PRE-VECTOR VARIATIONAL INEQUALITIES

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Existence theorems for pre-vector variational inequalities are established under different conditions on the operator \( T \) and the function \( \eta \). As an application, we establish the existence of a weak minimum of an optimisation problem on \( \eta \)-invex functions.

1. INTRODUCTION

Throughout this paper, let \( X, Z \) be Banach spaces, \((Y, D)\) be an ordered Banach spaces, ordered by a closed convex cone \( D \). Let \( L(X, Y) \) be the space of all bounded linear operators from \( X \) to \( Y \), \( E \subseteq X \) and \( C \subseteq Z \) be nonempty sets, \( \eta : E \times E \to E \) be a function, \( V : E \to 2^C \) and \( G : E \to 2^E \) be set-valued maps. We consider the following three problems:

PRE-VVIP. Find \( \bar{x} \in E \) such that

\[ \langle T(\bar{x}), \eta(y, \bar{x}) \rangle \not\preceq 0 \text{ for all } y \in E, \]

where \( T \) is a map from \( E \) to \( L(X, Y) \).

PRE-QVVIP. Find \( \bar{x} \in E \), \( \bar{y} \in V(\bar{x}) \) such that

\[ \langle H(\bar{x}, \bar{y}), \eta(y, \bar{x}) \rangle \not\preceq 0 \text{ for all } y \in G(\bar{x}), \]

where \( H \) is a map from \( E \times C \) to \( L(X, Y) \).

The Pre-VVIP has some relation with vector optimisation problems of \( \eta \)-invex function.

(P) \( \text{V-min } f(x) \text{ subject to } x \in E, \)

where \( f : E \to Y \) is a \( \eta \)-invex function [8].

It is easy to see that if \( \bar{x} \in E \), and \( T(\bar{x}) \) is the Fréchet derivative of \( f \) at \( \bar{x} \), and if \( \bar{x} \) is a solution of Pre-VVIP, then \( \bar{x} \) is a weak-minimum of (P).

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Hence sufficient conditions for the existence theorem of Pre-VVIP are also sufficient conditions for the existence of the weak minimum of (P). Therefore the study of Pre-VVIP is important in research concerning vector optimisation problems of $\eta$-invex functions.

In [7], F. Giannesi first introduced vector variational inequalities in a finite dimensional Euclidean space. Since then, many results have been obtained on the vector-variational inequality and vector complementary problems [2, 3, 4, 13]. In [2, 3, 13], Cheng, Yang and Cheng, considered the case $\eta(y, x) = y - x$ in Pre-VIIP and Pre-QVVIP. In [11], Parida, Sahoo and Kumar considered the case $Y = R$, $D = R_+$ and $X = R^n$ in Pre-VVIP. If $X = R^n$, $Y = R$, $D = R_+$, $\eta(y, x) = y - x$, then Pre-VVIP reduces to the well-known Hartman and Stampacchia variational inequality problem [9]. If $X = R^n$, $Z = R^m$, $Y = R$, $D = R_+$, $G(x) = E$ for all $x \in E$, then the Pre-Quasi VVIP reduces to the problem studied by Parida and Sen [10].

In this paper, we investigate existence theorems for Pre-VVIP, Pre-QVVIP and as a consequence of our results, we establish sufficient conditions for the existence theorem of a weak minima [3] of the problem (P).

2. PRELIMINARIES

Throughout this paper, let $D^*$ be the polar cone of $D$. Let $x, y \in Y$. We denote $x \leq y$ if $y - x \in D$ and $x < y$ if $y - x \notin \text{int}D$. If $D$ is a pointed, closed, convex cone and $D$ induces a partial order in $Y$, then $(Y, D)$ is called an ordered topological vector space.

DEFINITION 1: Let $T : X \to L(X,Y)$, $\eta : X \times X \to X$. Then $T$ is said to be $\eta$-monotone if $(T(x), \eta(x, y)) - (T(y), \eta(x, y)) \geq 0$ for all $x, y \in X$.

DEFINITION 2: [8] Let $f : X \to Y$ be Fréchet differentiable on $X$. Then $f$ is said to be $\eta$-invex on $X$ if there exists a function $\eta : X \times X \to Y$ such that for all $x, y \in X$,

$$f(y) - f(x) \geq \langle Df(x), \eta(y, x) \rangle,$$

where $Df(x)$ is the Fréchet derivative of $f$ at $x$.

DEFINITION 3: Let $T : E \subseteq X \to L(X,Y)$. Then $T$ is said to be pre-$\eta$-hemicontinuous if for all $x, y \in E$, the map $t \to \langle T(x + t(y - x)), \eta(y, x) \rangle$ is continuous at $t = 0$.

3. MAIN RESULTS

LEMMA 1. Let $E \subseteq X$ be a non-empty convex subset and $\eta : E \times E \to E$ be a map with $\eta(x, x) = 0$, for all $x \in E$. Suppose that $T : E \to L(X,Y)$ is $\eta$-monotone.
and pre-v-hemicontinuous and the map \((T(x), \eta(u, y))\) is convex with respect to \(u \in E\). Then the following two problems are equivalent.

(a) Find \(x \in E\) such that \(\langle T(x), \eta(y, x) \rangle \neq 0\) for all \(y \in E\).

(b) Find \(x \in E\) such that \(\langle T(y), \eta(y, x) \rangle \neq 0\) for all \(y \in E\).

**Proof:** (a) That implies (b) follows immediately from the \(\eta\)-monotonicity of \(T\).

Conversely, if (b) holds for each \(x \in E\), then

\[
\langle T(\lambda y + (1 - \lambda)z, \eta(\lambda y + (1 - \lambda)z, x)) \rangle \neq 0, \text{ for all } y \in E.
\]

Since \(\langle T(x), \eta(u, y) \rangle\) is convex with respect to \(u\) and \(\eta(x, x) = 0\), it follows that

\[
\langle T(x + \lambda(y - z), \eta(x + \lambda(y - z), x)) \rangle \leq \lambda \langle T(x + \lambda(y - z), \eta(y, x)) \rangle \text{ for all } 0 < \lambda < 1.
\]

(1) and (2) imply

\[
\langle T(x + \lambda(y - z), \eta(y, x)) \rangle \neq 0 \text{ for all } \lambda \in (0, 1).
\]

Since \(T\) is pre-v-hemicontinuous, it follows from (3) that

\[
\langle T(x), \eta(y, x) \rangle \neq 0 \text{ for all } y \in E.
\]

Hence (a) is true.

**Theorem 1.** Let \(\text{int}D \neq \emptyset\) and \(\text{int}D^* \neq \emptyset\). Let \(E\) be a nonempty, compact convex set in \(X\), \(\eta : E \times E \to E\) be a map, \(\eta(x, x) = 0\), for all \(x \in E\). Suppose \(T : E \to L(X, Y)\) is \(\eta\)-monotone, pre-v-hemicontinuous and \(\langle T(x), \eta(u, y) \rangle\) is convex with respect to \(u\), and for each fixed \(y \in E\), \(\eta(y, x)\) is continuous with respect to \(x\) on \(E\). Then there exists \(\bar{x} \in E\) such that

\[
\langle T(\bar{x}), \eta(x, \bar{x}) \rangle \neq 0 \text{ for all } x \in E.
\]

**Proof:** For each fixed \(y \in E\), let \(F_1(y) = \{x \in E \mid \langle T(x), \eta(y, x) \rangle \neq 0\}\). Then \(F_1 : E \to 2^E\). We prove that \(F_1\) is a KKM map [12]. If this is not the case, there exists a finite set \(A = \{x_1, \ldots, x_n\} \subseteq E\) such that \(\text{cov}A \not\subset \bigcup_{i=1}^n F_1(x_i)\), where \(\text{cov}A\) denotes the convex hull of \(A\). Hence there exist \(\alpha_i \geq 0\), for all \(i = 1, \ldots, n\), \(\sum_{i=1}^n \alpha_i = 1\) and \(x = \sum_{i=1}^n \alpha_i x_i\) such that \(x \not\in \bigcup_{i=1}^n F_1(x_i)\). Then \(x \not\in F_1(x_i)\) for all \(i = 1, \ldots, n\). Hence \(\langle T(x), \eta(x_i, x) \rangle < 0\) for all \(i = 1, \ldots, n\). Since \(\eta(x, x) = 0\) and \(T(x) \in L(X, Y)\), it follows that

\[
0 = \langle T(x), \eta(x, x) \rangle \leq \sum_{i=1}^n \alpha_i \langle T(x), \eta(x_i, x) \rangle < 0.
\]
This leads to a contradiction. Hence \( F_1 \) is a KKM map.

Let \( F_2(y) = \{ x \in E \mid \langle T(y), \eta(y, x) \rangle \leq 0 \} \).

Since \( T \) is \( \eta \)-monotone, it is easy to see that \( F_2 \) is also a KKM map on \( E \). By Lemma 1

\[
\bigcap_{y \in E} F_1(y) = \bigcap_{y \in E} F_2(y).
\]

Since for each fixed \( y \in E \), we have \( T(y) \in L(X, Y) \) and \( \eta(y, x) \) is continuous with respect to \( x \in E \) and \( Y \setminus \text{int } D \) is closed, it follows that \( F_2(y) \) is a compact subset in \( E \). By the F-KKM theorem [5].

\[
\bigcap_{y \in E} F_1(y) = \bigcap_{y \in E} F_2(y) \neq \emptyset.
\]

Hence there exists \( \bar{x} \in E \) such that

\[
\langle T(\bar{x}), \eta(x, \bar{x}) \rangle \leq 0 \quad \text{for all } x \in E.
\]

\[\]

**Lemma 2.** Let \( E \subseteq X \) be a nonempty convex set and \( \eta : E \times E \to E \) be a map with \( \eta(x, x) = 0 \) for all \( x \in E \). Suppose \( T = (T_1, \cdots, T_n) : E \to L(X, R^n) \) is \( \eta \)-monotone and \( \text{pre-}v\text{-hemicontinuous} \). Suppose further that for fixed \( x, y \in E \) and for each \( i = 1, \cdots, n \), the map \( \langle T_i(x), \eta(u, y) \rangle \) is strongly quasiconvex with respect to \( u \in E \) and \( R^n \) is ordered by \( R^n_+ = \{ x = (x_1, \cdots, x_n) : x_i \geq 0 \text{ for all } i = 1, \cdots, n \} \). Then the following two problems are equivalent.

(a) Find \( x \in E \) such that \( \langle T(x), \eta(y, x) \rangle \leq 0 \) for all \( y \in E \).

(b) Find \( x \in E \) such that \( \langle T(y), \eta(y, x) \rangle \leq 0 \) for all \( y \in E \).

**Proof:** That (a) \( \Rightarrow \) (b) is the same as Lemma 1. Conversely, suppose (b) holds. Then there exists \( x \in E \) such that \( \langle T(y), \eta(y, x) \rangle \leq 0 \) for all \( y \in E \). Let \( y \in E \), \( y \neq x \) and \( 0 < \lambda < 1 \), then \( \langle T(\lambda y + (1 - \lambda)x), \eta(\lambda y + (1 - \lambda)x, x) \rangle \leq 0 \). Hence there exists \( 1 \leq i \leq n \) such that

\[
\langle T_i(\lambda y + (1 - \lambda)x), \eta(\lambda y + (1 - \lambda)x, x) \rangle \geq 0.
\]

Since \( (T_i(x), \eta(u, y)) \) is strongly quasiconvex with respect to \( u \in E \),

\[
0 \leq \langle T_i(\lambda y + (1 - \lambda)x), \eta(\lambda y + (1 - \lambda)x, x) \rangle
\]

\[
< \max\{ \langle T_i(\lambda y + (1 - \lambda)x), \eta(y, x) \rangle, \langle T_i(\lambda y + (1 - \lambda)x), \eta(x, x) \rangle \}
\]

\[
= \max\{ \langle T_i(\lambda y + (1 - \lambda)x, \eta(y, x)) \rangle, 0 \}.
\]
Hence \( \langle T_i(\lambda y + (1 - \lambda)z, \eta(y, z)) \rangle > 0 \), and \( \langle T(\lambda y + (1 - \lambda)z), \eta(y, z) \rangle \not< 0 \). Then following the same argument as Lemma 1, we can show that

\[
\langle T(z), \eta(y, z) \rangle \not< 0 \text{ for all } y \in E.
\]

**Theorem 2.** Let \( E \subseteq X \) be a nonempty convex set in \( E \), \( \eta : E \times E \to E \) be a function, and for each fixed \( y \in E \), let the map \( \eta(y, z) \) be a continuous function of \( z \) on \( E \) which, \( \eta(z, z) = 0 \) for all \( z \in E \). Suppose that \( T = (T_1, \ldots, T_n) : E \to L(X, R^n) \) is \( \eta \)-monotone and pre-\( v \)-hemicontinuous. For fixed \( x, y \in E \) and for each \( i = 1, 2, \ldots, n \), suppose \( \langle T_i(z), \eta(u, y) \rangle \) is strongly quasiconvex with respect to \( u \). Suppose further that there exists a compact convex subset \( K \) of \( E \) such that for each \( y \in E \setminus K \) there exists \( z \in K \) with \( \langle T(y), \eta(z, y) \rangle < 0 \). Then there exists a \( x \in K \) such that \( \langle T(x), \eta(x, x) \rangle \not< 0 \) for all \( x \in E \).

By Lemma 2 and with the same argument as in the proof of Theorem 1, we can show that for every compact set \( M \subseteq E \) there exists an \( x \in M \) such that \( \langle T(x), \eta(x, x) \rangle \not< 0 \) for all \( x \in M \). For each \( y \in E \), let

\[
K(y) = \{ x \in K, \langle T(x), \eta(y, z) \rangle \not< 0 \}.
\]

Since \( T : E \to L(X, Y) \) is continuous and \( Y \setminus \text{int}D \) is a closed set, it follows that the set \( K(y) \) is closed in \( K \) and hence compact. Let \( \{ y_1, \ldots, y_m \} \subseteq E \) and let \( A = \text{cov}[K \cup \{ y_1, \ldots, y_m \}] \). Thus \( A \) is a compact and convex set in \( E \), so there exists an \( x \in E \) such that

\[
\langle T(x), \eta(y, x) \rangle \not< 0 \text{ for all } y \in A.
\]

Now \( x \in K \), for otherwise, there exists a \( y \in K \) such that \( \langle T(x), \eta(y, x) \rangle < 0 \), which contradicts (4). Since \( \langle T(x), \eta(y, x) \rangle \not< 0 \) for all \( x \in A \), it follows that \( x \in \bigcap_{i=1}^m K(y_i) \).

Thus the family of closed subsets \( \{ K(y) : y \in E \} \) has the finite intersection property. Since \( K \) is compact, it follows that \( \bigcap_{y \in E} K(y) \neq \emptyset \). So there exists an \( x_0 \in K(y) \) for all \( y \in E \). Therefore there exists a \( x_0 \in K \) such that \( \langle T(x_0), \eta(x_0, x_0) \rangle \not< 0 \) for all \( y \in E \).

**Lemma 3.** [1] Let \( G : X \to 2^Y \) and \( W \) be a real valued function defined on \( X \times Y \), \( V(x) = \sup_{y \in G(x)} W(x, y) \) and \( M(x) = \{ y \in G(x) \mid V(x) = W(x, y) \} \). Suppose that

(a) \( W \) is continuous on \( X \times Y \).

(b) \( G \) is continuous [1] with compact values [1].

Then the set-valued map \( M \) is upper semicontinuous [1].

**Theorem 3.** Let \( E \) be a nonempty compact convex set in \( X \) and \( C \) a compact convex set in \( Y \). Let \( V : E \to 2^C \) be upper semicontinuous, convex and closed valued and let \( \phi : E \times C \times E \to R \) be continuous. Suppose that

(a) \( \phi(x, y, z) \geq 0 \) for all \( x \in E \),
For each fixed \((x, y) \in E \times C\), \(\phi(x, y, u)\) is quasiconvex with respect to \(u \in E\).

(c) \(G : E \to 2^E\) is continuous with compact convex values.

Then there exists \(\bar{x} \in G(\bar{x})\) and \(\bar{y} \in V(\bar{x})\) such that

\[ \phi(\bar{x}, \bar{y}, x) \geq 0 \text{ for all } x \in G(\bar{x}). \]

**Proof:** For each \((x, y) \in E \times Y\), let

\[ \pi(x, y) = \{s \in G(x) \mid \phi(x, y, s) = \min_{u \in G(x)} \phi(x, y, u)\}. \]

Then it follows from Lemma 3 that \(\pi(x, y)\) is upper semicontinuous. Since \(\phi(x, y, u)\) is quasiconvex with respect to \(u\), it follows that \(\pi(x, y)\) is a convex subset of \(E\). The set-valued function \(F : E \times C \to 2^E \times 2^C\) is defined by \(F(x, y) = \{(\pi(x, y), V(x))\}\). Then \(F\) is nonempty, convex closed and upper semicontinuous. By the generalised Kakutani fixed point theorem [6], there exists \((\bar{x}, \bar{y}) \in E \times C\) such that \((\bar{x}, \bar{y}) \in F(\bar{x}, \bar{y})\). Hence there exist a \(\bar{x} \in G(\bar{x})\) and a \(\bar{y} \in V(\bar{x})\) such that

\[ \phi(\bar{x}, \bar{y}, x) \geq 0 \text{ for all } x \in G(\bar{x}). \]

**Theorem 4.** Let \(E\) be a nonempty convex set in \(X\) and \(C\) a closed convex set in \(Y\). Let \(V : E \to 2^C\) be an upper semicontinuous closed and convex valued map and let \(\phi : E \times C \times E \to \mathbb{R}\) be a continuous function. Suppose that

(a) \(\phi(x, y, x) \geq 0\) for all \(x \in E\).

(b) For each fixed \((x, y) \in E \times C\), \(\phi(x, y, u)\) is quasiconvex with respect to \(u \in E\).

(c) There exists nonempty compact convex set \(K \subseteq E\) such that for each \((x, y) \in E \times C\) with \(x \notin K\), there exists \(u \in K\) such that \(\phi(x, y, u) < 0\).

Then there exist a \(\bar{x} \in K\), and a \(\bar{y} \in V(\bar{x})\) such that

\[ \phi(\bar{x}, \bar{y}, u) \geq 0 \text{ for all } u \in E. \]

**Proof:** Let \(M\) be a compact and convex subset of \(C\). For each \(u \in E\), let \(K(u) = \{x \in K \mid \text{there exists } y \in V(x) \cap M \text{ such that } \phi(x, y, u) \geq 0\}\). It is easy to see that \(K(u)\) is a closed subset of \(K\). Let \(u_1, \ldots, u_m \in E\) and \(W(x) = V(x) \cap M\) and \(A = \text{conv}(K \cup \{u_1, \ldots, u_m\})\). Then \(A\) is a compact and convex subset of \(E\). By Theorem 3, there exist \(x_0 \in A\), \(y_0 \in W(x_0) = V(x_0) \cap M\) such that \(\phi(x_0, y_0, u) \geq 0\) for all \(u \in A\). By the assumption (c), we see that \(x_0 \in K\) and \(x_0 \in \bigcap_{i=1}^m K(u_i)\). Thus the collection \(\{K(u) : u \in E\}\) of closed sets in \(K\) has the finite intersection property.
We have \( \bigcap_{u \in E} K(u) \neq \emptyset \). Hence there exists \( \bar{z} \in K(u) \) for all \( u \in E \). This shows that there exist \( \bar{z} \in K \) and \( \bar{y} \in V(\bar{z}) \cap M \subset V(\bar{z}) \) such that \( \phi(\bar{z}, \bar{y}, u) \geq 0 \) for all \( u \in E \). \( \square \)

**THEOREM 5.** Let \( E \) be a nonempty compact convex set in \( X \) and \( C \) be a closed convex set in \( Z \). Let \( V : E \to 2^C \) be an upper semicontinuous closed convex valued map, \( H : E \times C \to L(X,Y) \) be continuous and \( \eta : E \times E \to E \) be continuous functions. Suppose that

(a) \( \eta(x, x) = 0 \).

(b) There exists \( 0 \neq y^* \in D^* \) such that for each \( (z, y) \in E \times C \), the function \( \langle y^* \circ H(x, y), \eta(u, z) \rangle \) is quasiconvex with respect to \( u \in E \).

(c) \( G : E \to 2^E \) is continuous with compact values.

Then there exist \( \bar{z} \in G(\bar{z}) \) and \( \bar{y} \in V(\bar{z}) \) such that

\( \langle H(\bar{z}, \bar{y}), \eta(u, \bar{z}) \rangle \not< 0 \) for all \( u \in G(\bar{z}) \).

**PROOF:** Let \( \phi(x, y, u) = \langle y^* \circ H(x, y), \eta(u, x) \rangle \). Then the theorem follows from Theorem 3 and the assumption \( 0 \neq y^* \in D^* \). \( \square \)

**COROLLARY 1.** Let \( E \) be a nonempty compact convex set in \( \mathbb{R}^n \), and \( C \) be a nonempty convex set in \( \mathbb{R}^m \). Let \( V : E \to 2^C \) be an upper semicontinuous, convex and closed valued map, let \( H : E \times C \to \mathbb{R}^n \) and \( \eta : E \times E \to E \) be continuous functions. Suppose that

(a) \( \eta(x, x) = 0 \).

(b) For each \( (z, y) \in E \times C \), the function \( \langle H(x, y), \eta(u, z) \rangle \) is quasiconvex in \( u \).

(c) \( G : E \to 2^E \) is continuous with compact values.

Then there exist \( \bar{z} \in G(\bar{z}) \), \( \bar{y} \in V(\bar{z}) \) such that

\( \langle H(\bar{z}, \bar{y}), \eta(u, \bar{z}) \rangle \geq 0 \) for all \( u \in G(\bar{z}) \).

**PROOF:** If we let \( X = \mathbb{R}^n \), \( Y = R \), \( Z = \mathbb{R}^m \), then \( H : E \times C \to L(\mathbb{R}^n, \mathbb{R}^m) = L(\mathbb{R}^n, \mathbb{R}) = \mathbb{R}^n \) and the Corollary follows immediately from Theorem 5. \( \square \)

**REMARK.** If \( G(x) = E \) for all \( x \in E \), then Corollary 1 reduces to Theorem 2 [11].

**THEOREM 6.** Let \( E \) be a nonempty, convex set in \( X \), \( \text{int}D = \phi \) and \( \text{int}D^* \neq \phi \). Let \( \eta : E \times E \to E \) be a function, \( \eta(x, x) = 0 \), \( \eta(x, y) = -\eta(y, x) \) for all \( x, y \in E \) and for each fixed \( y \in E \), let \( \eta(y, x) \) be continuous with respect to \( x \in E \). Suppose that \( f : E \to Y \) is \( \eta \)-invex on \( E \) with \( T(x) \) be the Fréchet derivative of \( f \) at \( x \). Suppose that \( T \) is \( \text{pre-v-hemicontinuous on E and } \langle T(x), \eta(u, y) \rangle \rangle \) is convex with respect to \( u \in E \). Then there exists a \( \bar{z} \in E \) such that \( \bar{z} \) is a weak minimum of problem (P).
PROOF: Let \( x, y \in E \). Since \( f \) is \( \eta \)-invex in \( E \) it is easy to see that \( T \) is \( \eta \)-monotone. Then by Theorem 1 and the \( \eta \)-invexity of \( f \), there exists \( \bar{x} \in E \) such that \( \bar{x} \) is a weak minimum of (P).

REFERENCES


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