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EXISTENCE AND UNIQUENESS OF POSITIVE EIGENFUNCTIONS FOR CERTAIN EIGENVALUE SYSTEMS

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Abstract. The existence and uniqueness of eigenvalues and positive eigenfunctions for some quasilinear elliptic systems are considered. Some necessary and sufficient conditions which guarantee the existence and uniqueness of eigenvalues and positive eigenfunctions are given.

§1. Introduction

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. For $p \in (1, \infty)$, we denote by Δ_p the $p-$Laplacian defined by $\Delta_p u = \mathrm{div}(|\bigtriangledown u|^{p-2} \bigtriangledown u)$. It is well-known that, when $a(x) \in L^\infty(\Omega)$ and $\max(a(x), 0) \not\equiv 0$, the eigenvalue problem

$$(1.1) \qquad -\Delta_p u = \lambda a(x)|u|^{p-2}u \quad \text{in} \quad \Omega,\, u = 0 \quad \text{on} \quad \partial\Omega$$

has a unique eigenvalue λ_0 with nonnegative eigenfunctions. More precisely, the eigenvalue λ_0 is simple, i.e., the set of all solutions of (1.1) with $\lambda = \lambda_0$ consists of $\{t\phi_0 : t \in \mathbf{R}^1\}$, where ϕ_0 is an eigenfunction of (1.1) corresponding to λ_0 such that $\phi_0 \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0, 1)$ and $\phi_0(x) > 0$ for all $x \in \Omega$ (see [1], [2]). In fact we have

$$(1.2) \qquad \lambda_0 = \inf\left\{\int_\Omega |\bigtriangledown u|^p dx : u \in W_0^{1,p}(\Omega), \int_\Omega a(x)|u|^p dx = 1\right\},$$

and the solutions of (1.1) for $\lambda = \lambda_0$ are the minimizers of (1.2) (see [2]).

When we consider the existence and uniqueness of nonnegative eigenfunctions for elliptic eigenvalue systems, the following example shows that the situation is different.

Example. Let $A(x)$ be a $L^\infty(\Omega)$ function satisfying $\max\{A(x), 0\} \not\equiv 0$ and $\max\{-A(x), 0\} \not\equiv 0$. For $d > 2$, we denote by k_1 and k_2 two positive

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constants satisfying $0 < k_1 < 1 < k_2$ and $k^{-1/2} + k^{1/2} = d$. Choose C_1 and C_2 such that (we assume $\alpha_2 + \beta_2 \neq 2$)

$$C_1 k_1^{1-\beta_2} - C_2 k_1^{\alpha_2 - 1} = C_1 k_2^{1-\beta_2} - C_2 k_2^{\alpha_2 - 1} = d.$$

Let λ_0 and ϕ_0 be the eigenvalue and the positive eigenfunction of

$$-\Delta u = \lambda A(x) u \quad \text{in} \quad \Omega,\ u = 0 \quad \text{on} \quad \partial\Omega.$$

We choose $a_1(x) = A(x)$, $b_1(x) = -A(x)$, $a_2(x) = C_2 A(x)$, $b_2(x) = C_1 A(x)$ and $\alpha_2 + \beta_2 \neq 2$. Then we have $\max\{a_j(x), 0\} \not\equiv 0$ and $\max\{b_j(x), 0\} \not\equiv 0$, and $(k_1\phi_0, \phi_0)$ and $(k_2\phi_0, \phi_0)$ are two positive eigenfunctions associated with eigenvalue λ_0/d of the following eigenvalue system

$$\begin{cases} -\Delta u = \lambda[a_1(x)u^{1/2}v^{1/2} + a_2(x)u^{\alpha_2}v^{1-\alpha_2}] \ \text{in } \Omega, \\ -\Delta v = \lambda[b_1(x)u^{1/2}v^{1/2} + b_2(x)u^{1-\beta_2}v^{\beta_2}] \ \text{in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega. \end{cases}$$

In this article, we shall prove that a similar uniqueness result holds for certain elliptic eigenvalue systems with nonnegative coefficients. Consider elliptic eigenvalue systems of the form

$$\begin{cases} -\Delta_p u = \lambda \sum_{i=1}^{K} a_i(x) u^{\alpha_i} v^{p-1-\alpha_i} \ \text{in } \Omega, \\ -\Delta_q v = \lambda \sum_{j=1}^{N} b_i(x) u^{q-1-\beta_j} v^{\beta_j} \ \text{in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

where $1 < p < +\infty, 1 < q < +\infty$, $0 \leq \alpha_i \leq p-1 (i = 1, 2, \cdots, K)$, $0 \leq \beta_j \leq q - 1 (j = 1, 2, \cdots, N)$, $a_i(x) (i = 1, 2, \cdots, K)$ and $b_j(x) (j = 1, 2, \cdots, N)$ are nonnegative $L^\infty(\Omega)$ functions.

The main theorems obtained in this article are

THEOREM 1.1. The eigenvalue of (1.3) with nontrivial nonnegative eigenfunctions is unique. Nontrivial nonnegative eigenfunctions of (1.3) are positive.

THEOREM 1.2. (1) When $\alpha_1 < p - 1$ and $\beta_1 < q - 1$, or $\alpha_i = p - 1$ and $\beta_j < q - 1$ for $i = 1, 2, \cdots, K$ and $j = 1, 2, \cdots, N$, (1.3) possesses a unique eigenvalue with positive eigenfunctions, the corresponding positive eigenfunction is unique up to a scalar multiple.

(2) When $\alpha_i = p-1$ for all $i = 1, 2, \cdots, K$, $\beta_1 = q-1$ and $\beta_j < q-1$ for $j = 2, \cdots, N$, (1.3) has a unique eigenvalue with positive eigenfunctions if and only if

$$\inf\left\{\int_\Omega |\nabla u|^p dx : \int_\Omega \sum_{i=1}^K a_i(x)|u|^p dx = 1, u \in W_0^{1,p}(\Omega)\right\}$$
$$< \inf\left\{\int_\Omega |\nabla v|^q dx : \int_\Omega b_1(x)|v|^q dx = 1, v \in W_0^{1,q}(\Omega)\right\}.$$

The corresponding positive eigenfunctions, if they exist, are unique up to a scalar multiple.

(3) When $\alpha_i = p-1$ and $\beta_j = q-1$ for $i = 1, 2, \cdots, K$, $j = 1, 2, \cdots, N$, (1.3) possesses an eigenvalue with positive eigenfunctions if and only if

$$\inf\left\{\int_\Omega |\nabla u|^p dx : \int_\Omega \left[\sum_{i=1}^K a_i(x)\right] |u|^p dx = 1, u \in W_0^{1,p}(\Omega)\right\}$$
$$= \inf\left\{\int_\Omega |\nabla v|^q dx : \int_\Omega \left[\sum_{j=1}^N b_j(x)\right] |v|^q dx = 1, v \in W_0^{1,q}(\Omega)\right\}.$$

In this case, the corresponding positive eigenfunctions, if they exist, are not unique up to a scalar multiple.

By a positive eigenfunction of the eigenvalue system (1.3) corresponding to an eigenvalue λ we mean a weak solution $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ satisfying $u, v \in L^\infty(\Omega)$ and $u > 0$, $v > 0$ in Ω. A nonnegative eigenfunction of (1.3) is a pair $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, which does not vanish identically in Ω and satisfies $u, v \in L^\infty(\Omega)$ and $u \geq 0$, $v \geq 0$ in Ω. A nontrivial nonnegative eigenfunction is a nonnegative eigenfunction of (1.3), each component of which does not vanish identically in Ω.

This article is organized as follows. In section 2 we recall some well-known results for the single equation with Δ_p. That (1.3) has a unique eigenvalue with nontrivial nonnegative eigenfunctions (Theorem 1.1) and that the positive eigenfunctions are unique up to a scalar multiple are proved in Section 3. In Section 4 we consider the existence of branches of nonnegative (or positive) solutions for some quasilinear elliptic systems. The existence of positive eigenfunctions (Theorem 1.2) is considered in section 5.

In this article, we shall write $(u_1, v_1) \geq (u_2, v_2)$ if $u_1 \geq u_2$ and $v_1 \geq v_2$, $(u_1, v_1) > (u_2, v_2)$ if $u_1 > u_2$ and $v_1 > v_2$. We also denote by $\|(u, v)\| = \sup_{x\in\Omega} [|u(x)| + |v(x)|]$.

§2. Some results for a single equation with Δ_p

In this section we recall some well-known results for the single equation with Δ_p. Consider the following Dirichlet problem

$$-\Delta_p u = f(x) \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{2.4}$$

where $p > 1$, $f(x) \in L^\infty(\Omega)$ with the norm $\|f\| \overset{def.}{=} \sup_{x\in\Omega} |f|$. By the variational method we know that the Dirichlet problem (2.4) has a unique solution $u \in W_0^{1,p}(\Omega)$. We first introduce the weak comparison principle, which follows from the same argument as that used in Lemma 4.1 in [3] and the fact that $[\psi_p(\vec{x}) - \psi_p(\vec{y})] \cdot (\vec{x} - \vec{y}) = 0$ implies $\vec{x} = \vec{y}$.

Lemma 2.1. Assume that $u_1, u_2 \in W^{1,p}(\Omega)$, respectively, are weak solutions of

$$-\Delta_p u_1 = f_1(x) \quad \text{in} \quad \Omega, \qquad u = g_1 \quad \text{on} \quad \partial\Omega, \tag{2.5}$$

$$-\Delta_p u_2 = f_2(x) \quad \text{in} \quad \Omega, \qquad u = g_2 \quad \text{on} \quad \partial\Omega, \tag{2.6}$$

with $f_1 \leq f_2$ in $L^q(\Omega)$ and $g_1 \leq g_2$ in $W^{\frac{1}{q},p}(\partial\Omega)$, where $q = \frac{p}{p-1}$. Then $u_1 \leq u_2$ almost everywhere in Ω.

Choose R_0 so large that $\overline{\Omega} \subset \{x : |x| < R_0\}$. Clearly,

$$0 \leq U(|x|) = n^{\frac{1}{1-p}} (R_0^{\frac{p}{p-1}} - |x|^{\frac{p}{p-1}}) \in W^{1,p}(|x| < R_0)$$

is a positive radial weak solution of the equation

$$-\Delta_p U = 1 \quad \text{for} \quad |x| \leq R_0, \quad U = 0 \quad \text{on} \quad |x| = R_0.$$

The following lemma comes from Lemma 2.1.

Lemma 2.2. Assume that there exists a constant M such that $0 \leq f(x) \leq M^{p-1}$ in Ω. Then the weak solution u of (2.4) satisfies $0 \leq u(x) \leq MU(|x|)$ almost everywhere in Ω.

The following strong comparison principle comes from [4].

Lemma 2.3. ([4]) Assume $u_1, u_2 \in W_0^{1,p}(\Omega)$, respectively, are weak solution of (2.5) and (2.6) with $g_1 = g_2 = 0$, $f_1, f_2 \in L^\infty(\Omega)$ satisfying $0 \le f_1 \le f_2$ in Ω. Then either $u_1 \equiv u_2 \ge 0$ in Ω or else

$$0 \le u_1 < u_2 \quad in \quad \Omega, \quad 0 \ge \frac{\partial u_1}{\partial \nu} > \frac{\partial u_2}{\partial \nu} \quad \text{on} \quad \partial\Omega, \tag{2.7}$$

where $\vec{\nu}$ denotes the outward unit normal vector at $\partial\Omega$.

Denote by $C(\overline{\Omega})$ the space of all continuous functions defined on $\overline{\Omega}$ with the standard norm

$$\|u\| \overset{def.}{=} \sup_{x\in\Omega} |u(x)|,$$

and let V_+ be the positive cone in $C(\overline{\Omega})$, $V_+ = \{u \in C(\overline{\Omega}), u \ge 0 \text{ in } \Omega\}$. Let $f(x,u,v), g(x,u,v)$ be nonnegative functions defined in $\Omega \times [0,+\infty) \times [0,+\infty)$ and satisfy, for any fixed positive constant M, $f, g \in L^\infty(\Omega \times [0,M] \times [0,M])$. For any $(u,v) \in V_+ \times V_+$, by variational methods there exists a unique weak solution $(\overline{u}, \overline{v}) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ satisfying

$$\begin{cases} -\Delta_p \overline{u} = f(x,u,v) \text{ in } \Omega, \\ -\Delta_q \overline{v} = g(x,u,v) \text{ in } \Omega, \\ \overline{u} = \overline{v} = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.8}$$

Lemma 2.2 and the regularity result proved in [5] mean there exists a constant $\beta \in (0,1)$ depending solely upon p, q and n, and another constant C depending solely upon p, q, n, $\|f(x,u,v)\|$ and $\|g(x,u,v)\|$, such that $\overline{u}, \overline{v} \in C_0^{1,\beta}(\overline{\Omega})$ with the Hölder norm $\|\overline{u}\|_{1+\beta,\overline{\Omega}} \le C$ and $\|\overline{v}\|_{1+\beta,\overline{\Omega}} \le C$. Define a mapping

$$T : (u,v) \longmapsto (\overline{u}, \overline{v}) = T(u,v). \tag{2.9}$$

Obviously, T is a self-mapping of $V_+ \times V_+$, and $T : V_+ \times V_+ \to V_+ \times V_+$ is continuous and relatively compact.

§3. Uniqueness of eigenvalues and eigenfunctions

In this section, we consider the uniqueness of eigenvalues with nontrivial nonnegative eigenfunctions of (1.3). We first prove the uniqueness of eigenvalues associated with nontrivial nonnegative eigenfunctions.

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Proof of Theorem 1.1. Suppose that (u_1, v_1), (u_2, v_2) are two nontrivial nonnegative eigenfunctions associated with eigenvalues Λ_1, Λ_2, respectively, for the eigenvalue system (1.3). Lemma 2.1, Lemma 2.3 and the regularity result in [5] imply that $\Lambda_1, \Lambda_2 > 0$; $u_j, v_j \in C^{1,\beta}(\overline{\Omega})$, $j = 1, 2$, are positive in Ω; and

$$\frac{\partial u_1}{\partial \nu} < 0, \frac{\partial u_2}{\partial \nu} < 0, \frac{\partial v_1}{\partial \nu} < 0 \quad \text{and} \quad \frac{\partial v_2}{\partial \nu} < 0 \quad \text{on } \partial\Omega. \tag{3.10}$$

Thus, without loss of generality, we may assume that $(0,0) < (u_1, v_1) \leq (u_2, v_2)$ and $\Lambda_1 \leq \Lambda_2$. Let us consider the following elliptic system:

(3.11)

$$\begin{cases} -\Delta_p u = \Lambda_1 \sum_{i=1}^{K} a_i(x) u^{\alpha_i} v^{p-1-\alpha_i} + (\Lambda_2 - \Lambda_1) \sum_{i=1}^{K} a_i(x) u_2^{\alpha_i v_2^{p-1-\alpha_i}} & \text{in } \Omega, \\ -\Delta_q v = \Lambda_1 \sum_{j=1}^{N} b_j(x) u^{q-1-\beta_j} v^{\beta_j} + (\Lambda_2 - \Lambda_1) \sum_{j=1}^{N} b_j(x) u_2^{q-1-\beta_j} v_2^{\beta_j} & \text{in } \Omega, \\ u = v = 0 \ \text{ on } \partial\Omega. \end{cases}$$

Observe that (u_2, v_2) is a solution of (3.11). By Lemma 2.1, we have $(\xi u_1, \xi v_1 \leq T(\xi u_1, \xi v_1)$ for all $\xi \in (0, +\infty)$ and $T(\xi u_2, \xi v_2) \leq (\xi u_2, \xi v_2)$ for all $\xi \in (1, +\infty)$, where T is the mapping from $V_+ \times V_+$ to $V_+ \times V_+$ defined in (2.9) for

$$\begin{aligned} f(x, u, v) &= \Lambda_1 \sum_{i=1}^{K} a_i(x) u^{\alpha_i} v^{p-1-\alpha_i} + (\Lambda_2 - \Lambda_1) \sum_{i=1}^{K} a_i(x) u_2^{\alpha_i} v_2^{p-1-\alpha_i}, \\ g(x, u, v) &= \Lambda_1 \sum_{j=1}^{N} b_j(x) u^{q-1-\beta_j} v^{\beta_j} + (\Lambda_2 - \Lambda_1) \sum_{j=1}^{N} b_j(x) u_2^{q-1-\beta_j} v_2^{\beta_j}. \end{aligned}$$

Making use of (3.10) and the fact that $(0,0) < (u_1, v_1) \leq (u_2, v_2)$, we can pick $\xi > 1$ so large that $(u_2, v_2) \leq \xi(u_1, v_1)$. Then, from Lemma 2.1 we arrive at

$$\begin{aligned} (u_2, v_2) &\leq \xi(u_1, v_1) \leq T(\xi u_1, \xi v_1) \leq \ldots \leq T^k(\xi u_1, \xi v_1) \\ &\leq T^k(\xi u_2, \xi v_2) \leq \ldots \leq T(\xi u_2, \xi v_2) \leq \xi(u_2, v_2). \end{aligned} \tag{3.12}$$

The compactness of T implies $T^k(\xi u_2, \xi v_2) \to (u_3, v_3)$ in V_+ as $k \to +\infty$ for some $(u_3, v_3) \in V_+ \times V_+$ and

$$(u_2, v_2) \leq (\xi u_1, \xi v_1) \leq T(u_3, v_3) = (u_3, v_3) \leq \xi(u_2, v_2). \tag{3.13}$$

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Hence (u_3, v_3) is a positive solution of (3.11).We claim that $(u_3, v_3) = (u_2, v_2)$. If this is the case, (3.13) implies that $(u_2, v_2) = \xi(u_1, v_1)$, and hence $\Lambda_1 = \Lambda_2$ as desired.

On the contrary, suppose $(u_3, v_3) \not\equiv (u_2, v_2)$. We assume that $(u_3, v_3) \leq (u_2, v_2)$ is false. Consequently, by (3.10) we can pick $t \in (1, +\infty)$ which is the smallest number satisfying

$$t^{-1}(u_3, v_3) \leq (u_2, v_2). \tag{3.14}$$

We have

$$\begin{aligned} t^{1-p} f(u_3, v_3) &\leq f(t^{-1}u_3, t^{-1}v_3) \\ &\leq f(u_2, v_2), t^{1-p} f(u_3, v_3) \not\equiv f(t^{-1}u_3, t^{-1}v_3) \end{aligned} \tag{3.15}$$

and

$$\begin{cases} -\Delta_p(t^{-1}u_3) = t^{1-p} f(u_3, v_3) \quad \text{in } \Omega, \\ -\Delta_p(u_2) = f(u_2, v_2) \quad \text{in } \Omega, \\ u_3 = u_2 = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.16}$$

By (3.16) and (3.16), Lemma 2.3 implies

$$0 < t^{-1}u_3 < u_2 \text{ in } \Omega, \text{ and } \frac{\partial u_2}{\partial \nu} < \frac{\partial (t^{-1}u_3)}{\partial \nu} < 0 \text{ on } \partial\Omega.$$

Then we find $\overline{t} \in (1, t)$ such that

$$0 < \overline{t}^{-1} u_3 \leq u_2 \text{ in } \Omega,$$

a contradiction to our choice of t. That nontrivial nonnegative eigenfunctions of (1.3) are positive follows from Lemma 2.3 directly.

Remark 3.1. It is obvious that (1.3) may have a unique eigenvalue for which the eigenfunctions are of the form $(u, 0)$ with $u > 0$, and a unique eigenvalue for which the eigenfunctions are of the form $(0, v)$ with $v > 0$. If (1.3) has an eigenvalue with nontrivial nonnegative eigenfunctions, we know that the eigenvalue is unique by Theorem A. Hence, (1.3) has at most three eigenvalues associated with nonnegative eigenfunctions. It is possible that (1.3) may have no eigenvalues with nontrivial nonnegative (positive) eigenfunctions.

THEOREM 3.1. Except the case that $\alpha_i = p-1 (i = 1, 2, \cdots, K)$ and $\beta_j = q-1 (j = 1, 2, \cdots, N)$, positive eigenfunctions of the eigenvalue problem (1.3) are unique up to scalar multiples.

Proof. By Theorem 1.1, an eigenvalue of (1.3) with positive eigenfunctions is positive and unique. Assume that (ϕ_1, ψ_1) and (ϕ_2, ψ_2) are two positive eigenfunctions of (1.3), associated with an eigenvalues λ. It is sufficient to prove that (ϕ_1, ψ_1) and (ϕ_2, ψ_2) are colinear. By Lemma 2.3 ,

$$\phi_i(x) > 0, \quad \psi_i(x) > 0 \quad \text{for} \quad x \in \Omega, i = 1, 2,$$

$$\frac{\partial \phi_i}{\partial \nu} < 0, \quad \frac{\partial \psi_i}{\partial \nu} < 0, \quad \text{for} \quad x \in \partial\Omega, i = 1, 2.$$

Thus, we can choose two positive numbers $C_1 < C_2$ such that

$$C_1(\phi_1, \psi_1) \leq (\phi_2, \psi_2) \leq C_2(\phi_1, \psi_1).$$

Without loss of generality, we assume that $C_1 = 1$ and $(\phi_1, \psi_1) \leq (\phi_2, \psi_2)$. We claim that (ϕ_1, ψ_1) and (ϕ_2, ψ_2) are linearly dependent. Indeed, otherwise there exists the smallest number $t_0 > 1$ such that

$$t_0(\phi_1, \psi_1) \geq (\phi_2, \psi_2), \quad t_0(\phi_1, \psi_1) \not\equiv (\phi_2, \psi_2). \tag{3.17}$$

Let

$$F(x, u, v) = \lambda \sum_{i=1}^{K} a_i(x) u^{\alpha_i} v^{p-1-\alpha_i},$$
$$G(x, u, v) = \lambda \sum_{j=1}^{N} b_j(x) u^{q-1-\beta_j} v^{\beta_j}.$$

They satisfy

$$F(x, t_0\phi_1, t_0\psi_1) \geq F(x, \phi_2, \psi_2), \quad G(x, t_0\phi_1, t_0\psi_1) \geq G(x, \phi_2, \psi_2).$$

We claim that

$$F(x, t_0\phi_1, t_0\psi_1) \not\equiv F(x, \phi_2, \psi_2), G(x, t_0\phi_1, t_0\psi_1) \not\equiv G(x, \phi_2, \psi_2). \tag{3.18}$$

In fact, if

$$F(x, t_0\phi_1, t_0\psi_1) \equiv F(x, \phi_2, \psi_2),$$

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we deduce from (3.10) that $t_0\phi_1 = \phi_2$. It is obvious that $t_0\psi_1$ and ψ_2 are two positive solutions of the following elliptic problem

$$(3.19) \qquad -\Delta_q v = G(x, \phi_2, v) \quad x \in \Omega, \quad v = 0 \quad x \in \partial\Omega.$$

When $\beta_j < q - 1$ for some $j \in \{1, 2, \cdots, N\}$, the same argument as that in the proof of Theorem 2.1 in [3] shows that (3.19) possesses at most one positive solution. Thus, we have $t_0\psi_1 = \psi_2$, hence $(t_0\phi_1, t_0\psi_1) = (\phi_2, \psi_2)$, which is impossible because of (3.17). When $\beta_j = q - 1$ for all $j = 1, 2, \cdots, N$, the uniqueness of nonnegative eigenfunctions for the following eigenvalue problem

$$-\Delta_q v = \lambda \left[\sum_{j=1}^{N} b_j(x)\right] v^{q-1} \quad x \in \Omega, \quad v = 0 \quad x \in \partial\Omega$$

shows that $t_0\psi_1$ and ψ_2 are linearly dependent, and then there exists C_0 such that $t_0\psi_1 = C_0\psi_2$. But in this case we must have $\alpha_i < p-1$ for some $i \in \{1, 2, \cdots, K\}$, so we obtain $C_0 = 1$ from the hypothesis $F(x, t_0\phi_1, t_0\psi_1) \equiv F(x, \phi_2, \psi_2)$. Thus $(t_0\phi_1, t_0\psi_1) = (\phi_2, \psi_2)$, which is impossible because of (3.17). Hence (3.18) holds.

By Lemma 2.2 and the regularity result of Lieberman [5],

$$(3.20) \qquad (\phi_i(x), \psi_i(x)) \in C^{1,\beta}(\overline{\Omega}) \times C^{1,\beta}(\overline{\Omega}), \quad i = 1, 2$$

with some $\beta \in (0, 1)$. A combination of (3.17) and (3.18) with Lemma 2.3 yields that

$$t_0(\phi_1, \psi_1) > (\phi_2, \psi_2) \quad x \in \Omega,$$

$$\Big(\frac{\partial(t_0\phi_1)}{\partial\nu}, \frac{\partial(t_0\psi_1)}{\partial\nu}\Big) < \Big(\frac{\partial\phi_2}{\partial\nu}, \frac{\partial\psi_2}{\partial\nu}\Big) < (0, 0) \quad x \in \partial\Omega.$$

Hence, there exists a positive number $t_1 \in (0, t_0)$ such that

$$t_1(\phi_1, \psi_1) \geq (\phi_2, \psi_2),$$

a contradiction to our choice of the number t_0. Hence (ϕ_1, ψ_1) and (ϕ_2, ψ_2) are linearly dependent. ∎

§4. Bifurcation properties of nonnegative solutions

In this section, we consider the bifurcation properties of nonnegative solutions of the following elliptic system

$$(4.21)\quad \begin{cases} -\Delta_p u = \lambda \sum_{i=1}^K a_i(x)u^{\alpha_i}v^{p-1-\alpha_i} + f(x,u,v,\lambda), & x \in \Omega, \\ -\Delta_q v = \lambda \sum_{j=1}^N b_j(x)u^{q-1-\beta_j}v^{\beta_j} + g(x,u,v,\lambda), & x \in \Omega, \\ u = v = 0, \qquad x \in \partial\Omega. \end{cases}$$

We assume that

(F). For any given $M > 0$, $f(x,u,v,\lambda)$ and $g(x,u,v,\lambda)$ are two nonnegative L^∞ functions defined in $\Omega\times[0,M]\times[0,M]\times[0,M]$, $f(x,u,v,0) = 0$, $g(x,u,v,0) = 0$, and

$$\lim_{t\to 0^+} f(x,tu,tv,\lambda)t^{1-p} = 0, \qquad \lim_{t\to 0^+} g(x,tu,tv,\lambda)t^{1-q} = 0,$$

uniformly with respect to $(x,u,v) \in \Omega \times (0,1] \times (0,1]$ and λ on bounded intervals.

THEOREM 4.1. Assume $f(x,u,v,\lambda)$ and $g(x,u,v,\lambda)$ satisfy (F). If $(\lambda_0,0,0)$ is a bifurcation point of nonnegative nontrivial solutions for (4.21), then λ_0 is an eigenvalue of (1.3) with nonnegative eigenfunctions.

Proof. Let

$$F(x,u,v,\lambda) = \lambda \sum_{i=1}^K a_i(x)u^{\alpha_i}v^{p-1-\alpha_i} + f(x,u,v,\lambda),$$
$$G(x,u,v,\lambda) = \lambda \sum_{j=1}^N b_j(x)u^{q-1-\beta_j}v^{\beta_j} + g(x,u,v,\lambda).$$

For a function $h \in L^\infty(\Omega)$, we denote by u the weak solution of the following elliptic boundary value problem

$$-\Delta_p u = h(x), \quad x \in \Omega, \qquad u = 0 \quad x \in \partial\Omega.$$

Let T_p be the mapping defined by $T_p(h(x)) = u(x)$. Lemma 2.1, Lemma 2.2 and the regularity results of Lieberman [5] imply $T_p : V_+ \to V_+$ is a completely continuous mapping. Let J be the mapping defined by

$$J[\lambda,u,v] = [T_p(F(x,u,v,\lambda)), T_q(G(x,u,v,\lambda))].$$

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and let $\{(\lambda_k, u_k, v_k)\}$ be a sequence of nonnegative nontrivial solutions of (4.21) satisfying $\lim_{k\to\infty} \lambda_k = \lambda_0$, $\lim_{k\to\infty}(u_k, v_k) = (0,0)$ in $V_+ \times V_+$ and

$$(u_k, v_k) = J[\lambda_k, u_k, v_k]. \tag{4.22}$$

Denote $t_k = \|(u_k, v_k)\|$. Let $(\overline{u}_k, \overline{v}_k) = t_k^{-1}(u_k, v_k)$. It follows from (4.22) that

$$(\overline{u}_k, \overline{v}_k) = (T_p[t_k^{1-p} F(x, t_k\overline{u}_k, t_k\overline{v}_k, \lambda_k)], T_q[t_k^{1-q} G(x, t_k\overline{u}_k, t_k\overline{v}_k, \lambda_k)]). \tag{4.23}$$

Notice that $\|(\overline{u}_k, \overline{v}_k)\| = 1$ and $t_k \to 0^+$ as $k \to \infty$. For any $\epsilon > 0$, the hypothesis (F) shows that there exists k_0 so large that

$$|t_k^{1-p} f(x, t_k\overline{u}_k, t_k\overline{v}_k, \lambda_k)| < \epsilon, |t_k^{1-q} g(x, t_k\overline{u}_k, t_k\overline{v}_k, \lambda_k)| < \epsilon, \tag{4.24}$$

as $k \geq k_0$. (4.23) and $\|(\overline{u}_k, \overline{v}_k)\| = 1$ imply that $F(x, t_k\overline{u}_k, t_k\overline{v}_k, \lambda_k)$ and $G(x, t_k\overline{u}_k, t_k\overline{v}_k, \lambda_k)$ are bounded sequences in V_+. It follows from the regularity results of Lieberman[5] that $\{(\overline{u}_k, \overline{v}_k)\}$ is a bounded sequence in the Banach space $C_0^{1,\beta}(\overline{\Omega})$ for some positive constant β. Thus there exist $(u_0, v_0) \in C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$ and a subsequence (still denoted by $\{(u_k, v_k)\}$) such that

$$\lim_{k\to\infty}(\overline{u}_k, \overline{v}_k) = (u_0, v_0) \quad \text{in} \quad C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega}), \tag{4.25}$$

and $\|(u_0, v_0)\| = \lim_{k\to\infty} \|(\overline{u}_k, \overline{v}_k)\| = 1$. Combining (4.24) with (4.25) yields

$$\begin{aligned} &\lim_{k\to\infty}(t_k^{1-p} F(x, t_k\overline{u}_k, t_k\overline{v}_k, \lambda_k), t_k^{1-q} G(x, t_k\overline{u}_k, t_k\overline{v}_k, \lambda_k)) \\ &= (\lambda_0 \sum_{i=1}^{K} a_i(x) u_0^{\alpha_i} v_0^{p-1-\alpha_i}, \lambda_0 \sum_{j=1}^{N} b_j(x) u_0^{q-1-\beta_j} v_0^{\beta_j}) \quad \text{in} \quad V_+ \times V_+. \end{aligned} \tag{4.26}$$

Since T_p and T_q are completely continuous operators mapping V_+ into itself, it follows from (4.23) and (4.27) that

$$(u_0, v_0) = (T_p[\lambda_0 \sum_{i=1}^{K} a_i(x) u_0^{\alpha_i} v_0^{p-1-\alpha_i}], T_q[\lambda_0 \sum_{j=1}^{N} b_j(x) u_0^{q-1-\beta_j} v_0^{\beta_j}]),$$

with $\|(u_0, v_0)\| = 1$. Hence λ_0 is an eigenvalue of (1.3) with a nonnegative eigenfunction (u_0, v_0). We complete the proof of Theorem 4.1. ∎

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For $x \in \Omega$, we denote $\frac{1}{d(x)} = \inf\{|x-y| : y \in \partial\Omega\}$. Let $\{(\lambda_k, u_k, v_k)\}$ be a branch of nonnegative solutions of (4.21) such that $\lim\limits_{k\to\infty} \lambda_k = \lambda_0$ and $\lim\limits_{k\to\infty}(u_k, v_k) = (0,0)$ in $V_+ \times V_+$. Denote by $(\overline{u}_k, \overline{v}_k) = \|(u_k, v_k)\|^{-1}(u_k, v_k)$. From the proof of Theorem 4.1 we know that

$$\lim_{k\to\infty}(\overline{u}_k, \overline{v}_k) = (u_0, v_0) \quad \text{in} \quad C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega}), \tag{4.27}$$

and that λ_0 is an eigenvalue of (1.3) with a nonnegative eigenfunction (u_0, v_0) satisfying $\|(u_0, v_0)\| = 1$. (4.27) implies

$$\lim_{k\to\infty}(d(x)\overline{u}_k, d(x)\overline{v}_k) = (d(x)u_0, d(x)v_0) \quad \text{in} \quad V_+ \times V_+. \tag{4.28}$$

When (u_0, v_0) is a nontrivial nonnegative eigenfunction of (1.3) and when there exists some $i \in \{1, 2, \cdots, K\}$ (or $j \in \{1, 2, \cdots, N\}$) such that $\alpha_i < p-1$ (or $\beta_j < q-1$), Theorem 1.1, Theorem 3.1 and Lemma 2.3 mean that there exists a positive constant $\delta > 0$, which is independent of $\{(\lambda_k, u_k, v_k)\}$, such that $\inf\{d(x)u_0(x) | x \in \Omega\} \geq \delta$ and $\inf\{d(x)v_0(x) | x \in \Omega\} \geq \delta$. By (4.28) we can choose two positive constant $C_1 < C_2$ satisfying

$$\begin{aligned} C_1 &\leq \frac{\delta - |d(x)(u_k(x) - u_0(x))|}{|d(x)v_0(x)| + |d(x)(v_k(x) - v_0(x))|} \leq \frac{u_k(x)}{v_k(x)} \\ &= \frac{d(x)u_k(x)}{d(x)v_k(x)} \leq \frac{|d(x)u_0(x)| + |d(x)(u_k(x) - u_0(x))|}{\delta - |d(x)(v_k(x) - v_0(x))|} \leq C_2 \end{aligned}$$

for k large enough.

When (u_0, v_0) is a nonnegative eigenfunction of (1.3) with $v_0 \equiv 0$, it is obvious that nonnegative eigenfunctions of (1.3) of the form $(u, 0)$ and $\|(u, 0)\| = 1$ are unique. As above we get

$$\frac{v_k(x)}{u_k(x)} = \frac{d(x)v_k(x)}{d(x)u_k(x)} \leq \frac{|d(x)v_k(x)|}{\delta - |d(x)(u_k(x) - u_0(x))|}$$

for k large enough, which shows that $v_k(x) = o(u_k(x))$ uniformly for $x \in \Omega$ as $k \to \infty$. Thus, we have

Remark 4.1. Assume that $(\lambda_0, 0, 0)$ is a bifurcation point of (4.21) and that $\{(\lambda_k, u_k, v_k)\}$ is a branch of nonnegative solutions bifurcating from $(\lambda_0, 0, 0)$. Then,

(1). If λ_0 is an eigenvalue of (1.3) with nonnegative eigenfunctions of the form $(u, 0)$, then $\lim_{k\to\infty} \frac{v_k(x)}{u_k(x)} = 0$ uniformly for $x \in \Omega$.

(2). If λ_0 is an eigenvalue of (1.3) with nonnegative eigenfunctions of the form $(0, v)$, then $\lim_{k\to\infty} \frac{u_k(x)}{v_k(x)} = 0$ uniformly for $x \in \Omega$.

(3). If λ_0 is an eigenvalue of (1.3) with nontrivial nonnegative eigenfunctions and there exists some $i \in \{1, 2, \cdots, K\}$ (or $j \in \{1, 2, \cdots, N\}$) such that $\alpha_i < p - 1$ (or $\beta_j < q - 1$), then there exist positive numbers $C_1 \leq C_2$, which are independent of the sequence $\{(\lambda_k, u_k, v_k)\}_{k=1}^{\infty}$, such that $C_1 v_k(x) \leq u_k(x) \leq C_2 v_k(x)$ for $x \in \Omega$ and k large enough.

In the sequel, we shall consider the existence of a branch of nonnegative solutions of (4.21). Denote by $P_\epsilon = \{(u, v) \in V_+ \times V_+ : \quad \|u\| + \|v\| \leq \epsilon\}$. Using an argument similar to that in proof of Lemma 2.4 of [7], we have

Lemma 4.2. Assume the functions $f(x, u, v, \lambda)$ and $g(x, u, v, \lambda)$ satisfy (F). Suppose that there is a positive number $\overline{\lambda}$ such that, for $\lambda > \overline{\lambda}$, $(\lambda, 0, 0)$ is not a bifurcation point for (4.21) and $\deg(I - J[\lambda, \cdot, \cdot], P_\epsilon, (0, 0)) = 0$ for ϵ small enough. Then there exists $\lambda_0 \in (0, \overline{\lambda}]$ such that the set of nontrivial nonnegative solutions of (4.21) contains an unbounded subcontinuum bifurcating from $(\lambda_0, 0, 0)$, where J is a completely continuous mapping defined in the proof of Theorem 4.1.

Let Λ_1 be the eigenvalue of the problem

$$-\Delta_p u = \lambda a_1(x)|u|^{p-2}u \quad \text{in} \quad \Omega, u = 0 \quad \text{on} \quad \partial\Omega,$$

with the positive eigenfunction $\phi(x)$ satisfying $\|\phi\| = 1$. Let Λ_2 be the eigenvalue of the problem

$$-\Delta_q v = \lambda b_1(x)|v|^{q-2}v \quad \text{in} \quad \Omega, v = 0 \quad \text{on} \quad \partial\Omega,$$

with the positive eigenfunction $\psi(x)$ satisfying $\|\psi\| = 1$. It is obvious that $\phi, \psi \in C^{1,\beta}(\overline{\Omega})$ and that there exist two positive constant $C_1 \leq C_2$ satisfying $0 < C_1\psi(x) \leq \phi(x) \leq C_2\psi(x)$ for all $x \in \Omega$. Let

$\Lambda_3 = \max\{\lambda : \lambda$ is an eigenvalue of (1.3) with nonnegative eigenfunctions$\}$

$$\Lambda = \max\{\Lambda_1 C_2^{p-1-\alpha_1}, \Lambda_2 C_1^{\beta_1+1-q}, \Lambda_3\}.$$

By Remark 3.1 we deduce that $0 < \Lambda < \infty$.

Lemma 4.3. Let $\lambda > \Lambda$. For any $\epsilon > 0$ small we have $\deg(I - J[\lambda, \cdot, \cdot], P_\epsilon, (0, 0)) = 0$.

Proof. Define

$$
\begin{aligned}
F_t(x,u,v) &= \lambda \sum_{i=1}^{K} a_i(x) u^{\alpha_i} v^{p-1-\alpha_i} + t f(x,u,v,\lambda),\\
G_t(x,u,v) &= \lambda \sum_{j=1}^{N} b_j(x) u^{q-1-\beta_j} v^{\beta_j} + t g(x,u,v,\lambda).\\
H[t,u,v] &= (T_p[F_t(x,u,v)], T_q[G_t(x,u,v)]).
\end{aligned}
$$

Clearly, $H : [0,1]\times V_+ \times V_+ \to V_+ \times V_+$ is a completely continuous mapping. We claim that the operator equation $(u,v) - H[t,u,v] = (0,0)$ has no solution on $\{(u,v) \in V_+ \times V_+, \|u\| + \|v\| = \epsilon\}$ for $t \in [0,1]$ and ϵ small. Indeed, otherwise there exist $\{(u_k, v_k)\}$ and $\{t_k\}$ such that $(0,0) \not\equiv (u_k, v_k) \to (0,0)$ in $V_+ \times V_+$, $t_k \to t_0 \in [0,1]$ and $(u_k, v_k) = H[t_k, u_k, v_k]$. Using the same argument as that in the proof of Theorem 4.1, we can prove that λ is an eigenvalue of (1.3) with nonnegative eigenfunctions. Thus $\lambda \le \Lambda$, which is impossible. We have

$$
\begin{aligned}
\deg(I - J[\lambda,\cdot,\cdot], P_\epsilon, (0,0)) &= \deg(I - H[1,\cdot,\cdot], P_\epsilon, (0,0))\\
&= \deg(I - H[0,\cdot,\cdot], P_\epsilon, (0,0)).
\end{aligned} \tag{4.29}
$$

Define

$$
\begin{aligned}
A_t(x,u,v) &= \lambda \sum_{i=1}^{K} a_i(x) u^{\alpha_i} v^{p-1-\alpha_i} + t,\\
B_t(x,u,v) &= \lambda \sum_{i=1}^{N} b_j(x) u^{q-1-\beta_j} v^{\beta_j} + t,\\
S[t,u,v] &= (T_p[A_t(x,u,v)], T_q[B_t(x,u,v)]).
\end{aligned}
$$

Clearly, $S : [0,1]\times V_+ \times V_+ \to V_+ \times V_+$ is a completely continuous mapping too. The choice of λ means that $(u,v) - S[0,u,v] = (0,0)$ has no solution on $\{(u,v) \in V \times V, \|u\| + \|v\| = \epsilon\}$ for ϵ small. We claim that $(u,v) - S[t,u,v] = (0,0)$ has no solution on P_ϵ for $t \in (0,1]$ and ϵ small. Indeed, otherwise there exist $t_0 \in (0,1]$ and $(u_0, v_0) \in P_\epsilon$ such that $(u_0, v_0) = S[t_0, u_0, v_0]$. Lemma 2.3 shows

$$
(u_0, v_0) > (0,0) \quad x \in \Omega, \qquad \Big(\frac{\partial u_0}{\partial \nu}, \frac{\partial v_0}{\partial \nu}\Big) < (0,0) \quad x \in \partial\Omega. \tag{4.30}
$$

Moreover, (u_0, v_0) is a supersolution of (1.3). The fact that $\lambda > \Lambda \ge \Lambda_1 C_2^{p-1-\alpha_1}$ and $\lambda > \Lambda \ge \Lambda_2 C_1^{\beta_1+1-q}$ means that (ϕ, ψ) is a subsolution of (1.3). Choose $\delta > 0$ so small that $\delta(\phi, \psi) \leq (u_0, v_0)$ for all $x \in \Omega$. Thus, $\delta(\phi, \psi)$ is a subsolution of (1.3), and by the supersolution-subsolution method, (1.3) possesses a nonnegative solution $(\overline{u}, \overline{v})$ satisfying $\delta(\phi, \psi) \leq (\overline{u}, \overline{v}) \leq (u_0, v_0)$ for all $x \in \Omega$. Hence, λ is an eigenvalue of (1.3) with a nonnegative eigenfunction $(\overline{u}, \overline{v})$. This is contrary to $\lambda > \Lambda$. Hence $(u, v) - S[t, u, v] = (0, 0)$ has no solution on P_ϵ for $t \in (0, 1]$ and ϵ small, by (4.30)

$$\begin{aligned} &\deg(I - J[\lambda, \cdot, \cdot], P_\epsilon, (0,0)) = \deg(I - H[0, \cdot, \cdot], P_\epsilon, (0,0)) \\ &= \deg(I - S[0, \cdot, \cdot], P_\epsilon, (0,0)) = \deg(I - S[1, \cdot, \cdot], P_\epsilon, (0,0)) = 0. \end{aligned}$$

$\square$

A combination of Theorem 4.1 with Lemma 4.2 and Lemma 4.3 yields

THEOREM 4.4. Assume $f(x, u, v, \lambda)$ and $g(x, u, v, \lambda)$ satisfy (F). Then (4.21) contains an unbounded component of nonnegative solutions bifurcating from $(\lambda_0, 0, 0)$, where λ_0 is one of eigenvalues of (1.3) associated with nonnegative eigenfunctions.

§5. The existence of positive eigenfunctions

In this section we shall consider the existence of positive eigenfunctions for the elliptic eigenvalue system (1.3).

Proof of Theorem 1.2. (1). Consider the following elliptic system

$$(5.31) \qquad \begin{cases} -\Delta_p u = \lambda \sum_{i=1}^K a_i(x) u^{\alpha_i} v^{p-1-\alpha_i} + \lambda v^{p+1}, & x \in \Omega, \\ -\Delta_q v = \lambda \sum_{j=1}^N b_j(x) u^{q-1-\beta_j} v^{\beta_j} + \lambda u^{q+1}, & x \in \Omega, \\ u = v = 0. \quad x \in \partial\Omega. \end{cases}$$

By Theorem 4.4 and Lemma 2.3 the system (5.31) possesses a sequence of positive solutions $\{(u_k, v_k, \lambda_k)\}$ such that $\lim_{k\to\infty} \lambda_k = \lambda_0$, $\lim_{k\to\infty} (u_k, v_k) = (0, 0)$ in $V_+ \times V_+$, and λ_0 is an eigenvalue of (1.3) with nonnegative eigenfunctions. Thus, $\lambda_0 > 0$. What we want to prove is to show that λ_0 is an eigenvalue associated with nontrivial nonnegative eigenfunctions of (1.3). Denote by

$$B_k(x) = \sum_{i=1}^K a_i(x) \left(\frac{v_k}{u_k}\right)^{p-1-\alpha_i} + v_k^{p+1} u_k^{1-p},$$

$$C_k(x) = \sum_{j=1}^N b_j(x) \left(\frac{u_k}{v_k}\right)^{q-1-\beta_j} + u_k^{q+1} v_k^{1-q}.$$

By Lemma 2.3, $B_k(x), C_k(x) \in L^\infty(\Omega)$ and satisfy

$$-\Delta_p u_k = \lambda_k B_k(x) u_k^{p-1} \quad x \in \Omega, \quad u_k = 0 \quad x \in \partial\Omega \tag{5.32}$$

and

$$-\Delta_q v_k = \lambda_k C_k(x) v_k^{q-1} \quad x \in \Omega, \quad v_k = 0 \quad x \in \partial\Omega. \tag{5.33}$$

The uniqueness of eigenvalues with nonnegative eigenfunctions for a signal equation shows (for example see [2]) that

$$\lambda_k = \inf\left\{ \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega B_k(x)|u|^p dx}, \quad u \in W_0^{1,p}(\Omega) \right\},$$
$$\lambda_k = \inf\left\{ \frac{\int_\Omega |\nabla v|^q dx}{\int_\Omega C_k(x)|v|^q dx}, \quad v \in W_0^{1,q}(\Omega) \right\}.$$

We claim that λ_0 is an eigenvalue of (1.3) with nontrivial nonnegative eigenfunctions. Otherwise, Remark 4.1 shows

$$\lim_{k\to\infty} \frac{u_k}{v_k} = 0 \quad \text{or} \quad \lim_{k\to\infty} \frac{v_k}{u_k} = 0 \quad \text{uniformly for } x \in \Omega,$$

hence, for any $\epsilon > 0$

$$\frac{u_k}{v_k} \le \epsilon \quad \text{or} \quad \frac{v_k}{u_k} \le \epsilon \quad \text{uniformly for } x \in \Omega \text{ and for } k \text{ large enough.}$$

When $\alpha_1 < p-1$ and $\beta_1 < q-1$, without loss of generality, we consider the case

$$\frac{v_k}{u_k} \le \epsilon \quad \text{uniformly for } x \in \Omega \text{ and } k \text{ large enough.}$$

We deduce from the definition of B_k and C_k

$$B_k \le 1 + \sum_{i=1}^{K} a_i(x) \text{ and } \quad C_k \ge b_1(x)\epsilon^{\beta_1 - q+1}$$

for all $x \in \Omega$ and k large enough. Thus we get

$$\begin{cases} \lambda_k \ge \inf\left\{ \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega [1+\sum_{i=1}^k a_i(x)]|u|^p dx}, \quad u \in W_0^{1,p}(\Omega) \right\} \\ \lambda_k \le \epsilon^{q-1-\beta_1} \inf\left\{ \frac{\int_\Omega |\nabla v|^q dx}{\int_\Omega b_1(x)|v|^q dx}, \quad v \in W_0^{1,q}(\Omega) \right\}, \end{cases}$$

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which is impossible for ϵ small enough and k large.

When $\alpha_i = p-1$ and $\beta_j < q-1$ for $i = 1, 2, \cdots, K$ and $j = 1, 2, \cdots, N$ (we assume $\beta_j \leq \beta_1$), we have

$$B_k(x) \leq 1 + \sum_{i=1}^{K} a_i(x), \quad C_k(x) \geq b_1(x)\epsilon^{\beta_1 - q + 1}$$

or

$$B_k(x) \geq \sum_{i=1}^{K} a_i(x), \quad C_k(x) \leq \epsilon^{q-1-\beta_1}\left(1 + \sum_{j=1}^{N} b_j(x)\right)$$

for $x \in \Omega$ and k large enough. We deduce that

$$\begin{cases} \lambda_k \geq \inf\left\{\frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \left[1+\sum_{i=1}^{K} a_i(x)\right]|u|^p dx}, \quad u \in W_0^{1,p}(\Omega)\right\} \\ \lambda_k \leq \epsilon^{q-1-\beta_1} \inf\left\{\frac{\int_\Omega |\nabla v|^q dx}{\int_\Omega b_1(x)|v|^q dx}, \quad v \in W_0^{1,q}(\Omega)\right\} \end{cases}$$

or

$$\begin{cases} \lambda_k \leq \inf\left\{\frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \sum_{i=1}^{K} a_i(x)|u|^p dx}, \quad u \in W_0^{1,p}(\Omega)\right\} \\ \lambda_k \geq \epsilon^{\beta_1-q+1} \inf\left\{\frac{\int_\Omega |\nabla v|^q dx}{\int_\Omega \left[1+\sum_{j=1}^{N} b_j(x)\right]|v|^q dx}, \quad v \in W_0^{1,q}(\Omega)\right\}, \end{cases}$$

which is impossible when k large enough and ϵ small enough. Hence we prove that λ_0 is an eigenvalue of (1.3) with nontrivial nonnegative eigenfunctions, and by Lemma 2.3 λ_0 is an eigenvalue with positive eigenfunctions. The uniqueness of eigenvalues with positive eigenfunctions and the uniqueness of positive eigenfunctions of (1.3) follow from Theorem 3.1 and Theorem 1.1.

(2). Suppose that (1.3) possesses an eigenvalue λ_0 with a positive eigenfunction (ϕ, ψ). It is obvious that $\lambda_0 > 0$ and λ_0 is an eigenvalue with positive eigenfunctions of the following eigenvalue problems

$$-\Delta_p u = \lambda \left[\sum_{i=1}^{K} a_i(x)\right] u^{p-1} \quad x \in \Omega, \quad u = 0 \quad x \in \partial\Omega, \tag{5.34}$$

and

$$-\Delta_q v = \lambda \left[\sum_{j=1}^{N} b_j(x)\phi^{q-1-\beta_j}\psi^{\beta_j+1-q}\right] v^{q-1} \quad x \in \Omega, \quad v = 0 \quad x \in \partial\Omega. \tag{5.35}$$

The uniqueness of eigenvalues with positive eigenfunctions for (5.34) or (5.35)(see [2]) shows

$$\begin{aligned}\lambda_0 =& \inf\left\{\int_\Omega |\nabla u|^p dx : \int_\Omega \sum_{i=1}^{K} a_i(x)|u|^p dx = 1, u \in W_0^{1,p}(\Omega)\right\}\\ =& \inf\left\{\int_\Omega |\nabla v|^q dx : \int_\Omega [\sum_{j=1}^{N} b_j(x)\psi^{\beta_j+1-q}\phi^{q-1-\beta_j}]v^q dx = 1, v \in W_0^{1,q}(\Omega)\right\}\\ <& \inf\left\{\int_\Omega |\nabla v|^q dx : \int_\Omega b_1(x)|v|^q dx = 1, v \in W_0^{1,q}(\Omega)\right\}.\end{aligned}$$

Now we prove that (1.3) possesses positive eigenfunctions. Denote by $\lambda_0 > 0$ the unique eigenvalue of (5.34) with positive eigenfunctions, and let $\phi(x) \in C^{1,\beta}(\overline{\Omega}) \subset L^\infty(\Omega)$ be the positive eigenfunction associated to λ_0 satisfying $\|\phi(x)\| = 1$. By the existence results in [2], the following problem

$$\begin{cases} -\Delta_q v = \lambda b_2(x)\phi^{q-1-\beta_2}v^{q-1} & x \in \Omega,\\ v = 0 & x \in \partial\Omega,\end{cases} \tag{5.36}$$

has a unique eigenvalue $\lambda_1 > 0$ associated with positive eigenfunctions. Let us denote by $\psi_1(x)$ the positive eigenfunction of (5.36) satisfying $\|\psi_1\| = 1$.

On the other hand, the condition implies

$$\lambda_0 < \inf\left\{\int_\Omega |\nabla v|^q dx : \int_\Omega b_1(x)v^q dx = 1, v \in W_0^{1,q}(\Omega)\right\},$$

and hence we can choose a positive constant b_0 so small that

$$\lambda_0 < \inf\left\{\int_\Omega |\nabla v|^q dx : \quad \int_\Omega [b_1(x) + b_0]v^q dx = 1, v \in W_0^{1,q}(\Omega)\right\}.$$

Then the following eigenvalue problem

$$\begin{cases} -\Delta_q v = \lambda[b_1(x) + b_0]v^{q-1} & x \in \Omega,\\ v = 0 & x \in \partial\Omega,\end{cases} \tag{5.37}$$

has an eigenvalue, $\lambda_2 > \lambda_0$, associated with positive eigenfunctions. We denote by $\psi_2(x)$ a positive eigenfunction of (5.37) associated to λ_2 satisfying $|\psi_2(x)| = 1$. By Lemma 2.3,

$$\phi(x) > 0, \psi_1(x) > 0, \psi_2(x) > 0 \quad \text{for} \quad x \in \Omega,$$

$$\frac{\partial \phi}{\partial \nu} < 0, \frac{\partial \psi_1}{\partial \nu} < 0, \frac{\partial \psi_2}{\partial \nu} < 0 \quad \text{for} \quad \in \partial\Omega,$$

and hence we can choose two positive constants C_1 and C_2 such that

$$\psi_1(x) \leq C_1\phi(x), \quad \psi_2(x) \geq C_2\phi(x) \quad \text{for} \quad x \in \Omega.$$

Choose $M > 0$ large enough and $\epsilon > 0$ small enough such that

$$\lambda_2 b_0 \geq \lambda_0 \sum_{j=2}^{N} b_j(x)(C_2 M)^{\beta_j+1-q}, \quad \lambda_1 \epsilon^{q-1-\beta_2} \psi_1^{q-1-\beta_2} \leq \lambda_0,$$

$$M\psi_2(x) \geq \epsilon\psi_1(x) \quad \text{for} \quad x \in \Omega.$$

Then $\epsilon\psi_1(x)$ and $M\psi_2(x)$ are a subsolution and a supersolution, respectively, of the following elliptic problem

$$(5.38) \qquad \begin{cases} -\Delta_q v = \lambda_0 \sum_{j=1}^{N} b_j(x)\phi^{q-1-\beta_j} v^{\beta_j} & \text{for} \quad x \in \Omega, \\ v = 0 \quad \text{for} \quad \in \partial\Omega. \end{cases}$$

The subsolution-supersolution method shows (5.38) possess a positive solution $\psi(x)$. Obviously, λ_0 and (ϕ, ψ) are the eigenvalue and the positive eigenfunction of (1.3), respectively. The uniqueness of the eigenvalue with positive eigenfunctions and the uniqueness of positive eigenfunctions of (1.3) follows from Theorem 3.1 and Theorem 1.1.

(3). The result is obvious, we omit its proof.

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