equiv_E 0 \text{ for monomials with } |f_i| + |f_j| \leqslant e. \tag{7}$$

We assume (7) holds for some fixed value of e and aim to show that

$$[W_i \circ f_i, W_j \circ f_j] \equiv_E 0 \text{ for } |f_i| + |f_j| = e + 1.$$

Lemma 7. *$W_i \circ \widetilde{f} \equiv_E 0$ for all $i \leqslant r_0$, $\widetilde{f} \in \mathrm{Ker}(\rho)_{e+2}$.*

https://doi.org/10.1017/S0004972700034900 Published online by Cambridge University Press

PROOF: By Lemma 4(1) $\operatorname{Ker}(\rho)_{e+2}$ is generated as a vector space by the elements of $\widetilde{R}$ of the form

$$\widetilde{p}_1(X_jX_r - X_rX_j)\widetilde{p}_2 \text{ and } \widetilde{p}_1(X_jq_k^\varepsilon - q_k^\varepsilon X_j)\widetilde{p}_2,$$

where $1 \leqslant j < r \leqslant n$, $1 \leqslant k \leqslant m$, $\varepsilon = \pm 1$, $\widetilde{p}_1, \widetilde{p}_2$ are monomials in $\widetilde{R}$ and $|\widetilde{p}_1| + |\widetilde{p}_2| \leqslant e$. Then we can assume that either $\widetilde{f} = \widetilde{p}_1(X_jX_r - X_rX_j)\widetilde{p}_2$ or

$$\widetilde{f} = \widetilde{p}_1(X_jq_k^\varepsilon - q_k^\varepsilon X_j)\widetilde{p}_2.$$

If $\widetilde{f} = \widetilde{p}_1(X_jX_r - X_rX_j)\widetilde{p}_2$ using that $|\widetilde{p}_1| \leqslant e$

$$W_i \circ \widetilde{f} = W_i \circ \big(\widetilde{p}_1(X_jX_r - X_rX_j)\widetilde{p}_2\big) \equiv_E \big[W_i \circ \widetilde{p}_1, W_{\alpha(j,r)}\big] \circ \widetilde{p}_2 \equiv_E 0 \circ \widetilde{p}_2 = 0.$$

If $\widetilde{f} = \widetilde{p}_1(X_jq_k^\varepsilon - q_k^\varepsilon X_j)\widetilde{p}_2$

$$W_i \circ \widetilde{f} = W_i \circ \big(\widetilde{p}_1(X_jq_k^\varepsilon - q_k^\varepsilon X_j)\widetilde{p}_2\big) = W_i \circ \big(\widetilde{p}_1(X_j - X_j^{q_k^{-\varepsilon}})q_k^\varepsilon\widetilde{p}_2\big) \equiv_E$$
$$\big[W_i \circ \widetilde{p}_1, -W_{\beta(j,k,-\varepsilon)}\big] \circ (q_k^\varepsilon\widetilde{p}_2) \equiv_E 0 \circ (q_k^\varepsilon\widetilde{p}_2) = 0. \qquad \square$$

Let W be the associative two-sided ideal of F generated by $W_1, \dots, W_{r_0}$ and

$$\chi : W \otimes W \to [W, W]$$

be the homomorphism of right $\widetilde{\Delta}(\widetilde{R})$-modules sending $m_1 \otimes m_2$ to $[m_1, m_2]$. We view W as a right $\widetilde{R}$-module via the adjoint action, $[W, W]$ is a $\widetilde{R}$-submodule of W and $W \otimes W$ is a right $\widetilde{\Delta}(\widetilde{R})$-module via the comultiplication $\widetilde{R} \to \widetilde{R} \otimes \widetilde{R}$. Abusing the notations we write $\circ$ for both the right adjoint action in F and in $F \otimes F$.

LEMMA 8. *Let $\widetilde{\zeta} \in \operatorname{Ker}(\rho \otimes \rho)_{e+e_0}$. Then for $1 \leqslant r, s \leqslant r_0$*

$$\chi\big((W_r \otimes W_s) \circ \widetilde{\zeta}\big) \equiv_E 0.$$

PROOF: By Lemma 4(2) $\operatorname{Ker}(\rho \otimes \rho)_{e+e_0}$ is spanned by

$$\widetilde{\mu}_1\big((X_jX_i - X_iX_j) \otimes 1\big)\widetilde{\mu}_2,\ \widetilde{\mu}_1\big(1 \otimes (X_jX_i - X_iX_j)\big)\widetilde{\mu}_2,\ \widetilde{\mu}_1\big((X_iq_j^\varepsilon - q_j^\varepsilon X_i) \otimes 1\big)\widetilde{\mu}_2$$

and

$$\widetilde{\mu}_1\big(1 \otimes (X_iq_j^\varepsilon - q_j^\varepsilon X_i)\big)\widetilde{\mu}_2,$$

where $\widetilde{\mu}_1, \widetilde{\mu}_2$ are monomials in $\widetilde{R} \otimes \widetilde{R}$ with $|\widetilde{\mu}_1| + |\widetilde{\mu}_2| \leqslant e + e_0 - 2$ and $\varepsilon = \pm 1$. As the problem is symmetric with respect to swapping the components in $W \otimes W$ it is sufficient to consider only the cases $\widetilde{\zeta} = \widetilde{\mu}_1\big((X_jX_i - X_iX_j) \otimes 1\big)\widetilde{\mu}_2$ and $\widetilde{\zeta} = \widetilde{\mu}_1\big((X_iq_j^\varepsilon - q_j^\varepsilon X_i) \otimes 1\big)\widetilde{\mu}_2$. We write $\widetilde{\mu}_1 = \widetilde{p}_1 \otimes \widetilde{q}_1$, $\widetilde{\mu}_2 = \widetilde{p}_2 \otimes \widetilde{q}_2$, where $\widetilde{p}_1, \widetilde{p}_2, \widetilde{q}_1, \widetilde{q}_2$ are monomials in $\widetilde{R}$.

First we consider the case $\widetilde{\zeta} = \widetilde{\mu}_1\big((X_jX_i - X_iX_j) \otimes 1\big)\widetilde{\mu}_2$. Note that by Corollary 1

$$\widetilde{p}_1(X_iX_j - X_jX_i) \equiv_E \sum_k (\pm W_{\theta'(i,j,k)} \circ \widetilde{t}_k)\widetilde{p}_{3,k}$$

where $\widetilde{t}_k, \widetilde{p}_{3,k}$ are monomials in $\widetilde{R}$ with $|\widetilde{t}_k| \leqslant \max\Big\{|\widetilde{p}_1| - e_0, 0\Big\} \leqslant \max\{e-2, 0\} \leqslant e$ and $|\widetilde{p}_{3,k}| \leqslant |\widetilde{p}_1|$. Then

$$\begin{aligned}\chi\Big((W_r \otimes W_s) \circ \widetilde{\mu}_1\big((X_iX_j - X_jX_i) \otimes 1\big)\widetilde{\mu}_2\Big)& \\ &= \Big[W_r \circ \big(\widetilde{p}_1(X_iX_j - X_jX_i)\widetilde{p}_2\big), W_s \circ (\widetilde{q}_1\widetilde{q}_2)\Big] \\ &\equiv_E \Big[W_r \circ \Big(\sum_k (\pm W_{\theta'(i,j,k)} \circ \widetilde{t}_k)\widetilde{p}_{3,k}\widetilde{p}_2\Big), W_s \circ (\widetilde{q}_1\widetilde{q}_2)\Big] \\ &= \Big[\sum_k [W_r, \pm W_{\theta'(i,j,k)} \circ \widetilde{t}_k] \circ (\widetilde{p}_{3,k}\widetilde{p}_2), W_s \circ (\widetilde{q}_1\widetilde{q}_2)\Big] \equiv_E 0.\end{aligned}$$

The latest equivalence holds since $|\widetilde{t}_k| \leqslant e$ and then $[W_r, \pm W_{\theta'(i,j,k)} \circ \widetilde{t}_k] \equiv_E 0$.

Now we consider the case $\widetilde{\zeta} = \widetilde{\mu}_1\big((X_iq_j^\varepsilon - q_j^\varepsilon X_i) \otimes 1\big)\widetilde{\mu}_2$. Using again Corollary 1

$$\widetilde{p}_1(X_iq_j^\varepsilon - q_j^\varepsilon X_i) \equiv_E \sum_k (\pm W_{\mu'(i,j,k)} \circ \widetilde{t}_k)\widetilde{p}_{3,k}$$

where $\widetilde{t}_k, \widetilde{p}_{3,k}$ are monomials in $\widetilde{R}$ with $|\widetilde{t}_k| \leqslant \max\Big\{|\widetilde{p}_1| - e_0 + 1, 0\Big\} \leqslant \max\{e-1, 0\} \leqslant e$ and $|\widetilde{p}_{3,k}| \leqslant |\widetilde{p}_1| + 1$. Then

$$\begin{aligned}\chi\Big((W_r \otimes W_s) \circ \widetilde{\mu}_1\big((X_iq_j^\varepsilon - q_j^\varepsilon X_i) \otimes 1\big)\widetilde{\mu}_2\Big)& \\ &= \Big[W_r \circ \big(\widetilde{p}_1(X_iq_j^\varepsilon - q_j^\varepsilon X_i)\widetilde{p}_2\big), W_s \circ (\widetilde{q}_1\widetilde{q}_2)\Big] \\ &\equiv_E \Big[W_r \circ \Big(\sum_k (\pm W_{\mu'(i,j,k)} \circ \widetilde{t}_k)\widetilde{p}_{3,k}\widetilde{p}_2\Big), W_s \circ (\widetilde{q}_1\widetilde{q}_2)\Big] \\ &= \Big[\sum_k \Big[W_r, \pm(W_{\mu'(i,j,k)} \circ \widetilde{t}_k)\Big] \circ (\widetilde{p}_{3,k}\widetilde{p}_2), W_s \circ (\widetilde{q}_1\widetilde{q}_2)\Big] \\ &\equiv_E \Big[\sum_k 0 \circ (\widetilde{p}_{3,k}\widetilde{p}_2), W_s \circ (\widetilde{q}_1\widetilde{q}_2)\Big] = 0.\end{aligned}$$

∎

Lemma 9. *If for every monomial $\widetilde{f}$ in $\widetilde{R}$ of length $e+1$ we have $[W_i \circ \widetilde{f}, W_j] \equiv_E 0$ then for all monomials $\widetilde{f}_i, \widetilde{f}_j$ in $\widetilde{R}$ with $|\widetilde{f}_i| + |\widetilde{f}_j| = e+1$ we have $[W_i \circ \widetilde{f}_i, W_j \circ \widetilde{f}_j] \equiv_E 0$.*

Proof: We use induction on $|\widetilde{f}_j|$. If $|\widetilde{f}_j| \geqslant 1$ then $\widetilde{f}_j = \widetilde{g}X_t$ or $\widetilde{f}_j = \widetilde{g}q_t^\varepsilon$ for some $\varepsilon = \pm 1$, $|\widetilde{g}| = |\widetilde{f}_j| - 1$. In the first case as X_t is a Lie element

$$\begin{aligned}\Big[W_i \circ \widetilde{f}_i, W_j \circ \widetilde{f}_j\Big] &= \Big[W_i \circ \widetilde{f}_i, W_j \circ (\widetilde{g}X_t)\Big] \\ &= [W_i \circ \widetilde{f}_i, W_j \circ \widetilde{g}] \circ X_t - \Big[W_i \circ (\widetilde{f}_iX_t), W_j \circ \widetilde{g}\Big] \equiv_E -\Big[W_i \circ (\widetilde{f}_iX_t), W_j \circ \widetilde{g}\Big],\end{aligned}$$

where the last equivalence follows from the fact that $|\widetilde{f}_i| + |\widetilde{g}| = e$, hence

$$[W_i \circ \widetilde{f}_i, W_j \circ \widetilde{g}] \equiv_E 0.$$

https://doi.org/10.1017/S0004972700034900 Published online by Cambridge University Press

In the second case

$$\left[W_i \circ \widetilde{f}_i, W_j \circ \widetilde{f}_j\right] = \left[W_i \circ \widetilde{f}_i, W_j \circ \left(\widetilde{g} q_t^{\varepsilon}\right)\right] = \left[W_i \circ \left(\widetilde{f}_i q_t^{-\varepsilon}\right), W_j \circ \widetilde{g}\right] \circ q_t^{\varepsilon},$$

hence $\left[W_i \circ \widetilde{f}_i, W_j \circ \widetilde{f}_j\right] \equiv_E 0$ if and only if $\left[W_i \circ \widetilde{f}_i q_t^{-\varepsilon}, W_j \circ \widetilde{g}\right] \equiv_E 0$. This completes the inductive step. ∎

Theorem 6. *Let f be a monomial in $\widetilde{R}$ such that $\left|\widetilde{f}\right| = e+1$. Then for $1 \leqslant r, s \leqslant r_0$*

$$\left[W_r \circ \widetilde{f}, W_s\right] \equiv_E 0.$$

Proof: By Lemma 5 if $\widetilde{f}_1$ is obtained from $\widetilde{f}$ by reordering of the entries then $\widetilde{f} - \widetilde{f}_1 = \widetilde{\lambda}_1 \in \operatorname{Ker}(\rho)_{e+1}$. By Lemma 7 $\left[W_r \circ \widetilde{\lambda}_1, W_s\right] \equiv_E 0$. Hence

$$\left[W_r \circ \widetilde{f}, W_s\right] \equiv_E 0 \text{ if and only if } \left[W_r \circ \widetilde{f}_1, W_s\right] \equiv_E 0.$$

Let z_i be the number of entries of X_i in f and m_i be the sum of all possible $\varepsilon = \pm 1$ such that q_i^{ε} is a subword of $\widetilde{f}$. Since $e_0 \geqslant l(n+m)+1$ either there is some

$$z_i \geqslant l+1$$

or there is some

$$|m_i| \geqslant l+1$$

or there is a reordering $\widetilde{f}_1$ of $\widetilde{f}$ such that a cancelation $q_i^{\varepsilon} q_i^{-\varepsilon}$ occurs in $\widetilde{f}_1$, hence

$$\left|\widetilde{f}_1\right| < \left|\widetilde{f}\right|.$$

In the last case we have $\left|\widetilde{f}_1\right| \leqslant e$, hence $\left[W_r \circ \widetilde{f}_1, W_s\right] \equiv_E 0$ and we are done. Then we can assume that we are in one of the first two cases and consider some reordering $\widetilde{f}_1$ of $\widetilde{f}$ such that

$$\widetilde{f}_1 = \widetilde{b}^{l+1} \widetilde{h}$$

for some monomial $\widetilde{h} \in Im(\eta) \subset \widetilde{R}$ that does not start with $\widetilde{b}^{-1}$, where

$$\widetilde{b} \in \{X_1, \ldots, X_n, q_1^{\pm 1}, \ldots, q_m^{\pm 1}\}.$$

We write h for $\rho(\widetilde{h})$ and b for $\rho(\widetilde{b})$. Note that $\widetilde{h} \in \widetilde{R}_{e-l}$, hence $h \in R_{e-l}$. Following [5] we multiply (3) or (4) or (5) depending on the value of b with $h \otimes 1$ and obtain for the fixed $r, s \in \{1, \ldots, r_0\}$

$$(8) \qquad (b^{l+1} h) \otimes 1 + \alpha + \sum_{j=1}^{c} (g_{rj} \otimes 1) \beta_{rskj} + \sum_{j=1}^{c} (1 \otimes g_{sj}) \gamma_{rskj} = 0,$$

https://doi.org/10.1017/S0004972700034900 Published online by Cambridge University Press

where

$$b = x_k, \alpha = \sum_{i=0}^{l}(b^i \otimes 1)\Delta(f_{rski})(h \otimes 1), \beta_{rskj} = \phi_{rskj}(h \otimes 1), \gamma_{rskj} = \psi_{rskj}(h \otimes 1) \quad \text{or}$$

$$b = q_k, \alpha = \sum_{i=0}^{l}(b^i \otimes 1)\Delta(\widehat{f}_{rski})(h \otimes 1), \beta_{rskj} = \widehat{\phi}_{rskj}(h \otimes 1), \gamma_{rskj} = \widehat{\psi}_{rskj}(h \otimes 1) \quad \text{or}$$

$$b = q_k^{-1}, \alpha = \sum_{i=-l}^{0}(b^i \otimes 1)\Delta(\underline{f}_{rski})(h \otimes 1), \beta_{rskj} = \underline{\phi}_{rskj}(h \otimes 1), \gamma_{rskj} = \underline{\psi}_{rskj}(h \otimes 1).$$

Note that the choice of e_0 together with Lemma 3 imply

(9) $$\alpha \in (R \otimes R)_l \Delta(R_{e_0/2})(R \otimes R)_{e-l} = (R \otimes R)_e \Delta(R_{e_0/2}) \subseteq (R \otimes R)_{e+e_0},$$
$$(g_{rj} \otimes 1)\beta_{rskj}, (1 \otimes g_{sj})\gamma_{rskj} \in (R \otimes R)_{e_0+e-l} \subseteq (R \otimes R)_{e+e_0}.$$

We lift (8) in $\widetilde{R} \otimes \widetilde{R}$ to find $\widetilde{\tau} \in \widetilde{R} \otimes \widetilde{R}$ such that

(10) $$\widetilde{\tau} = \widetilde{f}_1 \otimes 1 + \widetilde{\alpha} + \sum_j (\eta(g_{rj}) \otimes 1)\widetilde{\beta}_{rskj} + \sum_j \big(1 \otimes \eta(g_{sj})\big)\widetilde{\gamma}_{rskj},$$

where $\widetilde{\beta}_{rskj} = (\eta \otimes \eta)(\beta_{rskj})$, $\widetilde{\gamma}_{rskj} = (\eta \otimes \eta)(\gamma_{rskj})$ and by (9) we can choose

$$\widetilde{\alpha} \in \big(\widetilde{R} \otimes \widetilde{R}\big)_e \widetilde{\Delta}\big(\widetilde{R}_{e_0/2}\big) \subseteq \big(\widetilde{R} \otimes \widetilde{R}\big)_{e+e_0},$$

such that $(\rho \otimes \rho)(\widetilde{\alpha}) = \alpha$. Note that

$$(\eta(g_{rj}) \otimes 1)\widetilde{\beta}_{rskj}, \big(1 \otimes \eta(g_{sj})\big)\widetilde{\gamma}_{rskj} \in \big(\widetilde{R} \otimes \widetilde{R}\big)_{e+e_0}.$$

Then

$$\widetilde{\tau} \in \operatorname{Ker}(\rho \otimes \rho)_{e+e_0}.$$

Note that

$$\big[W_r \circ \widetilde{f}_1, W_s\big] \equiv_E 0$$

is equivalent to

$$\chi\Big((W_r \otimes W_s) \circ \big(\widetilde{f}_1 \otimes 1\big)\Big) \equiv_E 0.$$

Then by (10) it is sufficient to show that

$$\chi\big((W_r \otimes W_s) \circ \widetilde{\zeta}\big) \equiv_E 0 \text{ for } \widetilde{\zeta} \in \Big\{\widetilde{\tau}, \widetilde{\alpha}, (\eta(g_{rj}) \otimes 1)\widetilde{\beta}_{rskj}, \big(1 \otimes \eta(g_{sj})\big)\widetilde{\gamma}_{rskj}\Big\}.$$

These are covered by the following 3 cases:

1. $\widetilde{\zeta} \in \operatorname{Ker}(\rho \otimes \rho)_{e+e_0}$ for $\widetilde{\zeta} = \widetilde{\tau}$;
2. $\widetilde{\zeta} \in \big(\widetilde{R} \otimes \widetilde{R}\big)_e \widetilde{\Delta}\big(\widetilde{R}\big)$ for $\widetilde{\zeta} = \widetilde{\alpha}$;

https://doi.org/10.1017/S0004972700034900 Published online by Cambridge University Press

3. $\widetilde{\zeta} \in (\eta(g_{rj}) \otimes 1)\widetilde{R} \otimes \widetilde{R}$ or $\big(1 \otimes \eta(g_{sj})\big)\big(\widetilde{R} \otimes \widetilde{R}\big)$ for $\widetilde{\zeta} \in \big\{(\eta(g_{rj}) \otimes 1)\widetilde{\beta}_{rskj}, (1 \otimes \eta(g_{sj})\widetilde{\gamma}_{rskj}\big\}$.

Case 1 follows from Lemma 8. Case 2 follows by (7) and the fact that χ is a homomorphism of right $\widetilde{\Delta}(\widetilde{R})$-modules. Case 3 follows from the fact that $W_r \circ \eta(g_{rj}) \in E$, hence $W_r \circ \eta(g_{rj}) \equiv_E 0$. Similarly $W_s \circ \eta(g_{sj}) \equiv_E 0$. This completes the proof of Theorem 6. ∎

Note that Lemma 9 and Theorem 6 complete the proof of Theorem 5.

4. Proofs of Theorem 4 and Theorem 1

4.1. Proof of Theorem 4. We remind the reader that by Theorem 5 there is a finite subset E of $F = U(L_1)\#kQ$ such that

$$[W_i \circ f_i, W_j \circ f_j] \equiv_E 0$$

for all $i, j \leqslant r_0$ and all monomials f_i, f_j in $\widetilde{R}$. Let H_1 be the associative ring quotient of F modulo the two-sided ideal of F generated by E. Note that the elements of E are in fact Lie elements of F, that is, for the comultiplication Δ_F of F we have $\Delta_F(r) = r \otimes 1 + 1 \otimes r$ for $r \in E$. Then H_1 is a Hopf algebra in the category $\mathcal{X}$ that is, $H_1 \simeq U(L_0)\#kQ$, where L_0 is a Lie algebra quotient of L_1.

Let B_0 be the Lie subalgebra of L_0 such that B_0 is generated as a Lie algebra by the H_1-orbits of all images of $\{W_i\}_{1 \leqslant i \leqslant r_0}$ in H_1, where H_1 acts via the right adjoint action. Then by Theorem 5 B_0 is an Abelian Lie algebra. Furthermore note that H_1 is a Hopf extension of the universal enveloping algebra $U(B_0)$ by R, that is, there is a short exact sequence of Hopf algebras $U(B_0) \to H_1 \to R$ sending the image of X_i in H_1 to $x_i \in R$ and the image of q_j^ε in H_1 to $q_j^\varepsilon \in R$ for $\varepsilon = \pm 1$. Since H_1 acts on B_0 via the right adjoint action and B_0 is Abelian, B_0 acts trivially. This induces an action of R on B_0. Then there is a homomorphism of right R-modules

$$\pi : B_0 \to A$$

sending the image of W_i in B_0 to $w_i \in A$, where R acts on A via the right adjoint action induced from the short exact sequence of Hopf algebras $U(A) \to H \to R$. Since R is Noetherian $\operatorname{Ker} \pi$ is finitely generated as an R-module. Then there exists a finite subset $\widetilde{E}$ of L_1 such that the associative ring that is the quotient of F modulo the associative two-sided ideal generated by $E \cup \widetilde{E}$ is isomorphic to H. Then by the definition of F and Theorem 2(b) H is finitely presented in the category $\mathcal{X}$.

4.2. Proof of Theorem 1. By [7, Proposition 2] every finitely presented Hopf algebra in the category $\mathcal{X}$ is of homological type FP_2, thus 1. implies 2. By [7, Corollary 3] if H is a Hopf algebra in $\mathcal{X}$ of type FP_m such that H is a Hopf extension of $H_1 = U(L_1)$ by a Hopf algebra H_2 in $\mathcal{X}$, L_1 is an Abelian Lie algebra and H_2 is right Noetherian as an associative k-algebra then the mth homology $H_m(L_1) \simeq \wedge^m L_1$ of the Lie algebra L_1 is finitely generated as a right H_2-module via the iterated comultiplication $H_2 \to \otimes^m H_2$, where we view L_1 as a right H_2-module via the adjoint action. Applying this result for $m = 2$, $L_1 = A$ and $H_2 = R$ we get that 2. implies 3. Finally 3. implies 4. follows from Theorem 3 and 4. implies 1 follows from Theorem 4.

References

[1] G. Baumslag, 'Some remarks on finitely presented algebras', *J. Pure Appl. Algebra* **8** (1976), 187–196.

[2] M. Bestvina and N. Brady, 'Morse theory and finiteness properties of groups', *Invent. Math.* **129** (1997), 445–470.

[3] R. Bieri and J.R.J. Groves, 'Metabelian groups of type FP_∞ are virtually of type FP', *Proc. London Math. Soc. (3)* **45** (1982), 365–384.

[4] R. Bieri and R. Strebel, 'Valuations and finitely presented metabelian groups', *Proc. London Math. Soc. (3)* **41** (1980), 439–464.

[5] R. Bryant and J.R.J. Groves, 'Finite presentation of abelian-by-finite dimensional Lie algebras', *J. London Math. Soc. (2)* **60** (1999), 45–57.

[6] R. Bryant and J.R.J. Groves, 'Finitely presented Lie algebras', *J. Algebra* **218** (1999), 1–25.

[7] D.H. Kochloukova, 'Finite presentability and the homological type FP_m for a class of Hopf algebras', *Comm. Algebra* (to appear).

[8] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics **82** (American Mathematical Society, Providence, R.I., 1993).

IMECC-UNICAMP
Cx. P. 6065
13083-970 Campinas, SP
Brazil