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A GENERALIZATION OF UNIFORMLY ROTUND BANACH SPACES

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1. Introduction. Let X be a real Banach space. According to von Neumann's famous geometrical characterization X is a Hilbert space if and only if for all $x, y \in X$

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2.$$

Thus Hilbert space is distinguished among all real Banach spaces by a certain uniform behavior of the set of all two dimensional subspaces. A related characterization of real L^p spaces can be given in terms of uniform behavior of all two dimensional subspaces and a Boolean algebra of norm-1 projections [**16**]. For an arbitrary space X, one way of measuring the "uniformity" of the set of two dimensional subspaces is in terms of the real valued modulus of rotundity, i.e. for $\epsilon > 0$

$$\delta_X(\epsilon) \equiv \inf\{2 - ||x + y|| : \quad ||x||, ||y|| \leqq 1, ||x - y|| \geqq \epsilon\}.$$

The space is said to be *uniformly rotund* if for each $\epsilon > 0$ we have $\delta_X(\epsilon) > 0$. Uniformly rotund spaces share some of the properties of Hilbert space and an isomorphic characterization has been given by Enflo [**2**] and James [**4**].

In general terms our purpose in this paper is to study the extent to which Banach space properties can be obtained by requiring a uniform behavior for all n-dimensional subspaces for some fixed $n \geqq 2$. This idea originated with Mil'man [**11**] who discussed both smoothness and rotundity notions. Our approach, however, appears to be rather different from Mil'man's and the connections are not yet clear. The type of generalization which we shall consider can be motivated by the following restatement of the definition of uniformly rotund: A Banach space X is said to be $1 - \text{UR}$ if, for each $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that if x and y are norm-1 vectors with $||x + y|| \geqq 2 - \delta(\epsilon)$ then

$$\sup\left\{\begin{vmatrix} 1 & 1 \\ g(x) & g(y) \end{vmatrix} : g \in X^*, ||g|| \leqq 1\right\} < \epsilon.$$

Here, and throughout the sequel, the symbol $|\cdot|$ denotes the determinant. By analogy we say that X is $2 - \text{UR}$ if for each $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that

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for all norm-1 x, y, z if $||x + y + z|| > 3 - \delta(\epsilon)$ then

$$\sup\left\{\begin{vmatrix} 1 & 1 & 1 \\ f(x) & f(y) & f(z) \\ g(x) & g(y) & g(z) \end{vmatrix} : ||f||, ||g|| \leqq 1\right\} < \epsilon.$$

The notion of a K — UR space is defined in the obvious fashion.

Notice that in the definition of 2 — UR the quantity in the brackets can be thought of as twice the area of the triangle with vertices at x, y and z. This idea and its ramifications are studied in a sequence of papers by E. Silverman [**12**], [**13**], [**14**]. In geometric terms, if three points on the surface of a 2 — UR space enclose an area $\geqq \epsilon/2$, then the centroid of the triangle they determine lies a distance at least $\delta(\epsilon)/3$ beneath the surface of the ball.

In the following section we obtain some of the basic geometric properties of K — UR spaces. We show first that if a Banach space X is K — UR for some K then it is also $K + 1$ — UR. It is well known that a 1 — UR space is reflexive (and, in fact, super-reflexive). We show that a K — UR space is also super-reflexive. This fact was pointed out to us by Professor William Davis of the Ohio State University. We wish to thank him for allowing us to report it here. It is not hard to construct examples of spaces which are K — UR but not $(K - 1)$ — UR. However, from the above result and the work of Enflo and James, such spaces must be isomorphic to 1 — UR spaces. By "fixing" one variable we define the notion of a locally K — UR space and show that if X^{**} is locally 2 — UR then X is reflexive. In general, locally K — UR spaces need not be reflexive since locally 1 — UR is just the usual definition of an LUR space [**1**], [**10**].

Part of the motivation for studying the structure of Banach spaces from the geometric point of view is to determine the extent to which Hilbert space phenomena carry over to more general spaces. Of particular interest are questions concerning the behavior of approximations and existence of fixed points for non-linear operators. In many cases the appropriate generalization holds in 1 — UR spaces—or even characterizes this class of spaces [**17**]. Section 3 of this paper contains two "applications" of the notion of K — UR spaces. We show that if M is a Chebyshev subspace of a locally 2 — UR space then the nearest point map for M is continuous and that for each K, K — UR spaces have normal structure and hence have the fixed point property for weak compact convex sets. Both of these theorems generalize known properties of 1 — UR spaces.

2. K — UR Banach spaces and reflexivity.

We give first a simple result which shows that the notion of K — UR is "coherent."

THEOREM 1. *If for some K a Banach space X is K — UR then X is $K + 1$ — UR.*

Proof. Suppose that there are norm-1 sequences $(x_n^{(1)}), (x_n^{(2)}), \ldots, (x_n^{(K+2)})$

with $||x_n^{(1)} + x_n^{(2)} + \ldots + x_n^{(K+2)}|| \to K + 2$. Then from the triangle inequality for each j we have

(*) $||x_n^{(1)} + \ldots + x_n^{(j-1)} + x_n^{(j+1)} + \ldots + x_n^{(K+2)}|| \to K + 1.$

Now, let $f_1, f_2, \ldots, f_{K+1}$ be any norm-1 linear functionals and consider the determinant

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ f_1(x_n^{(1)}) & f_1(x_n^{(2)}) & \ldots & f_1(x_n^{(K+2)}) \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ f_{K+1}(x_n^{(1)}) & f_{K+1}(x_n^{(2)}) & \ldots & f_{K+1}(x_n^{(K+2)}) \end{vmatrix}.$$

Expanding in minors along the second row and using (*) and the fact that X is $K - \mathrm{UR}$ we conclude that X is $K + 1 - \mathrm{UR}$.

Generalizing our earlier definition somewhat we say that X is *locally* $2 - \mathrm{UR}$ if for each $||x|| = 1$ and $\epsilon > 0$ there is a $\delta \equiv \delta(x, \epsilon) > 0$ such that for all norm-1 y and z if $||x + y + x|| \geqq 3 - \delta$ then

$$\sup\left\{\begin{vmatrix} 1 & 1 & 1 \\ f(x) & f(y) & f(z) \\ g(x) & g(y) & g(z) \end{vmatrix} : ||f||, ||g|| \leqq 1\right\} < \epsilon.$$

THEOREM 2. *If X^{**} is locally $2 - \mathrm{UR}$ then X is reflexive.*

Proof. If X is not reflexive then for any $0 < \eta < 1/2$ there is an $||x^{**}|| = 1$ such that

$$\operatorname{dist}(x^{**}, X) = ||x^{**} + X|| \geqq 1 - \eta.$$

From the Bishop–Phelps Theorem we may even assume that there is a norm-1 $f \in X^*$ with $x^{**}(f) = 1$. Since $(X^{**}/X)^*$ is isometric to $X^\perp \subseteq X^{***}$ there is an $||x^\perp|| = 1$ with $x^\perp(x^{**}) \geqq 1 - \eta$.

From Goldstine's Theorem we find a net (x_α) on the unit ball of X with $x_\alpha \xrightarrow{*} x^{**}$. For a subsequence of this net $f(x_k) \to 1$. Hence for each $\epsilon > 0$ there is an $N(\epsilon)$ such that if $n, m \geqq N(\epsilon)$ then

$$||x^{**} + x_n + x_m|| \geqq x^{**}(f) + f(x_n) + f(x_m) > 3 - \delta(\epsilon)$$

and so

$$\begin{aligned} \epsilon > \sup&\left\{\begin{vmatrix} 1 & 1 & 1 \\ x^\perp(x^{**}) & x^\perp(x_n) & x^\perp(x_m) \\ G(x^{**}) & G(x_n) & G(x_m) \end{vmatrix} : \quad G \in X^{***}; ||G|| = 1\right\} \\ &= \sup\{x^\perp(x^{**})G(x_n - x_m) : \quad G \in X^{***}, ||G|| = 1\} \\ &\geqq (1 - \eta)||x_n - x_m|| > \tfrac{1}{2}||x_n - x_m||. \end{aligned}$$

Therefore, (x_k) is a norm Cauchy subsequence of (x_α) and so $x_k \to x^{**}$. This contradicts the fact that $\mathrm{dist}(x^{**}, X) > 1/2$.

Recall that if Y and X are Banach spaces then Y is said to be finitely representable in X if for each $\epsilon > 0$ and each finite dimensional subspace $M \subseteq Y$ there is an isomorphism $T: M \to X$ such that for all $m \in M$

$$(1 - \epsilon)||m|| \leqq ||Tm|| \leqq (1 + \epsilon)||m||.$$

The space X is *super-reflexive* if each Y which is finitely representable in X is reflexive. It is known that X is super-reflexive if and only if it is isomorphic to a $1 - \mathrm{UR}$ space [**2**], [**4**]. We shall show that every $k - \mathrm{UK}$ space is super-reflexive.

Using ideas similar to those in Theorem 2, James [**5**] showed that if X is not reflexive then for each $0 < \theta < 1$ there are sequences (x_i) and (x_j^*) in B and B^*, the unit balls of X and X^* respectively such that for all i, j

$$x_j^*(x_i) = \begin{cases} \theta & \text{if } j \leqq i \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 3. *If for any K, X is $K - \mathrm{UR}$ then X is reflexive.*

Proof. Suppose that X is $K - \mathrm{UR}$ but not reflexive. Using James' theorem for each $0 < \epsilon < 1$ we can choose $0 < \theta < 1$ so that $\theta > 1 - \delta(\epsilon)/(K + 1)$ and $\theta^K > \epsilon$ and vectors in B and B^* $(x_1, x_2, \ldots, x_{K+1})$, $(x_1^*, x_2^*, \ldots, x_{K+1}^*)$ so that

$$x_j^*(x_i) = \begin{cases} \theta & \text{if } j \leqq i \\ 0 & \text{if } j > i \end{cases}.$$

Here $\delta(\epsilon)$ is the function required in the definition of $K - \mathrm{UR}$.

Now we have that

$$\begin{aligned} ||x_1 + x_2 + \ldots + x_{K+1}|| &\geqq x_1^*(x_1 + x_2 + \ldots + x_{K+1}) \\ &= (K + 1)\theta > (K + 1) - \delta(\epsilon). \end{aligned}$$

On the other hand it is easy to check that

$$\epsilon < \theta^K = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_2^*(x_1) & x_2^*(x_2) & \ldots & x_2^*(x_{K+1}) \\ x_3^*(x_1) & x_3^*(x_2) & \ldots & x_3^*(x_{K+1}) \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ x_{K+1}^*(x_1) & x_{K+1}^*(x_2) & \ldots & x_{K+1}^*(x_{K+1}) \end{vmatrix}$$

which gives the required contradiction.

LEMMA 4. *If X is $K - \mathrm{UR}$ and Y is finitely representable in X then Y is also $K - \mathrm{UR}$.*

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Proof. If Y is not $K - \text{UR}$ then there are norm-1 sequences $(y_n{}^{(1)})$, $(y_n{}^{(2)}), \ldots,$ $(y_n{}^{(K+1)})$ such that

$$\lim_n ||y_n{}^{(1)} + y_n{}^{(2)} + \ldots + y_n{}^{(K+1)}|| \to K + 1$$

while the associated determinants remain bounded away from zero. Using the definitions of finite representability repeatedly gives norm-1 sequences in X such that

$$\lim_n ||x_n{}^{(1)} + x_n{}^{(2)} + \ldots + x_n{}^{(K+1)}|| \to K + 1.$$

Applying the Hahn–Banach theorem to the finite dimensional subspaces $X_n \equiv \text{span}[x_n{}^{(1)}, \ldots, x_n{}^{(K+1)}]$ gives determinants contradicting $K - \text{UR}$ in X.

THEOREM 5. *If for some* K, X *is* $K - \text{UR}$ *then* X *is super-reflexive.*

Proof. Combine Lemmas 5 and 6.

3. Applications. Recall that a closed subspace $M \subseteq X$ is called *Chebyshev* if each $x \in X$ has a unique nearest point in M. In case M is Chebyshev we denote by $P(M)$ the nearest point map $X \to M$. This map is non-linear in general and is characterized by the fact that for each x,

$$||x - P(M)x|| = \text{dist}(x, M) = ||x + M||.$$

It is immediate that for $m \in M$ and any $x \in X$, $P(M)(x + m) = P(M)x + m$ and for α a scalar $P(M)(\alpha x) = \alpha P(M)x$. If X is $1 - \text{UR}$ then it is reflexive and strictly convex. This is equivalent to having each closed subspace of X Chebyshev. In fact, for spaces which are $1 - \text{UR}$ the class of maps $\{P(M)| \quad M$ is a closed subspace of $X\}$ is known to be uniformly equicontinuous on bounded closed subsets of X [**17**], [**3**].

If X is $2 - \text{UR}$ then it is reflexive and so for each closed subspace M and each $x \in X$ there is at least one $m \in M$ with $||x - m|| = ||x + M||$. However, a $2 - \text{UR}$ X need not be strictly convex and so in general m will not be unique. In case best approximations are unique we have the following:

THEOREM 1. *If* M *is a Chebyshev subspace of a locally* $2 - \text{UR}$ *space* X, *then* $P(M)$ *is continuous.*

Proof. We begin with a few general observations. Let $x \in X$ and suppose that $x_n \to x$. If $x \in M$ then $P(M)x = x$ and also $||x_n - P(M)x_n|| = \text{dist}(x_n, M) \leqq ||x_n - x|| \to 0$ so that $P(M)x_n \to x = P(M)x$. Hence we may assume that $\text{dist}(x, M) > 0$.

Clearly $x_n' \equiv (x_n - P(M)x)/||x - P(M)x|| \to (x - P(M)x)/||x - P(M)x|| \equiv x'$ and $P(M)x_n \to P(M)x$ if and only if $P(M)x_n' \to 0 = P(M)x'$. Therefore, we may also assume that $||x|| = 1$ and $P(M)x = 0$ and we need only show that $P(M)x_n \to 0$. In fact, it is sufficient to show that $(P(M)x_n)$ converges because if $P(M)x_n \to m$ then

$$||x - m|| = \lim ||x_n - P(M)x_n|| \leqq \lim ||x_n|| = ||x||$$

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and so by uniqueness of nearest points, $m = 0$.

Finally, notice that for all n

$$||x|| \leqq ||x - P(M)x_n|| \leqq ||x_n - P(M)x_n|| + ||x - x_n|| \leqq ||x_n|| + ||x - x_n||$$

and so $||x - P(M)x_n|| \to ||x|| = 1$.

Let $\epsilon > 0$ be given and assume by passing to a subsequence, if necessary, that $||P(M)x_n|| \to a > 0$. For all n and m large $||x - P(M)x_n||$ and $||x - P(M)x_n||$ are close to 1 while

$$||x + (x - P(M)x_n) + (x - P(M)x_m)||/3 \geqq ||x|| = 1.$$

Let (f_n) be a sequence of norm-1 functionals such that for each $n, f_n(P(M)x_n) = ||P(M)x_n||$.

Since X is locally $2 - \text{UR}$, for n and m large we have for $\bar{\epsilon} < \epsilon/4$

$$a\bar{\epsilon}/2 > \sup_{||g||=1}\left\{\left|\begin{matrix} 1 & 1 & 1 \\ f_n(x) & f_n(x - P(M)x_n) & f_n(x - P(M)x_m) \\ g(x) & g(x - P(M)x_n) & g(x - P(M)x_m) \end{matrix}\right|\right\}.$$

The supremum is taken over all $||g|| = 1$ so expanding in minors along the last row gives

$$(*) \quad a\bar{\epsilon}/2 > ||f_n(P(M)x_m)P(M)x_n - ||P(M)x_n||P(M)x_m||.$$

Since $||P(m)x_k|| \to a$ for n or m large, equation (*) implies that

$$\bar{\epsilon} > ||P(M)x_n|| - |f_n(P(M)x_m)|.$$

In case $f_n(P(M)x_m) \geqq 0$, using (*) again gives

$$a\bar{\epsilon}/2 > ||P(M)x_n||\cdot||P(M)x_n - P(M)x_m|| - |\, ||P(M)x_n|| - f_n(P(M)x_m)|\, ||P(M)x_n||$$

so that

$$2\bar{\epsilon} > ||P(M)x_n - P(M)x_m||.$$

In case $f_n(P(M)x_m) \leqq 0$ we conclude by similar calculations that

$$2\bar{\epsilon} > ||P(M)x_n + P(M)x_m||.$$

Now, if $k > j > n$ and $f_n(P(M)x_k)$ and $f_n(P(M)x_j)$ have the same sign (say $\leqq 0$) then

$$||P(M)x_k - P(M)x_j|| \leqq ||P(M)x_k + P(M)x_n|| + ||P(M)x_n + P(M)x_j|| \leqq 2\bar{\epsilon} + 2\bar{\epsilon} = 4\bar{\epsilon} < \epsilon.$$

On the other hand if $f_j(P(M)x_k) \leqq 0$ then applying the above argument for j and k gives $||P(M)x_k + P(M)x_j|| < 2\bar{\epsilon} < \epsilon$. Combining the last two inequalities and using the triangle inequality we have that $||P(M)x_k|| < \epsilon$ which is impossible if we assume (as we may) that $\epsilon < a/2$.

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Hence the required convergent subsequence may be constructed from all those terms $P(M)x_k$ with $f_n(P(M)x_k) \geqq 0$ or from those with $f_n(P(M)x_k) \leqq 0$.

Let C be a closed convex subset of a Banach space X. The set C is said to have *normal structure* if for each closed convex bounded non-empty $K \subseteq C$ which is not a singleton, there exists an $x \in K$ such that

$$\sup\{||x - k|| \mid k \in K\} < \operatorname{diam}(K).$$

Here $\operatorname{diam}(K) \equiv \sup\{||k_1 - k_2|| \mid k_1, k_2 \in K\}$. If we may take $C = X$ then we say that the space X has normal structure.

It was proved by Kirk [**8**] that if X has normal structure, and K is any weakly compact convex subset of X, then each $T : K \to K$ which is non-expansive has a fixed point in K. T is said to be *non-expansive* if $||Tx - Ty|| \leqq ||x - y||$ for all $x, y \in K$.

It is not hard to show that a $1 -$ UR space has normal structure, and Smith [**15**] has shown that this property holds for any space having a very general sort of directional uniform rotundity.

THEOREM 2. *If, for some K, X is $K -$ UR then X has normal structure.*

Proof. Suppose that X is $K -$ UR and that C is a closed bounded convex subset of X which contradicts normal structure. We may assume that $\operatorname{diam} C = 1$ and using techniques of T. C. Lim [**9**] there exist sequences $(c_n^{(1)}), (c_n^{(2)}), \ldots, (c_n^{(K+2)})$ in C such that for all $1 \leqq j \leqq K + 1$

$$\lim_n \operatorname{dist}(c_n^{(j+1)}, \operatorname{co}(c_n^{(1)}, c_n^{(2)}, \ldots, c_n^{(j)})) = 1.$$

In particular, defining $x_n^{(j)} = c_n^{(j)} - c_n^{(K+2)}$ we have that $\lim_n||x_n^{(j)}|| = 1$ for all $1 \leqq j \leqq K + 1$ and also that

$$||x_n^{(1)} + x_n^{(2)} + \ldots + x_n^{(K+1)}||/(K + 1) \to 1.$$

Now, from the definition of $K -$ UR for any choices of sequences of norm-1 functionals $(f_n^{(1)}), (f_n^{(2)}), \ldots, (f_n^{(K-1)})$ we have

$$\lim_n \sup_{||g||=1} \begin{vmatrix} 1 & 1 & & 1 \\ f_n^{(1)}(x_n^{(1)}) & f_n^{(1)}(x_n^{(2)}) & \ldots & f_n^{(1)}(x_n^{(K+1)}) \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ f_n^{(K-1)}(x_n^{(1)}) & f_n^{(K-1)}(x_n^{(2)}) & \ldots & f_n^{(K-1)}(x_n^{(K+1)}) \\ g(x_n^{(1)}) & g(x_n^{(2)}) & \ldots & g(x_n^{(K+1)}) \end{vmatrix} = 0.$$

Expanding in minors along the last row and using the fact that the supremum is taken over all $||g|| = 1$ we conclude that

$$\lim_n||M_n^{(1)}x_n^{(1)} + M_n^{(2)}x_n^{(2)} + \ldots + M_n^{(K+1)}x_n^{(K+1)}|| = 0.$$

Since the above determinant is zero if the elements of the last row are re-

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placed by 1's we have that

$$\sum_{j=1}^{K+1} M_n^{(j)} = 0 \quad \text{for all } n.$$

Also we may assume that there is a choice of functionals $(f_n^{(1)}), \ldots, (f_n^{(K-1)})$ so that $|M_n^{(K+1)}|$ remains bounded away from zero. This requires some verification which we postpone until the end of the proof.

Notice that $\sum M_n^{(j)} = -\sum M_n^{(l)}$ where the sum on the left is over positive $M_n^{(j)}$'s and on the right over negative. Hence dividing by this sum and rearranging gives

$$\lim_n \|\sum \mu_n^{(j)} c_n^{(j)} - \sum \lambda_n^{(l)} c_n^{(l)}\| = 0$$

where $\sum \mu_n^{(j)} = \sum \lambda_n^{(l)} = 1$ and all $\mu_n^{(j)}, \lambda_n^{(l)} \geqq 0$. Without loss of generality assume that $c_n^{(K+1)}$ occurs among the right hand terms so that

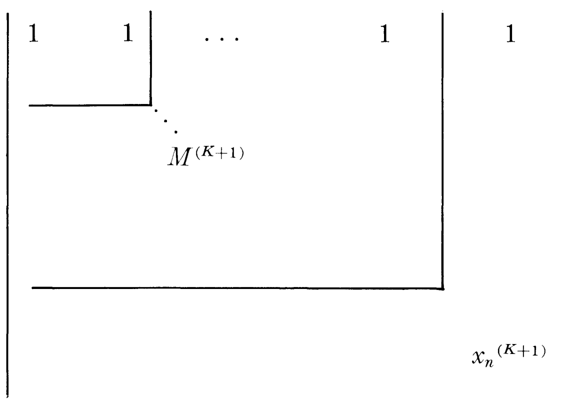
$$0 = \lim_n \|\sum \mu_n^{(j)} c_n^{(j)} - c_n^{(K+1)} - (\sum_{l<K+1} \lambda_n^{(l)} c_n^{(l)} - (\sum_{l<K+1} \lambda_n^{(l)}) c_n^{(K+1)})\|.$$

Let $\bar{\lambda}_n^{(l)} \equiv \lambda_n^{(l)} / \sum_{l<K+1} \lambda_n^{(l)}$ for $l < K + 1$ and use the triangle inequality to obtain

$$\begin{aligned} 0 &\geqq \lim_n [\|\sum \mu_n^{(j)} c_n - c_n^{(K+1)}\| - (\sum_{l<K+1} \lambda_n^{(l)}) \|\sum_{l<K+1} \bar{\lambda}_n^{(l)} c_n^{(l)} - c_n^{(K+1)}\|] \\ &= \lim_n [1 - \sum_{l<K+1} \lambda_n^{(l)}] = \lim_n (M_n^{(K+1)} / \sum M_n^{(l)}). \end{aligned}$$

However, since $\sum M_n^{(l)}$ is clearly bounded, this contradicts the fact that $M_n^{(K+1)}$ is bounded away from zero.

The preceding argument shows that either C has normal structure or else the $K \times K$ upper left hand determinants go to zero for all choices of functionals. However, the same argument applied to this determinant either contradicts the abnormality of C or implies that the upper left hand $(K - 1) \times (K - 1)$ determinants go to zero, etc., e.g.



Continuing in this manner we conclude that

$$\lim \sup_{\|g\|=1} \left\{ \left| \begin{matrix} 1 & 1 \\ g(x_n^{(1)}) & g(x_n^{(2)}) \end{matrix} \right| \right\} = 0$$

which is impossible since

$$||x_n^{(2)} - x_n^{(1)}|| = ||(c_n^{(2)} - c_n^{(K+2)}) - (c_n^{(1)} - c_n^{(K+2)})||$$
$$= ||c_n^{(2)} - c_n^{(1)}|| \to 1.$$

It has been pointed out by James [**6**] that there is an equivalent norm for l^2 which fails normal structure. Because of the preceding result this example is super-reflexive but not K − UR for any K. However, Karlovitz [**7**] has shown that l^2 in this equivalent norm still has the fixed point property, i.e. the conclusion of Kirk's Theorem holds. In view of this we would like to end with the following question: Does every super-reflexive space have the fixed point property?

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