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ASYMPTOTIC MONOTONICITY OF THE RELATIVE EXTREMA OF JACOBI POLYNOMIALS

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ABSTRACT. If $\mu_{k,n}(\alpha,\beta)$ denotes the relative extrema of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, ordered so that $\mu_{k+1,n}(\alpha,\beta)$ lies to the left of $\mu_{k,n}(\alpha,\beta)$, then R. A. Askey has conjectured twenty years ago that for $\alpha > \beta > -\frac{1}{2}$, $|\mu_{k,n+1}(\alpha,\beta)| < |\mu_{k,n}(\alpha,\beta)|$ for $k = 1,\ldots,n-1$ and $n = 1,2,\ldots$. In this paper, we give an asymptotic expansion for $\mu_{k,n}(\alpha,\beta)$ when k is fixed and $n \longrightarrow \infty$, which corrects an earlier result of R. Cooper (1950). Furthermore, we show that Askey's conjecture is true at least in the asymptotic sense.

1. **Introduction.** Let $-1 < y_{n-1,n} < \cdots < y_{1,n} < 1$ denote the critical points of the Legendre polynomial $P_n(x)$, *i.e.*, $P_n'(y_{k,n}) = 0$, and put $y_{0,n} = 1$ and $y_{n,n} = -1$. If $\mu_{k,n} = P_n(y_{k,n})$, then it was observed by Todd [11] that

$$|\mu_{k,n}| < |\mu_{k,n-1}|, \quad k = 1,\ldots,n-1. \tag{1.1}$$

Cooper [3] was the first to study this problem by using asymptotics. He showed that

$$\mu_{k,n} \sim J_0(j_{1,k}) + \frac{j_{1,k}^2}{12n^2} J_0(j_{1,k}) + \cdots, \quad \text{as } n \longrightarrow \infty, \tag{1.2}$$

for each fixed k, where $j_{1,k}$ is the k-th positive zero of $J_1(x)$. From (1.2), it is evident that $\mu_{k,n}$ is asymptotically decreasing. The general case of (1.1) was proved by Szegö [10], and extended to the ultraspherical polynomial by Szász [8].

Now let $P_n^{(\alpha,\beta)}(x)$ denote the Jacobi polynomial, and $y_{k,n}^{(\alpha,\beta)}$ be the location of the relative extrema of $P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$ ordered by $-1 = y_{n,n}^{(\alpha,\beta)} < y_{n-1,n}^{(\alpha,\beta)} < \cdots < y_{1,n}^{(\alpha,\beta)} < y_{0,n}^{(\alpha,\beta)} = 1$. Set

$$\mu_{k,n}(\alpha,\beta) = \frac{P_n^{(\alpha,\beta)}(y_{k,n}^{(\alpha,\beta)})}{P_n^{(\alpha,\beta)}(1)}, \quad k = 1,\ldots,n-1. \tag{1.3}$$

In [9, p. 190], it is conjectured that for $\alpha > \beta > -\frac{1}{2}$,

$$|\mu_{k,n+1}(\alpha,\beta)| < |\mu_{k,n}(\alpha,\beta)|, \quad k = 1,\ldots,n, \tag{1.4}$$

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and that the inequalities are reversed for the function

$$P_n^{(0,-1)}(x) = \frac{P_n(x) + P_{n-1}(x)}{2};$$

that is

$$(1.5) \qquad |\mu_{k,n+1}(0,-1)| > |\mu_{k,n}(0,-1)|, \quad k = 1, \ldots, n.$$

(These conjectures were made by Askey.) The inequalities in (1.5) have been recently verified for all n by using asymptotic methods [12].

The purpose of this paper is to show that (1.4) holds for sufficiently large n. This has been attempted by Cooper [3] more than forty years ago, but Cooper's asymptotic expansion of $\mu_{k,n}(\alpha,\beta)$ is incorrect, as pointed out by Askey [1, p. 32]. The problem to reconsider (1.4) from the asymptotic point of view is suggested also by Askey [1]. In Section 2, we will show that for each fixed $k = 1, 2, \ldots,$

$$(1.6) \qquad \mu_{k,n}(\alpha,\beta) = \Gamma(\alpha+1)\Big(\frac{2}{j_{\alpha+1,k}}\Big)^{\alpha} J_\alpha(j_{\alpha+1,k})\Big[1 + \frac{\alpha+3\beta+2j^2_{\alpha+1,k}}{24}\frac{}{N^2} + O(N^{-4})\Big],$$

as $n \longrightarrow \infty$, where $N = n + \frac{1}{2}(\alpha+\beta+1)$ and $j_{\alpha+1,k}$ is the k-th positive zero of the Bessel function $J_{\alpha+1}(x)$. Our approach differs completely from that of Cooper. We shall make use of the uniform asymptotic expansions of the Jacobi polynomial given in [5]. From (1.6), it is evident that $\mu_{k,n}(\alpha,\beta)$ decreases for sufficiently large n as long as k is fixed. However, the integer k in (1.4) may depend on n. Consequently, expansion (1.6) is not sufficient to prove the conjecture in (1.4) even in the sense of asymptotics. To overcome this difficulty, we shall first prove that (1.4) holds for all $k \geq K_n^{(1)}$, where $K_n^{(1)}$ is the smallest positive integer satisfying

$$(1.7) \qquad y_{k,n+1}^{(\alpha,\beta)} \leq x_0, \quad \text{where } x_0 = -\frac{\alpha-\beta}{\alpha+\beta+1}.$$

This is done in Section 3. (Note that $-1 < x_0 < 1$ when $\alpha > -\frac{1}{2}$ and $\beta > -\frac{1}{2}$.) We then use a uniform asymptotic approximation of the Jacobi polynomial given by Baratella and Gatteschi [2], which is sharper than that in [5], to show that (1.4) is true in asymptotic sense when $k \leq K_n^{(2)}$, where $K_n^{(2)}$ is the largest positive integer satisfying

$$(1.8) \qquad \cos\eta_0 \leq y_{k,n}^{(\alpha,\beta)}, \quad \text{where } \eta_0 = \cos^{-1} x_0 + \frac{1}{4}\cdot\frac{2\beta+1}{\alpha+\beta+1}.$$

(Using the Mean Value Theorem, it is easily seen that $0 < \eta_0 < \pi$.) This is done in Sections 4 and 5. The asymptotic monotonicity of $\mu_{k,n}(\alpha,\beta)$ is established in Section 6, where we prove that

$$(1.9) \qquad K_n^{(1)} \leq K_n^{(2)} \text{ for all sufficiently large } n.$$

2. **Asymptotic expansions of $\mu_{k,n}(\alpha,\beta)$.** From the differentiation formula

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{2.1}$$

it is evident that the critical points $y_{k,n}^{(\alpha,\beta)}$ of $P_n^{(\alpha,\beta)}(x)$ are exactly the zeros $x_{k,n-1}^{(\alpha+1,\beta+1)}$ of $P_{n-1}^{(\alpha+1,\beta+1)}(x)$. Thus the relative extrema $\mu_{k,n}(\alpha,\beta)$ given in (1.3) can also be expressed as

$$\mu_{k,n}(\alpha,\beta) = \frac{P_n^{(\alpha,\beta)}(\cos\theta_{k,n-1}^{(\alpha+1,\beta+1)})}{P_n^{(\alpha,\beta)}(1)}, \quad k=1,\ldots,n-1, \tag{2.2}$$

where $x_{k,n-1}^{(\alpha+1,\beta+1)} = \cos\theta_{k,n-1}^{(\alpha+1,\beta+1)}$. As in [9, p. 121], we enumerate the zeros of Jacobi polynomials in decreasing order:

$$-1 < x_{n,n}^{(\alpha,\beta)} < \cdots < x_{1,n}^{(\alpha,\beta)} < 1; \quad 0 < \theta_{1,n}^{(\alpha,\beta)} < \cdots < \theta_{n,n}^{(\alpha,\beta)} < \pi.$$

In [5], it is shown that for $\alpha > \beta > -\frac{1}{2}$, we have

$$\begin{aligned} P_n^{(\alpha,\beta)}(\cos\theta) &= \frac{\Gamma(n+\alpha+1)}{n!}\Big(\sin\frac{\theta}{2}\Big)^{-\alpha}\Big(\cos\frac{\theta}{2}\Big)^{-\beta}\Big(\frac{\theta}{\sin\theta}\Big)^{1/2} \\ &\quad\cdot\Big[\sum_{\ell=0}^{m-1}A_\ell(\theta)\frac{J_{\alpha+\ell}(N\theta)}{N^{\alpha+\ell}} + \theta^m O(N^{-m-\alpha})\Big], \end{aligned} \tag{2.3}$$

where

$$N = n + \frac{1}{2}(\alpha+\beta+1) \tag{2.4}$$

and the O-term is uniform with respect to $\theta \in [0,\pi-\varepsilon]$, $\varepsilon > 0$; see also the comments in [6, p. 396]. The coefficients $A_\ell(\theta)$ are analytic functions in $0 \le \theta \le \pi-\varepsilon$, and are $O(\theta^\ell)$ in that interval. In particular, $A_0(\theta) = 1$ and

$$A_1(\theta) = \Big(\alpha^2 - \frac{1}{4}\Big)\Big(\frac{1-\theta\cot\theta}{2\theta}\Big) - \frac{\alpha^2-\beta^2}{4}\tan\frac{\theta}{2}. \tag{2.5}$$

It is also shown in [5] that the zeros $\theta_{k,n}^{(\alpha,\beta)}$ of $P_n^{(\alpha,\beta)}(\cos\theta)$ satisfy

$$\theta_{k,n}^{(\alpha,\beta)} = \frac{j_{\alpha,k}}{N} + A_1(t)\frac{1}{N^2} + t^2 O\Big(\frac{1}{N^3}\Big), \tag{2.6}$$

where $t = j_{\alpha,k}/N$. The O-term is uniformly bounded for all values of $k = 1,2,\ldots,[\gamma n]$, where $\gamma \in (0,1)$ is a constant.

For simplicity, we introduce the function

$$g(\theta) \equiv \Big(\sin\frac{\theta}{2}\Big)^{-\alpha}\Big(\cos\frac{\theta}{2}\Big)^{-\beta}\Big(\frac{\theta}{\sin\theta}\Big)^{\frac{1}{2}}, \tag{2.7}$$

and suppress the dependence of the zeros $\theta_{k,n-1}^{(\alpha+1,\beta+1)}$ on α and β; *i.e.*, we write

$$\theta_{k,n-1} = \theta_{k,n-1}^{(\alpha+1,\beta+1)}, \quad k=1,\ldots,n-1. \tag{2.8}$$

Since

$$P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}, \tag{2.9}$$

coupling (2.2) and (2.3) gives
(2.10)

$$\mu_{k,n}(\alpha,\beta) = \frac{\Gamma(\alpha+1)}{N^\alpha} g(\theta_{k,n-1})\Big[J_\alpha(N\theta_{k,n-1}) + A_1(\theta_{k,n-1})\frac{J_{\alpha+1}(N\theta_{k,n-1})}{N} + \theta_{k,n-1}^2 O(N^{-2})\Big]$$

for all k satisfying $\theta_{k,n-1} \le \pi - \varepsilon$, $\varepsilon > 0$. Let

$$\tau_{k,n} = \frac{j_{\alpha+1,k}}{N} \tag{2.11}$$

and

$$\tilde{A}_1(\tau) = \Big[(\alpha+1)^2 - \frac{1}{4}\Big]\Big(\frac{1-\tau\cot\tau}{2\tau}\Big) - \frac{1}{4}[(\alpha+1)^2 - (\beta+1)^2]\tan\frac{\tau}{2}. \tag{2.12}$$

Then (2.6) gives

$$\theta_{k,n-1} = \tau_{k,n} + \tilde{A}(\tau_{k,n})\frac{1}{N^2} + \tau_{k,n}^2 O\Big(\frac{1}{N^3}\Big). \tag{2.13}$$

For each fixed k, $\tau_{k,n} \longrightarrow 0$ as $n \longrightarrow \infty$. Thus from the Maclaurin expansions

$$\frac{1}{\theta} - \cot\theta = \frac{\theta}{3} + \frac{\theta^3}{45} + \cdots + \frac{(-1)^{n-1}2^{2n}B_{2n}}{(2n)!}\theta^{2n-1} + \cdots, \quad |\theta| < 2\pi, \tag{2.14}$$

$$\tan\frac{\theta}{2} = \frac{\theta}{2} + \frac{\theta^3}{24} + \cdots + \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}\Big(\frac{\theta}{2}\Big)^{2n-1} + \cdots, \quad |\theta| < \pi, \tag{2.15}$$

we have

$$\tilde{A}_1(\tau_{k,n}) = \frac{j_{\alpha+1,k}}{N}\Big\{\frac{1}{6}\Big[(\alpha+1)^2 - \frac{1}{4}\Big] - \frac{1}{8}[(\alpha+1)^2 - (\beta+1)^2]\Big\} + \cdots \tag{2.16}$$

and

$$\theta_{k,n-1} = \frac{j_{\alpha+1,k}}{N} + \frac{j_{\alpha+1,k}}{N^3}\Big\{\frac{1}{6}\Big[(\alpha+1)^2 - \frac{1}{4}\Big] - \frac{1}{8}[(\alpha+1)^2 - (\beta+1)^2]\Big\} + O(N^{-5}). \tag{2.17}$$

From (2.7) it follows

$$\begin{aligned} g(\theta_{k,n-1}) = \Big(\frac{j_{\alpha+1,k}}{2N}\Big)^{-\alpha} \cdot \Big[1 - \frac{\alpha}{N^2}\Big\{\frac{1}{6}\Big[(\alpha+1)^2 - \frac{1}{4}\Big] - \frac{(\alpha+1)^2 - (\beta+1)^2}{8}\Big\} \\ + \frac{\alpha+3\beta+2}{6}\cdot\frac{j_{\alpha+1,k}^2}{4N^2} + O(N^{-4})\Big]. \end{aligned} \tag{2.18}$$

By Taylor's theorem,
(2.19)

$$\begin{aligned} J_{\alpha+\ell}(N\theta_{k,n-1}) = J_{\alpha+\ell}(j_{\alpha+1,k}) \\ + J'_{\alpha+\ell}(j_{\alpha+1,k})\Big\{\frac{1}{6}\Big[(\alpha+1)^2 - \frac{1}{4}\Big] - \frac{(\alpha+1)^2 - (\beta+1)^2}{8}\Big\}\frac{j_{\alpha+1,k}}{N^2} \\ + O(N^{-4}) \end{aligned}$$

for $\ell = 0, 1, 2, \ldots$. Since

$$j_{\alpha+1,k} J'_\alpha(j_{\alpha+1,k}) = \alpha J_\alpha(j_{\alpha+1,k}) \text{ and } J'_{\alpha+1}(j_{\alpha+1,k}) = J_\alpha(j_{\alpha+1,k}), \tag{2.20}$$

and since $A_1(\theta) = O(\theta)$, inserting (2.18) and (2.19) in (2.10), we obtain the desired result (1.6).

For our discussions in Sections 4 and 5, we need a uniform asymptotic expansion of the Jacobi polynomial given by Baratella and Gatteschi [2], which is quite different from the one stated in (2.3). As in [2], we let

$$A = 1 - 4\alpha^2, \quad B = 1 - 4\beta^2, \tag{2.21}$$

$$a(\theta) = \frac{2}{\theta} - \cot\frac{\theta}{2}, \quad b(\theta) = \tan\frac{\theta}{2}, \tag{2.22}$$

$$f(\theta) = N\theta + \frac{1}{16N}[Aa(\theta) + Bb(\theta)], \tag{2.23}$$

and

$$C_0 = 2^{-\frac{1}{2}} N^{-\alpha} \frac{\Gamma(n+\alpha+1)}{n!}\Big[1 + \frac{1}{16N^2}\Big(\frac{A}{6} + \frac{B}{2}\Big)\Big]^{-\alpha}. \tag{2.24}$$

For $\alpha, \beta > -\frac{1}{2}$ and $0 < \theta \le \pi - \varepsilon$, we have

$$\Big(\sin\frac{\theta}{2}\Big)^{\alpha+\frac{1}{2}} \Big(\cos\frac{\theta}{2}\Big)^{\beta+\frac{1}{2}} P_n^{(\alpha,\beta)}(\cos\theta) = \Big[\frac{f(\theta)}{f'(\theta)}\Big]^{\frac{1}{2}} \{C_0 J_\alpha[f(\theta)] - I\}, \tag{2.25}$$

where

$$I = \frac{\Gamma(n+\alpha+1)}{n!} \cdot O(N^{-\alpha-\frac{7}{2}}). \tag{2.26}$$

Baratella and Gatteschi actually considered only the case $-\frac{1}{2} \le \alpha, \beta \le \frac{1}{2}$. However, their argument can be extended to allow $\alpha, \beta > -\frac{1}{2}$, if we are willing to accept the order estimate (2.26), instead of the numerical bound which they obtained for the error term I. An important consequence of (2.25) is the following uniform asymptotic approximation of the zeros $\theta_{k,n}^{(\alpha,\beta)}$:

$$\theta_{k,n}^{(\alpha,\beta)} = \frac{j_{\alpha,k}}{N} - \frac{1}{16N^2}[Aa(t) + Bb(t)] + \varepsilon_1, \tag{2.27}$$

for all k satisfying $\theta_{k,n}^{(\alpha,\beta)} \le \pi - \varepsilon$, where $t = j_{\alpha,k}/N$ and

$$\varepsilon_1 = \sqrt{j_{\alpha,k}} \cdot O(N^{-9/2}). \tag{2.28}$$

Note that for fixed k, (2.27)–(2.28) is weaker than (2.6), but if k is allowed to grow with n then (2.27)–(2.28) is stronger than (2.6). A combination of (2.2), (2.25) and (2.27) gives

$$\begin{aligned}\mu_{k,n}(\alpha,\beta) = \Gamma(\alpha+1) \cdot \frac{g(\theta_{k,n-1})}{\sqrt{\theta_{k,n-1}}} \cdot \Big[\frac{f(\theta_{k,n-1})}{f'(\theta_{k,n-1})}\Big]^{1/2} \cdot N^{-\alpha} \\ \cdot \Big\{J_\alpha[f(\theta_{k,n-1})]\Big[1 + \frac{1}{16N^2}\Big(\frac{A}{6} + \frac{B}{2}\Big)\Big]^{-\alpha} + \varepsilon_2\Big\},\end{aligned} \tag{2.29}$$

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where

$$\varepsilon_2 = O(N^{-7/2}) \tag{2.30}$$

for all k satisfying $\theta_{k,n-1} \leq \pi - \varepsilon < \pi$. As in [12, eq. (4.13)], it can be shown that

$$\left[\frac{f(\theta_{k,n-1})}{f'(\theta_{k,n-1})}\right]^{1/2} = \left[1 + \frac{g_1(\theta_{k,n-1})}{32N^2} + \varepsilon_3\right] \cdot \sqrt{\theta_{k,n-1}}, \tag{2.31}$$

where

$$g_1(\theta) = A\left[\frac{a(\theta)}{\theta} - a'(\theta)\right] + B\left[\frac{b(\theta)}{\theta} - b'(\theta)\right] \tag{2.32}$$

and

$$\varepsilon_3 = O(N^{-4}) \tag{2.33}$$

for all $\theta_{k,n-1} \leq \pi - \varepsilon$. Substituting (2.31) in (2.29), we obtain

$$\begin{aligned} \mu_{k,n}(\alpha,\beta) &= \Gamma(\alpha+1)g(\theta_{k,n-1})N^{-\alpha}\left[1 + \frac{g_1(\theta_{k,n-1})}{32N^2} + \varepsilon_3\right] \\ &\quad \cdot \left\{J_\alpha[f(\theta_{k,n-1})] \cdot \left[1 + \frac{1}{16N^2}\left(\frac{A}{6} + \frac{B}{2}\right)\right]^{-\alpha} + \varepsilon_2\right\}; \end{aligned} \tag{2.34}$$

compare the O-term in (2.10) with the O-terms in (2.30) and (2.33).

3. **Monotonicity of $|\mu_{k,n}(\alpha,\beta)|$ when $k \geq K_n^{(1)}$.** Motivated by the arguments in [9, p. 168] and [1, p. 19], we let

$$R_n^{(\alpha,\beta)}(x) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} \tag{3.1}$$

and

$$f_n(x) = [R_n^{(\alpha,\beta)}(x)]^2 + \frac{(1-x^2)[\frac{d}{dx}R_n^{(\alpha,\beta)}(x)]^2}{n(n+\alpha+\beta+1)}. \tag{3.2}$$

Using the Jacobi differential equation, we obtain

$$f_n'(x) = \frac{2[\alpha-\beta+(\alpha+\beta+1)x]}{n(n+\alpha+\beta+1)}\left[\frac{d}{dx}R_n^{(\alpha,\beta)}(x)\right]^2; \tag{3.3}$$

see [9, p. 168, (7.32.4)]. Thus $f_n(x)$ is an increasing function in $x_0 \leq x \leq 1$, and a decreasing function in $-1 \leq x \leq x_0$, where x_0 is the point given in (1.7). The following result shows that $\{f_n(x)\}$ is a monotonically decreasing in n for $-1 \leq x \leq 1$.

LEMMA 1. *For $\alpha > \beta > -\frac{1}{2}$, we have*

$$f_{n+1}(x) \le f_n(x), \quad -1 \le x \le 1.$$

PROOF. Let

$$r(x) = [R_n^{(\alpha,\beta)}(x)]^2 - [R_{n+1}^{(\alpha,\beta)}(x)]^2.$$

From (3.1) and (2.9), it follows that

$$r(x) = \left[\frac{n!\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)}\right]^2 \cdot [(n+\alpha+1)P_n^{(\alpha,\beta)}(x) - (n+1)P_{n+1}^{(\alpha,\beta)}(x)] \\ \cdot [(n+\alpha+1)P_n^{(\alpha,\beta)}(x) + (n+1)P_{n+1}^{(\alpha,\beta)}(x)].$$

The recurrence relations [4, p. 173]

$$(3.4)\quad \left(n+\frac{\alpha}{2}+\frac{\beta}{2}+1\right)(1-x)P_n^{(\alpha+1,\beta)}(x) = (n+\alpha+1)P_n^{(\alpha,\beta)}(x) - (n+1)P_{n+1}^{(\alpha,\beta)}(x),$$

$$(3.5)\quad \left(n+\frac{\alpha}{2}+\frac{\beta}{2}+1\right)(1+x)P_n^{(\alpha,\beta+1)}(x) = (n+\beta+1)P_n^{(\alpha,\beta)}(x) + (n+1)P_{n+1}^{(\alpha,\beta)}(x)$$

then give

$$\begin{aligned} r(x) &= \left[\frac{n!\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)}\right]^2 \cdot \frac{2n+\alpha+\beta+2}{2} \cdot (1-x)P_n^{(\alpha+1,\beta)}(x) \\ &\quad \cdot \{[(n+\beta+1)P_n^{(\alpha,\beta)}(x) + (n+1)P_{n+1}^{(\alpha,\beta)}(x)] + (\alpha-\beta)P_n^{(\alpha,\beta)}(x)\} \\ &= \left[\frac{n!\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)}\right]^2 \left(\frac{2n+\alpha+\beta+2}{2}\right)^2 (1-x^2)P_n^{(\alpha+1,\beta)}(x)P_n^{(\alpha,\beta+1)}(x) \\ &\quad + \left[\frac{n!\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)}\right]^2 (1-x)(\alpha-\beta)P_n^{(\alpha+1,\beta)}(x) \cdot \frac{2n+\alpha+\beta+2}{2} P_n^{(\alpha,\beta)}(x). \end{aligned}$$

We also recall that

$$(3.6)\quad (2n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x) = (n+\alpha+\beta+1)P_n^{(\alpha+1,\beta)}(x) - (n+\beta)P_{n-1}^{(\alpha+1,\beta)}(x),$$

$$(3.7)\quad (2n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x) = (n+\alpha+\beta+1)P_n^{(\alpha,\beta+1)}(x) + (n+\alpha)P_{n-1}^{(\alpha,\beta+1)}(x);$$

see [4, p. 173]. From these equations, we obtain

$$\begin{aligned} r(x) &= \left[\frac{n!\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)}\right]^2 \frac{1-x^2}{4} \cdot [(n+\alpha+\beta+2)P_n^{(\alpha+1,\beta+1)}(x) + (n+\alpha+1)P_{n-1}^{(\alpha+1,\beta+1)}(x)] \\ &\quad \cdot [(n+\alpha+\beta+2)P_n^{(\alpha+1,\beta+1)}(x) - (n+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x)] \\ &\quad + \left[\frac{n!\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)}\right]^2 \cdot \frac{2n+\alpha+\beta+2}{2}(\alpha-\beta)(1-x)P_n^{(\alpha,\beta)}(x)P_n^{(\alpha+1,\beta)}(x) \end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{n!\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)}\right]^2 \cdot \frac{1-x^2}{4} \\
&\qquad \cdot \left[(n+\alpha+\beta+2)P_n^{(\alpha+1,\beta+1)}(x) + (n+\alpha+1)P_{n-1}^{(\alpha+1,\beta+1)}(x)\right] \\
&\qquad \cdot \left[(n+\alpha+\beta+2)P_n^{(\alpha+1,\beta+1)}(x) - (n+\alpha+1)P_{n-1}^{(\alpha+1,\beta+1)}(x)\right] \\
&\quad + \left[\frac{n!\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)}\right]^2 \cdot \frac{1-x^2}{4} \cdot \left[(n+\alpha+\beta+2)P_n^{(\alpha+1,\beta+1)}(x)\right. \\
&\qquad \left. + (n+\alpha+1)P_{n-1}^{(\alpha+1,\beta+1)}(x)\right] \cdot (\alpha-\beta)P_{n-1}^{(\alpha+1,\beta+1)}(x) \\
&\quad + \left[\frac{n!\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)}\right]^2 \frac{2n+\alpha+\beta+2}{2}(\alpha-\beta)(1-x)P_n^{(\alpha,\beta)}(x)P_n^{(\alpha+1,\beta)}(x),
\end{aligned}
$$

which in view of (2.1) can be written as

$$
r(x) = \frac{1-x^2}{(n+1)^2}\left[\frac{d}{dx}R_{n+1}^{(\alpha,\beta)}(x)\right]^2 - \frac{1-x^2}{(n+\alpha+\beta+1)^2}\left[\frac{d}{dx}R_n^{(\alpha,\beta)}(x)\right]^2 + r_1(x),
$$

where

$$
\begin{aligned}
r_1(x) &= \left[\frac{n!\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)}\right]^2 \cdot \frac{1-x^2}{4} \cdot (\alpha-\beta)P_{n-1}^{(\alpha+1,\beta+1)}(x) \\
&\qquad \cdot \left[(n+\alpha+\beta+2)P_n^{(\alpha+1,\beta+1)}(x) + (n+\alpha+1)P_{n-1}^{(\alpha+1,\beta+1)}(x)\right] \\
&\quad + \left[\frac{n!\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)}\right]^2 \cdot \frac{2n+\alpha+\beta+2}{2} \cdot (\alpha-\beta)\cdot(1-x)P_n^{(\alpha,\beta)}(x)P_n^{(\alpha+1,\beta)}(x).
\end{aligned}
$$

From (3.7), we also have

(3.8)

$$
\begin{aligned}
r_1(x) &= \left[\frac{n!\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)}\right]^2 \cdot \frac{1-x^2}{4} \cdot (\alpha-\beta)\cdot(2n+\alpha+\beta+2)\cdot P_n^{(\alpha+1,\beta)}(x)P_{n-1}^{(\alpha+1,\beta+1)}(x) \\
&\quad + \left[\frac{n!\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)}\right]^2 \cdot (1-x)\cdot(\alpha-\beta)\frac{2n+\alpha+\beta+2}{2}P_n^{(\alpha+1,\beta)}(x)P_n^{(\alpha,\beta)}(x) \\
&= \left[\frac{n!\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)}\right]^2 \cdot (\alpha-\beta)\cdot(2n+\alpha+\beta+2)P_n^{(\alpha+1,\beta)}(x) \\
&\qquad \cdot \left\{\frac{1-x^2}{4}P_{n-1}^{(\alpha+1,\beta+1)}(x) + \frac{1-x}{2}P_n^{(\alpha,\beta)}(x)\right\}.
\end{aligned}
$$

Adding (3.4) and (3.5) gives

$$
(1-x)P_n^{(\alpha+1,\beta)}(x) + (1+x)P_n^{(\alpha,\beta+1)}(x) = 2P_n^{(\alpha,\beta)}(x),
$$

which together with (3.6) yields

$$
\begin{aligned}
&\frac{1-x^2}{4}P_{n-1}^{(\alpha+1,\beta+1)}(x) + \frac{1-x}{2}P_n^{(\alpha,\beta)}(x) \\
&\quad = \frac{1-x^2}{4}\left[P_{n-1}^{(\alpha+1,\beta+1)}(x) + P_n^{(\alpha,\beta+1)}(x)\right] + \frac{(1-x)^2}{4}P_n^{(\alpha+1,\beta)}(x), \\
&\quad = \frac{1-x^2}{4(2n+\alpha+\beta+2)} \cdot \left[(n+\alpha+\beta+2)P_n^{(\alpha+1,\beta+1)}(x) + (n+\alpha+1)P_{n-1}^{(\alpha+1,\beta+1)}(x)\right] \\
&\qquad + \frac{(1-x)^2}{4}P_n^{(\alpha+1,\beta)}(x).
\end{aligned}
$$

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From (3.7), it follows that

$$\frac{1-x^2}{4}P_{n-1}^{(\alpha+1,\beta+1)}(x)+\frac{1-x}{2}P_n^{(\alpha,\beta)}(x)=\frac{1-x^2+(1-x)^2}{4}P_n^{(\alpha+1,\beta)}(x).$$

Inserting this in (3.8), we obtain

$$r_1(x)=\left[\frac{n!\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)}\right]^2(\alpha-\beta)(2n+\alpha+\beta+2)\frac{1-x}{2}[P_n^{(\alpha+1,\beta)}(x)]^2\geq 0.$$

Consequently,

$$\begin{aligned}f_n(x)&\geq[R_n^{(\alpha,\beta)}(x)]^2+\frac{(1-x^2)[\frac{d}{dx}R_n^{(\alpha,\beta)}(x)]^2}{(n+\alpha+\beta+1)^2}\\&=[R_{n+1}^{(\alpha,\beta)}(x)]^2+\frac{(1-x^2)[\frac{d}{dx}R_{n+1}^{(\alpha,\beta)}(x)]^2}{(n+1)^2}+r_1(x)\\&\geq[R_{n+1}^{(\alpha,\beta)}(x)]^2+\frac{(1-x^2)[\frac{d}{dx}R_{n+1}^{(\alpha,\beta)}(x)]^2}{(n+1)(n+\alpha+\beta+2)}+r_1(x)\\&\geq f_{n+1}(x).\end{aligned}$$

■

THEOREM 1. *For all $k\geq K_n^{(1)}$, we have*

$$|\mu_{k,n}(\alpha,\beta)|>|\mu_{k,n+1}(\alpha,\beta)|.$$

PROOF. For simplicity, we write $y_{k,n}=y_{k,n}^{(\alpha,\beta)}$. By Lemma 1,

$$|\mu_{k,n}(\alpha,\beta)|=\sqrt{f_n(y_{k,n})}>\sqrt{f_{n+1}(y_{k,n})}.$$

Since $k\geq K_n^{(1)}$ implies $y_{k,n+1}\leq x_0$, we have from [1, pp. 17–18]

$$y_{k,n}<y_{k,n+1}\leq x_0.$$

Since $f_n(x)$ is decreasing in $-1\leq x\leq x_0$, it follows that

$$\sqrt{f_{n+1}(y_{k,n})}>\sqrt{f_{n+1}(y_{k,n+1})}=|\mu_{k,n+1}(\alpha,\beta)|,$$

thus proving the theorem. ■

4. **Expansions for $g(\theta_{k,n-1})$ and $J_\alpha[f(\theta_{k,n-1})]$.** Let

(4.1) $$\tilde{A}=1-4(\alpha+1)^2\quad\text{and}\quad\tilde{B}=1-4(\beta+1)^2,$$

and recall the notations in (2.8) and (2.11). By (2.27) and Taylor's theorem,

(4.2) $$g(\theta_{k,n-1})=g(\tau_{k,n})-\frac{g'(\tau_{k,n})}{16N^2}[\tilde{A}a(\tau_{k,n})+\tilde{B}b(\tau_{k,n})]+\varepsilon_4,$$

where

$$\varepsilon_4 = g'(\tau_{k,n})\varepsilon_1 + \frac{g''(\zeta_1)}{2}\Big\{-\frac{1}{16N^2}[\tilde{A}a(\tau_{k,n}) + \tilde{B}b(\tau_{k,n})] + \varepsilon_1\Big\}^2, \tag{4.3}$$

ζ_1 being an arbitrary point between $\tau_{k,n}$ and $\theta_{k,n-1}$.

We first estimate the leading term in (4.3). Straightforward differentiation gives

$$g'(\theta) = g(\theta)\Big[-\frac{\alpha}{\theta} + \frac{\alpha+\frac{1}{2}}{2}a(\theta) + \frac{\beta+\frac{1}{2}}{2}b(\theta)\Big], \tag{4.4}$$

where $a(\theta)$ and $b(\theta)$ are as given in (2.22). Hence

$$g'(\theta)\varepsilon_1 = g(\theta)\Big[-\alpha + \frac{\alpha+\frac{1}{2}}{2}\theta a(\theta) + \frac{\beta+\frac{1}{2}}{2}\theta b(\theta)\Big]\frac{\varepsilon_1}{\theta}.$$

From (2.14), it is evident that $0 \le \theta a(\theta) \le 2$ for $0 \le \theta \le \pi$. Since $\theta b(\theta)$ is also bounded on $0 \le \theta \le \pi - \varepsilon$, it follows from (2.28) that

$$g'(\tau_{k,n})\varepsilon_1 = g(\tau_{k,n}) \cdot O(N^{-7/2}) \tag{4.5}$$

for all k satisfying $\theta_{k,n-1} \le \eta_0 < \pi$, where η_0 is given in (1.8).

We next estimate the second term in (4.3). Clearly

$$\sin\frac{\zeta_1}{2} = \sin\frac{\tau_{k,n}}{2} + (\cos\zeta_2)\Big(\frac{\zeta_1}{2} - \frac{\tau_{k,n}}{2}\Big)$$

for some ζ_2 between $\frac{1}{2}\zeta_1$ and $\frac{1}{2}\tau_{k,n}$. By (2.27),

$$\begin{aligned}\sin\frac{\zeta_1}{2} &\ge \Big(\sin\frac{\tau_{k,n}}{2}\Big)\Big\{1 - \frac{1}{2}\Big(\sin\frac{\tau_{k,n}}{2}\Big)^{-1}\Big|\frac{1}{16N^2}[\tilde{A}a(\tau_{k,n}) + \tilde{B}b(\tau_{k,n})] + \varepsilon_1\Big|\Big\}\\ &\ge \Big(\sin\frac{\tau_{k,n}}{2}\Big)\Big\{1 - \frac{\pi}{2\tau_{k,n}}\Big|\frac{1}{16N^2}[\tilde{A}a(\tau_{k,n}) + \tilde{B}b(\tau_{k,n})] + \varepsilon_1\Big|\Big\}.\end{aligned}$$

Since $a(\tau_{k,n})$ and $b(\tau_{k,n})$ are bounded for all $\tau_{k,n} \le \pi - \varepsilon$, we have

$$\sin\frac{\zeta_1}{2} \ge \sin\frac{\tau_{k,n}}{2} \cdot [1 + O(N^{-2})]$$

for all k satisfying $\theta_{k,n-1} \le \eta_0$. In a similar manner, we obtain

$$\cos\frac{\zeta_1}{2} \ge \cos\frac{\tau_{k,n}}{2} \cdot [1 + O(N^{-2})].$$

Therefore

$$\frac{g(\zeta_1)}{g(\tau_{k,n})} = O(1) \tag{4.6}$$

for all k satisfying $\theta_{k,n-1} \le \eta_0$. Again by differentiation

$$g''(\theta) = g(\theta)g_2(\theta),$$

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where

$$g_2(\theta) = \frac{(\alpha+\frac{1}{2})^2}{4}\cot^2\frac{\theta}{2} + \frac{\alpha+\frac{1}{2}}{4}\csc^2\frac{\theta}{2} - \frac{1}{4\theta^2} - \frac{\alpha+\frac{1}{2}}{2\theta}\cot\frac{\theta}{2}$$
$$- \frac{(\alpha+\frac{1}{2})(\beta+\frac{1}{2})}{2} + \frac{\beta+\frac{1}{2}}{2\theta}\tan\frac{\theta}{2} + \frac{(\beta+\frac{1}{2})^2}{4}\tan^2\frac{\theta}{2} + \frac{\beta+\frac{1}{2}}{4}\sec^2\frac{\theta}{2}.$$

From the Maclaurin expansions (2.14) and (2.15), it is easily seen that $\theta^2 g_2(\theta)$ is bounded on $[0, \pi - \varepsilon]$. The second term in (4.3) can be written as

$$\frac{1}{2}g(\zeta_1)g_2(\zeta_1)\zeta_1^2\left(\frac{\tau_{k,n}}{\zeta_1}\right)^2\left\{-\frac{1}{16N^2}\left[\tilde{A}\frac{a(\tau_{k,n})}{\tau_{k,n}} + \tilde{B}\frac{b(\tau_{k,n})}{\tau_{k,n}}\right] + \frac{\varepsilon_1}{\tau_{k,n}}\right\}^2.$$

Since $a(\theta)/\theta$ and $b(\theta)/\theta$ are bounded on $[0, \pi-\varepsilon]$, it is equal to $g(\zeta_1)\cdot O(N^{-4})$ for all $k \le K_n^{(2)}$. Consequently, we have by (4.6)

$$\text{(4.7)} \qquad \frac{1}{2}g''(\zeta_1)\left\{-\frac{1}{16N^2}[\tilde{A}a(\tau_{k,n}) + \tilde{B}b(\tau_{k,n})] + \varepsilon_1\right\}^2 = g(\tau_{k,n})\cdot O(N^{-4}).$$

Inserting (4.5) and (4.7) in (4.3) yields

$$\text{(4.8)} \qquad \varepsilon_4 = g(\tau_{k,n})\cdot O(N^{-7/2}).$$

for all $k \le K_n^{(2)}$. We summarize the above results in the following lemma.

LEMMA 2. *The function $g(\theta)$ defined in (2.7) has the asymptotic approximation*

$$g(\theta_{k,n-1}) = g(\tau_{k,n}) - \frac{g'(\tau_{k,n})}{16N^2}[\tilde{A}a(\tau_{k,n}) + \tilde{B}b(\tau_{k,n})] + \varepsilon_4,$$

where $\tilde{A}$ and $\tilde{B}$ are given in (4.1) and ε_4 satisfies (4.8).

We now turn to the consideration of $J_\alpha[f(\theta_{k,n-1})]$. From (2.23) and (2.27), we have

$$\text{(4.9)} \qquad \begin{aligned} f(\theta_{k,n-1}) &= N\theta_{k,n-1} + \frac{1}{16N}[Aa(\theta_{k,n-1}) + Bb(\theta_{k,n-1})] \\ &= j_{\alpha+1,k} + \frac{1}{16N}[(A-\tilde{A})a(\tau_{k,n}) + (B-\tilde{B})b(\tau_{k,n})] + \varepsilon_5, \end{aligned}$$

where

$$\varepsilon_5 = N\varepsilon_1 + \frac{1}{16N}[Aa'(\zeta) + Bb'(\zeta)]\cdot\left\{-\frac{1}{16N^2}[\tilde{A}a(\tau_{k,n}) + \tilde{B}b(\tau_{k,n})] + \varepsilon_1\right\},$$

ζ being an arbitrary point between $\theta_{k,n-1}$ and $\tau_{k,n}$. Since $a(\theta) = O(\theta)$ and $b(\theta) = O(\theta)$ for $0 \le \theta \le \pi - \varepsilon$, it follows that

$$\text{(4.10)} \qquad \varepsilon_5 = \sqrt{j_{\alpha+1,k}}\cdot O(N^{-7/2})$$

for all $k \le K_n^{(2)}$. By Taylor's theorem,

$$\begin{aligned} J_\alpha[f(\theta_{k,n-1})] &= J_\alpha(j_{\alpha+1,k}) + \frac{J'_\alpha(j_{\alpha+1,k})}{16N}[(A-\tilde{A})a(\tau_{k,n}) + (B-\tilde{B})b(\tau_{k,n})] \\ &\quad + \frac{J''_\alpha(j_{\alpha+1,k})}{512N^2}[(A-\tilde{A})a(\tau_{k,n}) + (B-\tilde{B})b(\tau_{k,n})]^2 + \varepsilon_6, \end{aligned}$$

where

$$\varepsilon_6 = J'_\alpha(j_{\alpha+1,k})\varepsilon_5 + \frac{J''_\alpha(j_{\alpha+1,k})}{16N}[(A-\tilde{A})a(\tau_{k,n}) + (B-\tilde{B})b(\tau_{k,n})]\varepsilon_5 + \frac{1}{2}J''_\alpha(j_{\alpha+1,k})\varepsilon_5^2$$
$$+ \frac{1}{6}J'''_\alpha(j_{\alpha+1,k})\Big\{\frac{1}{16N}[(A-\tilde{A})a(\tau_{k,n}) + (B-\tilde{B})b(\tau_{k,n})] + \varepsilon_5\Big\}^3$$
$$+ \frac{1}{24}J^{(4)}_\alpha(\zeta)\Big\{\frac{1}{16N}[(A-\tilde{A})a(\tau_{k,n}) + (B-\tilde{B})b(\tau_{k,n})] + \varepsilon_5\Big\}^4$$

and ζ is between $j_{\alpha+1,k}$ and $f(\theta_{k,n-1})$. Since

$$J''_\alpha(j_{\alpha+1,k}) = \Big(\frac{\alpha^2}{j^2_{\alpha+1,k}} - \frac{\alpha}{j^2_{\alpha+1,k}} - 1\Big)J_\alpha(j_{\alpha+1,k}), \tag{4.11}$$

$$J'''_\alpha(j_{\alpha+1,k}) = \Big(\frac{1-\alpha}{j_{\alpha+1,k}} + \frac{\alpha^3 - 3\alpha^2 + 2\alpha}{j^3_{\alpha+1,k}}\Big)J_\alpha(j_{\alpha+1,k}), \tag{4.12}$$

and $J^{(4)}_\alpha(\zeta) = O(1)$, it follows from (2.20) that

$$\varepsilon_6 = \sqrt{j_{\alpha+1,k}} \cdot J_\alpha(j_{\alpha+1,k}) \cdot O(N^{-7/2}). \tag{4.13}$$

Here we have again used the fact that $a(\theta) = O(\theta)$ and $b(\theta) = O(\theta)$ on $0 \le \theta \le \pi - \varepsilon$. The following lemma summarizes the above results.

LEMMA 3. *For all $k \le K_n^{(2)}$, we have*

$$J_\alpha[f(\theta_{k,n-1})] = J_\alpha(j_{\alpha+1,k}) + \frac{J'_\alpha(j_{\alpha+1,k})}{16N}[(A-\tilde{A})a(\tau_{k,n}) + (B-\tilde{B})b(\tau_{k,n})]$$
$$+ \frac{J''_\alpha(j_{\alpha+1,k})}{512N^2}[(A-\tilde{A})a(\tau_{k,n}) + (B-\tilde{B})b(\tau_{k,n})]^2 + \varepsilon_6,$$

where ε_6 satisfies (4.13).

5. **Asymptotic monotonicity of $|\mu_{k,n}(\alpha,\beta)|$ when $k \le K_n^{(2)}$.** From the asymptotic expansion (2.34), it is clear that the difference

$$D = \mu_{k,n}(\alpha,\beta) - \mu_{k,n+1}(\alpha,\beta) \tag{5.1}$$

can be written as

$$\begin{aligned} D = {} & \Gamma(\alpha+1)\cdot D_1 \cdot \Big[1 + \frac{g_1(\theta_{k,n-1})}{32N^2} + \varepsilon_3(n)\Big] \\ & \cdot \Big\{J_\alpha[f(\theta_{k,n-1})]\Big[1 + \frac{1}{16N^2}\Big(\frac{A}{6} + \frac{B}{2}\Big)\Big]^{-\alpha} + \varepsilon_2(n)\Big\} \\ & + \Gamma(\alpha+1)g(\theta_{k,n})(N+1)^{-\alpha}D_2 \\ & \cdot \Big\{J_\alpha[f(\theta_{k,n-1})]\Big[1 + \frac{1}{16N^2}\Big(\frac{A}{3} + \frac{B}{2}\Big)\Big]^{-\alpha} + \varepsilon_2(n)\Big\} \\ & + \Gamma(\alpha+1)g(\theta_{k,n})(N+1)^{-\alpha}\Big[1 + \frac{g_1(\theta_{k,n})}{32(N+1)^2} + \varepsilon_2(n+1)\Big]\cdot D_3, \end{aligned} \tag{5.2}$$

where

$$D_1 = g(\theta_{k,n-1})N^{-\alpha} - g(\theta_{k,n})(N+1)^{-\alpha}, \tag{5.3}$$

$$D_2 = \frac{g_1(\theta_{k,n-1})}{32N^2} + \varepsilon_3(n) - \frac{g_1(\theta_{k,n})}{32(N+1)^2} - \varepsilon_3(n+1) \tag{5.4}$$

and

$$\begin{aligned} D_3 = J_\alpha[f(\theta_{k,n-1})]\Big[1 + \frac{1}{16N^2}\Big(\frac{A}{3}+\frac{B}{2}\Big)\Big]^{-\alpha} + \varepsilon_2(n) \\ - J_\alpha[f(\theta_{k,n})]\Big[1 + \frac{1}{16(N+1)^2}\Big(\frac{A}{3}+\frac{B}{2}\Big)\Big]^{-\alpha} - \varepsilon_2(n+1). \end{aligned} \tag{5.5}$$

In the above equations, we have indicated the dependence of ε_2 and ε_3 on n. We shall now estimate each of the values of D_1, D_2 and D_3.

By Lemma 2, we have

$$D_1 = D_{11} + D_{12} + D_{13}, \tag{5.6}$$

where

$$D_{11} = g(\tau_{k,n})N^{-\alpha} - g(\tau_{k,n+1})(N+1)^{-\alpha}, \tag{5.7}$$

$$\begin{aligned} D_{12} = -\frac{g'(\tau_{k,n})}{16}[\tilde{A}a(\tau_{k,n}) + \tilde{B}b(\tau_{k,n})]N^{-\alpha-2} \\ + \frac{g'(\tau_{k,n+1})}{16}[\tilde{A}a(\tau_{k,n+1}) + \tilde{B}b(\tau_{k,n+1})](N+1)^{-\alpha-2} \end{aligned} \tag{5.8}$$

and

$$D_{13} = \varepsilon_4(n)\cdot N^{-\alpha} - \varepsilon_4(n+1)\cdot(N+1)^{-\alpha}. \tag{5.9}$$

We first deal with D_{11}. Put

$$g_3(\theta) = \theta^\alpha g(\theta).$$

Then

$$D_{11} = (j_{\alpha+1,k})^{-\alpha}[g_3(\tau_{k,n}) - g_3(\tau_{k,n+1})] = (j_{\alpha+1,k})^{-\alpha} g_3'(\zeta)(\tau_{k,n} - \tau_{k,n+1})$$

for some $\zeta \in (\tau_{k,n+1}, \tau_{k,n})$. Since by (2.14) and (2.15)

$$\begin{aligned} g_3'(\theta) &= \theta^{\alpha-1} g(\theta)\Big\{\Big(\alpha+\frac{1}{2}\Big)\Big(1 - \frac{\theta}{2}\cot\frac{\theta}{2}\Big) + \Big(\beta+\frac{1}{2}\Big)\frac{\theta}{2}\tan\frac{\theta}{2}\Big\} \\ &\geq \frac{\alpha+\frac{1}{2}+3(\beta+\frac{1}{2})}{12}\theta^{\alpha+1}g(\theta) \geq 0, \quad 0 \leq \theta \leq \eta_0, \end{aligned}$$

$g_3(\theta)$ is increasing and

$$D_{11} \geq \frac{(\alpha+\frac{1}{2})+3(\beta+\frac{1}{2})}{12}\cdot\frac{j_{\alpha+1,k}^2}{N(N+1)^{\alpha+2}}\cdot g(\tau_{k,n+1}) > 0. \tag{5.10}$$

To estimate D_{12}, we put

$$g_4(\theta) = \theta^2\Big[-\frac{\alpha+\frac{1}{2}}{2}\cot\frac{\theta}{2} + \frac{\beta+\frac{1}{2}}{2}\tan\frac{\theta}{2} + \frac{1}{2\theta}\Big]\cdot[\tilde{A}a(\theta)+\tilde{B}b(\theta)],$$

From (4.4) and (5.8), we have

$$D_{12} = -\frac{D_{11}}{16}\cdot\frac{g_4(\tau_{k,n})}{\tau_{k,n}^2}N^{-2} - \frac{g(\tau_{k,n+1})}{16j_{\alpha+1,k}^2}(N+1)^{-\alpha}[g_4(\tau_{k,n}) - g_4(\tau_{k,n+1})]. \tag{5.11}$$

Since

$$g_4'(\theta) = -\Big(\frac{\tilde{A}}{3}+\tilde{B}\Big)\alpha\theta + \theta^3 g_5(\theta), \tag{5.12}$$

where $g_5(\theta)$ can be shown to be a continuous function on $[0,\eta_0]$, $g_4'(\theta) = O(\theta)$ and $g_4(\theta) = O(\theta^2)$. Therefore it follows from (5.11) that

$$D_{12} = D_{11}\cdot O(N^{-2}) - \frac{g(\tau_{k,n+1})}{16j_{\alpha+1,k}^2(N+1)^\alpha}\cdot g_4'(\zeta_1)\cdot(\tau_{k,n}-\tau_{k,n+1})$$

for some $\zeta_1 \in (\tau_{k,n+1},\tau_{k,n})$. Inserting (5.12) in the last equation gives

$$\begin{aligned} D_{12} &= D_{11}\cdot O(N^{-2}) - \frac{g(\tau_{k,n+1})}{16j_{\alpha+1,k}^2(N+1)^\alpha}\Big[-\Big(\frac{\tilde{A}}{3}+\tilde{B}\Big)\alpha\zeta_1 + \zeta_1^3 g_5(\zeta_1)\Big]\frac{j_{\alpha+1,k}}{N(N+1)} \\ &= D_{11}\cdot O(N^{-2}) + \frac{\alpha}{16}\Big(\frac{1}{3}\tilde{A}+\tilde{B}\Big)\frac{g(\tau_{k,n+1})}{N(N+1)^{\alpha+2}} + \varepsilon_7, \end{aligned} \tag{5.13}$$

where

$$\begin{aligned} \varepsilon_7 &= \Big(\frac{1}{3}\tilde{A}+\tilde{B}\Big)\frac{g(\tau_{k,n+1})}{16j_{\alpha+1,k}^2(N+1)^\alpha}\cdot\alpha\cdot(\zeta_1-\tau_{k,n+1})\frac{j_{\alpha+1,k}}{N(N+1)} \\ &\quad - \frac{g(\tau_{k,n+1})}{16j_{\alpha+1,k}^2(N+1)^\alpha}\cdot\zeta_1^3 g_5(\zeta_1)\frac{j_{\alpha+1,k}}{N(N+1)} \\ &= g(\tau_{k,n+1})j_{\alpha+1,k}^2\cdot O(N^{-\alpha-7/2}). \end{aligned}$$

The estimate of D_{13} follows immediately from (4.8), and we have

$$D_{13} = g(\tau_{k,n+1})j_{\alpha+1,k}^2 O(N^{-\alpha-7/2}); \tag{5.14}$$

cf. the argument leading to (4.6). A combination of (5.10), (5.13) and (5.14) yields the following lemma.

LEMMA 4. *For all $k \le K_n^{(2)}$, the difference D_1 in (5.3) satisfies*

$$D_1 \ge \frac{(\alpha+\frac{1}{2})+3(\beta+\frac{1}{2})}{12}\cdot\frac{j_{\alpha+1,k}^2}{N(N+1)^{\alpha+2}}g(\tau_{k,n+1}) + \frac{\alpha}{16}\cdot\Big(\frac{1}{3}\tilde{A}+\tilde{B}\Big)\frac{g(\tau_{k,n+1})}{N(N+1)^{\alpha+2}} + \varepsilon_8,$$

where

$$\varepsilon_8 = j_{\alpha+1,k}^2\cdot g(\tau_{k,n+1})\cdot O(N^{-\alpha-7/2}).$$

To estimate D_2 in (5.4), we return to (2.32) and observe that

$$g_1'(\theta) = A\left[\frac{a'(\theta)\theta - a(\theta)}{\theta^2} - a''(\theta)\right] + B\left[\frac{b'(\theta)\theta - b(\theta)}{\theta^2} - b''(\theta)\right],$$

where $a(\theta)$ and $b(\theta)$ are given in (2.22). From (2.14) and (2.15), it is easily seen that $g_1'(\theta)$ is bounded on $[0, \eta_0]$. Hence by (2.27) and the Mean-Value Theorem,

$$g_1(\theta_{k,n-1}) = g_1(\tau_{k,n}) + \varepsilon_9, \tag{5.15}$$

where

$$\varepsilon_9 = g_1'(\zeta)\left\{-\frac{1}{16N^2}[\tilde{A}a(\tau_{k,n}) + \tilde{B}b(\tau_{k,n})] + \varepsilon_1\right\} = O(N^{-2})$$

for all $k \le K_n^{(2)}$. Put

$$\tilde{g}_1(\theta) = \theta^2 g_1(\theta).$$

Differentiation gives

$$\tilde{g}_1'(\theta) = \theta^3\left\{\frac{A}{\theta}\left[\frac{a(\theta)}{\theta^2} - \frac{a'(\theta)}{\theta} - a''(\theta)\right] + \frac{B}{\theta}\left[\frac{b(\theta)}{\theta^2} - \frac{b'(\theta)}{\theta} - b''(\theta)\right]\right\}.$$

Since each of the two terms inside the curly bracket is in absolute value an increasing function on $[0, \pi)$, we have

$$\tilde{g}_1'(\theta) = O(\theta^3), \quad 0 \le \theta \le \eta_0; \tag{5.16}$$

cf. (2.14) and (2.15). By (5.15), D_2 can be written as

$$\begin{aligned} D_2 &= \frac{1}{32 j_{\alpha+1,k}^2}[\tilde{g}_1(\tau_{k,n}) - \tilde{g}_1(\tau_{k,n+1})] + \frac{1}{32}\left[\frac{\varepsilon_9(n)}{N^2} - \frac{\varepsilon_9(n+1)}{(N+1)^2}\right] + \varepsilon_3(n) - \varepsilon_3(n+1) \\ &= \frac{1}{32 j_{\alpha+1,k}^2} \cdot \tilde{g}_1'(\zeta)\frac{j_{\alpha+1,k}}{N(N+1)} + \frac{1}{32}\left[\frac{\varepsilon_9(n)}{N^2} - \frac{\varepsilon_9(n+1)}{(N+1)^2}\right] + \varepsilon_3(n) - \varepsilon_3(n+1) \end{aligned}$$

for some $\zeta \in \big(j_{\alpha+1,k}/(N+1), j_{\alpha+1,k}/N\big)$. Therefore it follows from (5.16) that

$$D_2 = j_{\alpha+1,k}^2 \cdot O(N^{-4}). \tag{5.17}$$

We finally come to the estimation of D_3 given in (5.5), which we shall rewrite as

$$D_3 = D_{31} \cdot J_\alpha[f(\theta_{k,n-1})] + \left[1 + \frac{1}{16(N+1)^2}\left(\frac{A}{6} + \frac{B}{2}\right)\right]^{-\alpha} D_{32} + D_{33}, \tag{5.18}$$

where

$$D_{31} = \left[1 + \frac{1}{16N^2}\left(\frac{A}{6} + \frac{B}{2}\right)\right]^{-\alpha} - \left[1 + \frac{1}{16(N+1)^2}\left(\frac{A}{6} + \frac{B}{2}\right)\right]^{-\alpha}, \tag{5.19}$$

$$D_{32} = J_\alpha[f(\theta_{k,n-1})] - J_\alpha[f(\theta_{k,n})] \tag{5.20}$$

and

$$D_{33} = \varepsilon_2(n) - \varepsilon_2(n+1).$$

It is easily seen that

$$D_{31} = -\frac{\alpha}{16}\left(\frac{A}{3} + B\right) \cdot \frac{1}{N(N+1)^2} + O(N^{-4}) \tag{5.21}$$

and by (2.30)

$$D_{33} = O(N^{-7/2}) = j_{\alpha+1,k}^2 \cdot J_\alpha(j_{\alpha+1,k}) \cdot O(N^{-7/2}). \tag{5.22}$$

For D_{32}, we have the following result.

LEMMA 5. *For* $k \leq K_n^{(2)}$,

$$D_{32} = \frac{\alpha}{6}\Big[\Big(\alpha + \frac{1}{2}\Big) + 3\Big(\beta + \frac{1}{2}\Big)\Big]\frac{J_\alpha(j_{\alpha+1,k})}{N(N+1)^2} + \varepsilon_{10},$$

where

$$\varepsilon_{10} = j_{\alpha+1,k}^2 \cdot J_\alpha(j_{\alpha+1,k}) \cdot O(N^{-7/2}). \tag{5.23}$$

PROOF. Let

$$g_6(\theta) = \theta[(A - \tilde{A})a(\theta) + (B - \tilde{B})b(\theta)]$$

and

$$g_7(\theta) = \theta^2[(A - \tilde{A})a(\theta) + (B - \tilde{B})b(\theta)]^2.$$

Differentiation gives

$$g_6'(\theta) = 8\theta\Big[\frac{1}{3}\Big(\alpha + \frac{1}{2}\Big) + \Big(\beta + \frac{1}{2}\Big)\Big] + 8\theta^3\Big[\Big(\alpha + \frac{1}{2}\Big)g_8(\theta) + \Big(\beta + \frac{1}{2}\Big)g_9(\theta)\Big] \tag{5.24}$$

and

$$\begin{aligned} g_7'(\theta) = {} & 128\theta^3\Big[\Big(\alpha + \frac{1}{2}\Big)\frac{a(\theta)}{\theta} + \Big(\beta + \frac{1}{2}\Big)\frac{b(\theta)}{\theta}\Big] \\ & \cdot \Big\{\Big(\alpha + \frac{1}{2}\Big)\Big[\frac{a(\theta)}{\theta} + a'(\theta)\Big] + \Big(\beta + \frac{1}{2}\Big)\Big[\frac{b(\theta)}{\theta} + b'(\theta)\Big]\Big\}, \end{aligned} \tag{5.25}$$

where

$$g_8(\theta) = \frac{1}{\theta^2}\Big[\frac{a(\theta)}{\theta} + a'(\theta) - \frac{1}{3}\Big],$$

and

$$g_9(\theta) = \frac{1}{\theta^2}\Big[\frac{b(\theta)}{\theta} + b'(\theta) - 1\Big].$$

Using (2.14) and (2.15), it is readily shown that both $g_8(\theta)$ and $g_9(\theta)$ are bounded on $[0, \eta_0]$. By Lemma 3 and (5.20), we have

$$\begin{aligned} D_{32} = {} & \frac{J_\alpha'(j_{\alpha+1,k})}{16 j_{\alpha+1,k}}[g_6(\tau_{k,n}) - g_6(\tau_{k,n+1})] \\ & + \frac{J_\alpha''(j_{\alpha+1,k})}{512 j_{\alpha+1,k}^2}[g_7(\tau_{k,n}) - g_7(\tau_{k,n+1})] + \varepsilon_6(n) - \varepsilon_6(n+1). \end{aligned} \tag{5.26}$$

The first term on the right of (5.26) is equal to

$$\frac{\alpha}{16}\frac{J_\alpha(j_{\alpha+1,k})}{j_{\alpha+1,k}^2}g_6'(\zeta_2)\frac{j_{\alpha+1,k}}{N(N+1)}$$

on account of (2.20) and (2.11), where

$$\zeta_2 = \zeta_1\tau_{k,n} + (1 - \zeta_1)\tau_{k,n+1} = \frac{j_{\alpha+1,k}}{N+1} + \zeta_1\frac{j_{\alpha+1,k}}{N(N+1)}, \quad 0 < \zeta_1 < 1.$$

Applying this and (5.24) to (5.26) gives

$$D_{32} = \frac{\alpha}{6}\Big[\Big(\alpha+\frac{1}{2}\Big)+3\Big(\beta+\frac{1}{2}\Big)\Big]\frac{J_\alpha(j_{\alpha+1,k})}{N(N+1)^2}+\varepsilon_{10},$$

where

$$\begin{aligned}\varepsilon_{10} = {} & \frac{\alpha}{2}\cdot\frac{\zeta_1}{N^2(N+1)^2}\Big[\Big(\alpha+\frac{1}{2}\Big)+3\Big(\beta+\frac{1}{2}\Big)\Big]J_\alpha(j_{\alpha+1,k}) \\ & +\frac{\alpha}{2}\cdot j^2_{\alpha+1,k}\cdot J_\alpha(j_{\alpha+1,k})\cdot\frac{(N+\zeta_1)^3}{N^4(N+1)^4}\Big[\Big(\alpha+\frac{1}{2}\Big)g_8(\zeta_2)+\Big(\beta+\frac{1}{2}\Big)g_9(\zeta_2)\Big] \\ & +\frac{J''_\alpha(j_{\alpha+1,k})}{512 j^2_{\alpha+1,k}}\cdot g'_7(\zeta_3)\cdot\frac{j_{\alpha+1,k}}{N(N+1)}+\varepsilon_6(n)-\varepsilon_6(n+1),\end{aligned}$$

$\zeta_3 \in (\tau_{k,n+1}, \tau_{k,n})$. The desired order estimate (5.23) now follows from (4.11), (4.13) and (5.25). ∎

A combination of (5.18), (5.21), (5.22) and Lemma 5 gives

(5.27)

$$\begin{aligned}D_3 = {} & \Big[-\frac{\alpha}{16}\Big(\frac{A}{3}+B\Big)\frac{1}{N(N+1)^2}+O(N^{-4})\Big]\cdot J_\alpha[f(\theta_{k,n-1})] \\ & +\Big[1+\frac{1}{16(N+1)^2}\Big(\frac{A}{6}+\frac{B}{2}\Big)\Big]^{-\alpha}\cdot\Big\{\frac{\alpha}{6}\Big[\Big(\alpha+\frac{1}{2}\Big)+3\Big(\beta+\frac{1}{2}\Big)\Big]\frac{J_\alpha(j_{\alpha+1,k})}{N(N+1)^2}+\varepsilon_{10}\Big\} \\ & +j^2_{\alpha+1,k}\cdot J_\alpha(j_{\alpha+1,k})\cdot O(N^{-7/2}).\end{aligned}$$

Since

$$\Big[1+\frac{1}{16(N+1)^2}\Big(\frac{A}{6}+\frac{B}{2}\Big)\Big]^{-\alpha} = 1+O(N^{-2})$$

and

$$J_\alpha[f(\theta_{k,n-1})] = J_\alpha(j_{\alpha+1,k})[1+O(N^{-1})]$$

by Lemma 3, we obtain

$$\begin{aligned}D_3 = {} & \Big\{-\frac{\alpha}{16N(N+1)^2}\Big(\frac{A}{3}+B\Big)+\frac{\alpha}{16N(N+1)^2}\Big[\Big(\alpha+\frac{1}{2}\Big)+3\Big(\beta+\frac{1}{2}\Big)\Big]\Big\} \\ & \cdot J_\alpha(j_{\alpha+1,k})+j^2_{\alpha+1,k}J_\alpha(j_{\alpha+1,k})\cdot O(N^{-7/2}).\end{aligned} \tag{5.28}$$

We now return to (5.2), and consider the quantity $[J_\alpha(j_{\alpha+1,k})]^{-1}D$. First we replace D_1, D_2 and D_3 by their respective estimates given in Lemma 4, (5.17) and (5.28). To the resulting expression, we then apply Lemmas 2 and 3. This leads to the inequality

$$\begin{aligned}[J_\alpha(j_{\alpha+1,k})]^{-1}D \geq {} & \frac{\Gamma(\alpha+1)}{N(N+1)^{\alpha+2}}\cdot\Big\{\frac{(\alpha+\frac{1}{2})+3(\beta+\frac{1}{2})}{12}+\frac{\alpha}{16j^2_{\alpha+1,k}}\cdot\Big(\frac{\tilde{A}}{3}+\tilde{B}\Big) \\ & -\frac{\alpha}{16j^2_{\alpha+1,k}}\Big(\frac{A}{3}+B\Big)+\frac{\alpha}{6j^2_{\alpha+1,k}}\Big[\Big(\alpha+\frac{1}{2}\Big)+3\Big(\beta+\frac{1}{2}\Big)\Big]\Big\} \\ & \cdot j^2_{\alpha+1,k}\cdot g(\tau_{k,n+1})+\varepsilon_{10} \\ = {} & \frac{\Gamma(\alpha+1)}{N(N+1)^{\alpha+2}}\frac{(\alpha+\frac{1}{2})+3(\beta+\frac{1}{2})}{12}j^2_{\alpha+1,k}g(\tau_{k,n+1})+\varepsilon_{11},\end{aligned}$$

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where

$$\varepsilon_{11} = j^2_{\alpha+1,k} g(\tau_{k,n+1}) \cdot O(N^{\alpha-\frac{7}{2}}).$$

THEOREM 2. *For all $k \le K_n^{(2)}$ and for all n sufficiently large, we have*

$$|\mu_{k,n}(\alpha,\beta)| > |\mu_{k,n+1}(\alpha,\beta)|.$$

PROOF. Let

$$E = \frac{(-1)^k}{12} J_\alpha(j_{\alpha+1,k}) \frac{\Gamma(\alpha+1)}{N(N+1)^{\alpha+2}} \Big[\Big(\alpha+\frac{1}{2}\Big)+3\Big(\beta+\frac{1}{2}\Big)\Big] j^2_{\alpha+1,k} \cdot g(\tau_{k,n+1}).$$

We first note that $J_\alpha(j_{\alpha+1,k}) = J'_{\alpha+1}(j_{\alpha+1,k})$, and that the slope of $J_{\alpha+1}(x)$ alters in sign at its zeros $j_{\alpha+1,k}$. Since $J'_{\alpha+1}(j_{\alpha+1,1}) < 0$, it follows that

$$\operatorname{sgn}\{J_\alpha(j_{\alpha+1,k})\} = (-1)^k.$$

Consequently, $E > 0$. We next observe that

$$\operatorname{sgn}\{\mu_{k,n}(\alpha,\beta)\} = (-1)^k,$$

which can be proved in a manner similar to that given in [12, Theorem 6a]; *cf.* (1.6). Therefore

$$|\mu_{k,n}(\alpha,\beta)| - |\mu_{k,n+1}(\alpha,\beta)| = (-1)^k D \ge E\{1+O(N^{-\frac{1}{2}})\}. \qquad \blacksquare$$

6. **Proof of (1.9).** By Theorem 1, we know that conjecture (1.4) is true for all $k \ge K_n^{(1)}$. By Theorem 2, we also know that (1.4) holds in the asymptotic sense when $k \le K_n^{(2)}$. Thus, to show that (1.4) is asymptotically true for all $k = 1,\ldots,n$, it suffices to prove (1.9):

$$K_n^{(1)} \le K_n^{(2)} \text{ for all sufficiently large } n. \tag{6.1}$$

We shall establish this by contradiction. Suppose that there exists a sequence of positive integers $\{\ell_m\}$ such that

$$\lim_{m\to\infty} \ell_m = +\infty$$

and

$$K_{\ell_m}^{(1)} > K_{\ell_m}^{(2)}.$$

Then we can choose a sequence of positive integer k_m such that either

$$K_{\ell_m}^{(1)} > k_m \ge K_{\ell_m}^{(2)}$$

or

$$K_{\ell_m}^{(1)} \ge k_m > K_{\ell_m}^{(2)}.$$

Since $y_{k,n+1}^{(\alpha,\beta)} = \cos\theta_{k,n}^{(\alpha+1,\beta+1)}$ and $\theta_{k,n} = \theta_{k,n}^{(\alpha+1,\beta+1)}$ by (2.8), we obtain from (1.7) and (1.8)

$$\theta_{k_m-1,\ell_m} < \cos^{-1} x_0, \quad \theta_{k_m+1,\ell_m-1} > \eta_0. \tag{6.2}$$

In view of the wellknown asymptotic approximation [7, p. 247]

$$j_{\alpha,k} \sim \left(k + \frac{1}{2}\alpha - \frac{1}{4}\right)\pi - \frac{4\alpha^2 - 1}{8(k + \frac{1}{2}\alpha - \frac{1}{4})\pi} - \cdots$$

equation (2.27) gives

$$\theta_{k_m-1,\ell_m} = \frac{(k_m - 1)\pi + O(1)}{\ell_m + 1}$$

and

$$\theta_{k_m+1,\ell_m-1} = \frac{(k_m + 1)\pi + O(1)}{\ell_m}.$$

Consequently,

$$\theta_{k_m-1,\ell_m} - \theta_{k_m+1,\ell_m-1} = \frac{k_m\pi}{\ell_m(\ell_m + 1)} + \frac{O(1)}{\ell_m}$$

and

$$\lim_{m\to\infty} (\theta_{k_m-1,\ell_m} - \theta_{k_m+1,\ell_m-1}) = 0. \tag{6.3}$$

But, from (6.2) and (1.8), we have

$$\theta_{k_m+1,\ell_m-1} - \theta_{k_m-1,\ell_m} > \eta_0 - \cos^{-1} x_0 = \frac{1}{4}\frac{2\beta + 1}{\alpha + \beta + 1}.$$

This contradicts (6.3), and therefore (6.1) holds.

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